Chap 4

$$\gamma = \frac{1}{(1-\beta^2)^{1/2}} \qquad (1-\beta^2) = \frac{1}{\gamma^2}$$

$$\gamma \frac{d}{dt}\left[\gamma(c, \underset{\sim}{v})\right] = \gamma^4 \frac{\underset{\sim}{v}\cdot\underset{\sim}{a}}{c^2}\left[(c, \underset{\sim}{v})\right] + \gamma^2 \frac{d}{dt}(c, \underset{\sim}{v})$$

$$= \gamma^4 \frac{\underset{\sim}{v}\cdot\underset{\sim}{a}}{c^2}(c, \underset{\sim}{v}) + \gamma^2 (0, \underset{\sim}{a})$$

$$= \gamma^2\left(\gamma^2 \frac{\underset{\sim}{v}\cdot\underset{\sim}{a}}{c}, \underset{\sim}{a} + \gamma^2(\underset{\sim}{v}\cdot\underset{\sim}{a})\frac{\underset{\sim}{v}}{c^2}\right)$$

$$A_\alpha A^\alpha = -\gamma^4\left(\gamma^4 \frac{(\underset{\sim}{v}\cdot\underset{\sim}{a})^2}{c^2}\right) + \gamma^4\left(a^2 + 2(\underset{\sim}{a}\cdot\underset{\sim}{v})\frac{\gamma^2(\underset{\sim}{v}\cdot\underset{\sim}{a})}{c^2} + \gamma^4(\underset{\sim}{a}\cdot\underset{\sim}{v})^2\frac{v^2}{c^2}\right.$$

$$= \gamma^4\left\{-\gamma^4\frac{(\underset{\sim}{a}\cdot\underset{\sim}{v})^2}{c^2} + a^2 + 2\gamma^2\frac{(\underset{\sim}{a}\cdot\underset{\sim}{v})^2}{c^2} + \gamma^4(\underset{\sim}{a}\cdot\underset{\sim}{v})^2\frac{v^2}{c^4}\right\}$$

$$= \gamma^4\left\{a^2 + \frac{(\underset{\sim}{a}\cdot\underset{\sim}{v})^2}{c^2}\gamma^2\left[\gamma^2\beta^2 - \gamma^2 + 2\right]\right\}$$

$$= \gamma^4\left\{a^2 + \frac{\gamma^2(\underset{\sim}{a}\cdot\underset{\sim}{v})^2}{c^2}\left[\gamma^2(\beta^2-1) + 2\right]\right\}$$

$$\underline{-\gamma^2(1-\beta^2) + 2}$$
$$-1 + 2$$

$$= \gamma^4\left\{a^2 + \frac{\gamma^2(\underset{\sim}{a}\cdot\underset{\sim}{v})^2}{c^2}\right\}$$

Section 40 Material Stress - Energy Tensor

$$(\rho v^i)_{,t} + \Pi^{ij}{}_{;j} = f^i$$

$$V^\alpha = (-c, \underset{\sim}{v}^{(1)}, ~~v^{(2)}, v^{(3)}~~)$$

$$V^\beta = (-c, \underset{\sim}{v})$$

$$(V^\alpha V^\beta)_0 = (+c^2, 0)$$

$$(P^{\alpha\beta})_0 = \eta^{\alpha\beta} + c^{-2}(V^\alpha V^\beta)_0 = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$M_0 = \begin{pmatrix} \rho_{00} c^2 & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$$

$$\rho_{00} = \rho_0 (1 + e/c^2) = \rho_0 + \rho_0 e/c^2$$

$$\rho_{00} c^2 = \rho_0 c^2 + \rho_0 e$$

$$M_0 = ~~\rho_{00}~~ \rho_{00} (V^\alpha V^\beta)_0 + p\left[\eta^{\alpha\beta} + c^{-2}(V^\alpha V^\beta)_0\right]$$

$$M_0 = (V^\alpha V^\beta)_0 (\rho_{00} + p/c^2) + \eta^{\alpha\beta} p$$

$$V^\alpha = \gamma(c, \underset{\sim}{v}) \qquad V_\alpha = \gamma(-c, \underset{\sim}{v})$$

$$V^i = \gamma v^i \qquad V^i V^j = \gamma^2 v^i v^j$$

$$\rho_{000} = \rho_{00} + p/c^2 = \rho_0(1+e/c^2) + p/c^2$$
$$= \rho_0(1 + e/c^2 + (p/\rho_0)/c^2)$$

$$\rho_{000} = \rho_0(1 + h/c^2)$$

$$M^{\alpha\beta} = \rho_{000} V^\alpha V^\beta + p\eta^{\alpha\beta}$$

$$= \rho_0(1 + e/c^2 + P/\rho_0 c^2) V^\alpha V^\beta + p\eta^{\alpha\beta}$$

$$M^{ij} = \rho_0(1 + e/c^2 + P/\rho_0 c^2)\gamma^2 v^i v^j + p\,\cancel{\eta}\,\delta^{ij}$$

$$= \gamma(\gamma\rho_0) v^i v^j (1 + e/c^2 + P/\rho_0 c^2) + p\delta^{ij}$$

$$= \gamma\rho v^i v^j + \gamma\rho(v^i v^j/c^2)(e + P/\rho_0) + p\delta^{ij}$$

$$\gamma^2 = \frac{1}{(1-\beta^2)}$$

$$\gamma^2 - 1 = \frac{1}{(1-\beta^2)} - 1 = \frac{1-(1-\beta^2)}{1-\beta^2} = \frac{\beta^2}{1-\beta^2}$$

$$\beta^2$$

$$\gamma = (1 - v^2/c^2)^{-1/2}$$
$$= 1 + \frac{1}{2}\frac{v^2}{c^2} + \cdots$$

$$M^{0j} = (\rho_{00} + P/c^2)(V^0 \gamma v^i) + \cancel{p(-1)\delta^{0i}}$$

$$= \rho_0(1 + e/c^2 + P/\rho_0 c^2)\gamma^2 c v^i$$

$$= \gamma c \rho v^i (1 + e/c^2 + P/\rho_0 c^2)$$

$$M^{00} = (\rho_{00} + p/c^2)\gamma^2 c^2 + p(-1)$$
$$= \gamma^2 c^2 \rho_0 (1 + e/c^2) + \gamma^2 c^2 p/c^2 - p = \gamma^2 c^2 \rho_0 (1 + e/c^2)$$
$$+(\gamma^2 - 1)p$$

$$M^{00} = \gamma^2 c^2 \rho_0 (1 + e/c^2) + \beta^2 \gamma^2 p$$

$$= \gamma^2 \rho_0 (c^2 + e + p/\rho_0 \beta^2)$$

$$= \gamma^2 \rho_0 (c^2 + e + \beta^2 p/\rho_0)$$

$$= (1 + \tfrac{1}{2} v^2/c^2) \rho (c^2 + e + (p/\rho_0) v^2/c^2)$$

$$= \rho c^2 + \rho e + \tfrac{1}{2} \rho v^2 +$$

Section 42

$$F^\mu \equiv \Phi^\mu/\delta V_0 = (\gamma/\delta V_0)(\underset{\sim}{\phi}\cdot\underset{\sim}{v}/c, \underset{\sim}{\phi})$$

$$F^\mu = (\underset{\sim}{f}\cdot\underset{\sim}{v}/c, \underset{\sim}{f})$$

$$M^{\alpha\beta}{}_{;\beta} = F^\alpha$$

$$(\rho_{000} V^\alpha V^\beta + p g^{\alpha\beta})_{;\beta} = F^\alpha$$

Cartesian $(\rho_{000} V^\alpha V^\beta + p\eta^{\alpha\beta})_{;\beta} = F^\alpha$

$$M^{ij} = \rho_0 \gamma^2 v^i v^j (1 + e/c^2 + p/\rho_0 c^2) + p\delta^{ij}$$

$$M^{0i} = M^{i0} = \gamma^2 \rho_0 c v^i (1 + e/c^2 + p/\rho_0 c^2)$$

$$M^{00} = \gamma^2 \rho_0 (c^2 + e + \beta^2 p/\rho_0)$$

or $M^{ij} = \rho_{000}\gamma^2 v^i v^j + p\delta^{ij} = \rho_1 v^i v^j + p\delta^{ij}$

$$M^{i0} = M^{0i} = \gamma^2 c v^i \rho_{000} = \rho_1 c v^i$$

$$M^{00} = \rho_{000}(+\gamma^2 c^2) - p = c^2\rho_1 - p$$

$$(M^{0\beta})_{;\beta} = (M^{00})_{,0} + (M^{0i})_{;i} = \frac{1}{c}(M^{00})_{,t} + (M^{0i})_{;i}$$

$$= \frac{1}{c}(c^2\rho_1 - p)_{,t} + (\rho_1 c v^i)_{;i} = \underset{\sim}{f}\cdot\underset{\sim}{v}/c$$

$$= (c\rho_1 - p/c)_{,t} + (\rho_1 c v^i)_{;i} = c\left[(\rho_1 - p/c^2)_{,t} + (\rho_1 v^i)_{;i}\right] = \underset{\sim}{f}\cdot\underset{\sim}{v}/c$$

$$(m^{i\beta})_{,\beta} = (m^{i0})_{,ct} + (m^{ij})_{,j}$$

$$= \frac{1}{c}(\rho_1 c v^i)_{,t} + (\rho_1 v^i v^j + p\delta^{ij})_{,j}$$

(42.4a) $$= (\rho_1 v^i)_{,t} + (\rho_1 v^i v^j + p\delta^{ij})_{,j} = f^i$$

$$\rho_1 = \gamma^2 \rho_{000} = \gamma^2 \rho_0 (1 + e/c^2 + p/\rho_0 c^2)$$

$$= \gamma^2(\rho_0 + \rho_0 e/c^2 + p/c^2)$$

$$c^2\left[(\rho_1 - p/c^2)_{,t} + (\rho_0 v^j)_{,j}\right] = \underset{\sim}{f} \cdot \underset{\sim}{v}$$

42.5) $$\Rightarrow \left[\gamma^2(\rho_0 c^2 + \rho_0 e + p) - p\right]_{,t} + \left[\gamma^2(\rho_0 c^2 + \rho_0 e + p)v^j\right]_{,j} = \underset{\sim}{f} \cdot \underset{\sim}{v}$$

$$(\rho_1 v^i)_{,t} + (\rho_1 v^i v^j + p\delta^{ij})_{,j} = f^i$$

$$\left[\gamma^2(\rho_0 c^2 + \rho_0 e + p)v^i\right]_{,t} + \left[\gamma^2(\rho_0 c^2 + \rho_0 e + p)v^i v^j + c^2 p\delta^{ij}\right]_{,j} = f^i c^2$$

42.6) $$'' \quad + \left[\gamma^2(\rho_0 c^2 + \rho_0 e + p)v^i v^j\right]_{,j} + c^2 p_{,i} = f^i c^2$$

$$\eta^{\alpha\beta} V_\alpha = (-V^0, +\underset{\sim}{V})$$

Gas Energy Eqn

$$(42.2) \quad (\rho_{000} V^\alpha V^\beta + p\eta^{\alpha\beta})_{;\beta} = F^\alpha$$

$$V_\alpha (\rho_{000} V^\alpha V^\beta + p\eta^{\alpha\beta})_{;\beta} = V_\alpha F^\alpha$$

$$= V_\alpha V^\alpha (\rho_{000} V^\beta)_{;\beta} + V_\alpha \rho_{000} V^\beta V^\alpha_{;\beta} + V_\alpha (p g^{\alpha\beta})_{;\beta} = V_\alpha F^\alpha$$

$$= -c^2 (\rho_{000} V^\beta)_{;\beta} + \rho_{000} V^\beta V_\alpha V^\alpha_{;\beta} + \eta^{\alpha\beta} V_\alpha P_{;\beta} = V_\alpha F^\alpha$$

$$(42.8) \quad = -c^2 (\rho_{000} V^\alpha)_{;\alpha} + V^\alpha P_{;\alpha} = V_\alpha F^\alpha$$

$$(\rho_0 V_\alpha)_{;\alpha} (c^2 + e + p/\rho_0) = 0$$

$$-c^2 (\rho_{000} V^\alpha)_{;\alpha} + (\rho_0 V_\alpha)_{;\alpha} (c^2 + e + p/\rho_0) + V^\alpha P_{,\alpha} = V_\alpha F^\alpha$$

$$= -[\rho_0 (c^2 + e + p/\rho_0) V^\alpha)_{;\alpha} + (c^2 + e + p/\rho_0)(\rho_0 V_\alpha)_{;\alpha} + V^\alpha P_{,\alpha} = V_\alpha F^\alpha$$

$$= -\cancel{(c^2 + e + p/\rho_0)(\rho_0 V^\alpha)_{;\alpha}} - \rho_0 V^\alpha (c^2 + e + p/\rho_0)_{;\alpha} + \cancel{(c^2 + e + p/\rho_0)(\rho_0 V_\alpha)_{;\alpha}} + V^\alpha P_{,\alpha} = V_\alpha F^\alpha$$

$$-\rho_0 V^\alpha (\cancel{c^2} + e + p/\rho_0)_{;\alpha} + V^\alpha P_{,\alpha} = V_\alpha F^\alpha$$

$$-\rho_0 V^\alpha e_{;\alpha} - \rho_0 V^\alpha \frac{\partial}{\partial x^\alpha}\left(\frac{p}{\rho_0}\right) + V^\alpha \frac{\partial}{\partial x^\alpha}(P) = V_\alpha F^\alpha$$

$$-\rho_0 V^\alpha \frac{\partial e}{\partial x^\alpha} - \cancel{V^\alpha \frac{\partial p}{\partial x^\alpha}} - \frac{\rho_0 V^\alpha}{-\rho_0^2} p \frac{\partial \rho_0}{\partial x^\alpha} + \cancel{V^\alpha \frac{\partial P}{\partial x^\alpha}} = V_\alpha F^\alpha$$

$$(42.10) \quad -\rho_0 V^\alpha \frac{\partial e}{\partial x^\alpha} + \frac{p}{\rho_0} V^\alpha \frac{\partial \rho_0}{\partial x^\alpha} = V_\alpha F^\alpha = \rho_0 V^\alpha \left[-\frac{\partial e}{\partial x^\alpha} + \frac{p}{\rho_0^2} \frac{\partial \rho_0}{\partial x^\alpha} \right]$$

$$(\rho_1 v_i)_{,t} + (\rho_1 v_i v^j)_{,j} + p_{,i} = f_i$$

$$-\left\{ v_i (\rho_1 - p/c^2)_{,t} + v_i (\rho_1 v^j)_{,j} = (\underset{\sim}{f} \cdot \underset{\sim}{v}/c^2)\, v_i \right\}$$

$$(\rho_1 v_i)_{,t} - v_i (\rho_1)_{,t} + v_i (p/c^2)_{,t} + p_{,i}$$

$$+ (\rho_1 v_i v^j)_{,j} - v_i (\rho_1 v^j)_{,j} = f_i - v_i \underset{\sim}{f} \cdot \underset{\sim}{v}/c^2$$

$$p_{,i} + \rho_1 v_{i,t} + v_i (p/c^2)_{,t} + \rho_1 v^j v_{i,j} = f_i - v_i \underset{\sim}{f} \cdot \underset{\sim}{v}/c^2$$

(42.12) $$\rho_1 (v_{i,t} + v^j v_{i,j}) + \frac{v_i}{c^2} p_{,t} + p_{,i} = f_i - v_i \underset{\sim}{f} \cdot \underset{\sim}{v}/c^2$$

now $$V^\alpha \frac{\partial}{\partial x^\alpha} = \frac{D}{D\tau} = \gamma c \frac{\partial}{\partial ct} + \gamma v_i \frac{\partial}{\partial x^i} = \gamma \frac{\partial}{\partial t} + \gamma v_i \frac{\partial}{\partial x^i}$$

and $$\rho_1 \left(\frac{\partial v_i}{\partial t} + v^j \frac{\partial}{\partial x^j} v_i \right) = \rho_1 \left(\frac{\partial}{\partial t} + v^j \frac{\partial}{\partial x^j} \right) v_i = \frac{\rho_1}{\gamma} \frac{D v_i}{D\tau} = \rho_* \frac{D v_i}{D\tau}$$

hence

(42.13) $$\rho_* \frac{D v_i}{D\tau} + \frac{v_i}{c^2} p_{,t} \; \nabla_i p = f_i - v_i \underset{\sim}{f} \cdot \underset{\sim}{v}/c^2$$

$$\rho_* \frac{D\underset{\sim}{v}}{D\tau} = -\underset{\sim}{\nabla} p + \underset{\sim}{f} - \underset{\sim}{v}(\underset{\sim}{f} \cdot \underset{\sim}{v})/c^2 - \underset{\sim}{v}\, p_{,t}/c^2$$

(43.13) $$= -\underset{\sim}{\nabla} p + \underset{\sim}{f} - (\underset{\sim}{v}/c^2)\left[\underset{\sim}{f} \cdot \underset{\sim}{v} + \partial p/\partial t \right]$$

$$\int \underset{\sim}{p}_0 f(\underset{\sim}{p}_0)\, d\underset{\sim}{p}_0 = \,?$$ for isotropic $f(\underset{\sim}{p}_0)$

$$d\underset{\sim}{p} = \frac{E}{E_0}\, d\underset{\sim}{p}_0$$

$$\int E\, f(\underset{\sim}{p})\, d\underset{\sim}{p} =$$

$$\int (E^2/E_0)\, f(\underset{\sim}{p}_0)\, d\underset{\sim}{p}_0$$

$$= \gamma^2 \int [E_0 + \underset{\sim}{v}\cdot\underset{\sim}{p}_0/\cancel{E_{m_0}}]^2/E_0\; f(\underset{\sim}{p}_0)\, d\underset{\sim}{p}_0$$

$$= \gamma^2 \int E_0 [1 + \underset{\sim}{v}\cdot\underset{\sim}{p}_0/E_0]^2 f(\underset{\sim}{p}_0)\, d\underset{\sim}{p}_0$$

$$= \gamma^2 \int [[E_0 + \underset{\sim}{v}\cdot\underset{\sim}{p}_0]^2/E_0]\, f(\underset{\sim}{p}_0)\, d\underset{\sim}{p}_0$$

$$= \gamma^2 \int [E_0^2 + 2E_0(\underset{\sim}{v}\cdot\underset{\sim}{p}_0) + (\underset{\sim}{v}\cdot\underset{\sim}{p}_0)^2]\, E_0^{-1} f(\underset{\sim}{p}_0)\, d\underset{\sim}{p}_0$$

43.29 $$= \gamma^2 \int \{E_0 + (\underset{\sim}{v}\cdot\underset{\sim}{p}_0^2)/E_0\}\, f(\underset{\sim}{p}_0)\, d\underset{\sim}{p}_0$$

$$\underset{\sim}{p} = \gamma m_0 \underset{\sim}{v} \qquad E = \gamma m_0 c^2$$

$$\underset{\sim}{p} = E\underset{\sim}{v}/c^2$$

or $$p_0^i = E_0 U_0^i / c^2$$

$$\underset{\sim}{p}_0 = E_0 \underset{\sim}{U}_0 / c^2$$

$$p = p_0 + \underset{\sim}{v}\left[\gamma(E_0/c^2) + (\gamma-1)\underset{\sim}{v}\cdot p_0/v^2\right]$$

$$\begin{pmatrix} E/c \\ p \end{pmatrix} = \begin{pmatrix} \gamma & \mp\gamma\beta \\ \pm\cancel{\gamma}\beta & \underset{\sim}{I} + (\gamma-1)\beta^{-2}\beta\beta \end{pmatrix} \begin{pmatrix} E_0/c \\ p_0 \end{pmatrix}$$

$$p = (+\gamma\beta)E_0/c + \left[\underset{\sim}{I} + (\gamma-1)\beta^{-2}\beta\beta\right]p_0$$

$$= p_0 + [$$ [illegible]

$$+ \underset{\sim}{v}\,\gamma E_0/c^2 + (\gamma-1)\frac{\cancel{c^2}}{v^2}\frac{(\underset{\sim}{v}\cdot p_0)}{\cancel{c^2}}\underset{\sim}{v}$$

(43.35)
(43.11) $$\underset{\sim}{p} = p_0 + \underset{\sim}{v}\left[\gamma E_0/c^2 + (\gamma-1)(\underset{\sim}{v}\cdot p_0)/v^2\right] \quad ($$

$$(43.34)\ MD = \int p\, f(p)\,dp = \int f(p_0)\left[1 + (\underset{\sim}{v}\cdot p_0)/E_0\right]$$

$$\cdot\left\{p_0 + \underset{\sim}{v}\left[\gamma E_0/c^2 + (\gamma-1)(\underset{\sim}{v}\cdot p_0)/v^2\right]\right\}dp_0$$

$$=$$

$$\rho_{00}c^2 = \int E_0 f_0\, dp_0$$

$$p_m = \int(\underset{\sim}{U}_0\cdot \underset{\sim}{l})(p_0\cdot \underset{\sim}{l})f_0\, dp_0$$

$$MD = \frac{\gamma}{c^2}\int\left[\gamma E_0\underset{\sim}{v} + (\gamma-1)v^{-2}(\underset{\sim}{U}_0\cdot\underset{\sim}{v})(p_0\cdot\underset{\sim}{v})\underset{\sim}{v} + (\underset{\sim}{U}_0\cdot\underset{\sim}{v})p_0\right]f_0\,dp_0$$

$$= \frac{\gamma}{c^2}\left\{\gamma\underset{\sim}{v}\int E_0 f_0\, dp_0 + (\gamma-1)\underset{\sim}{v}\int(\underset{\sim}{U}_0\cdot\underset{\sim}{l})(p_0\cdot\underset{\sim}{l}) + v\int(\underset{\sim}{U}_0\cdot\underset{\sim}{l})\,p_0 f_0\, dp_0\right.$$

$$= \frac{\gamma}{c^2}\left\{\gamma v\,\rho_{00}c^2 + (\gamma-1)v\,p_m + \cancel{\gamma}\, v\, p_m\right\}$$

$$\boxed{\frac{\partial p}{\partial z} \quad \rho}$$

~~$\int (\underset{\sim}{U}_0 \cdot \underset{\sim}{v})\, p_0 f_0\, dp_0$~~

~~$= v \int (\underset{\sim}{U}_0 \cdot \underset{\sim}{\ell})\, p_0 f_0\, d\underline{p}_0$~~

$$\frac{\gamma}{c^2}\left\{ \gamma \underset{\sim}{v} \rho_{00} c^2 + \gamma \underset{\sim}{v}\, p_m \right\} = \gamma^2 \left\{ \underset{\sim}{v} \rho_{00} + \underset{\sim}{v}\, p_m / c^2 \right\}$$

(43.34)
$$= \gamma^2 \rho_{000} \underset{\sim}{v}$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial z} \mp \rho g'$$

$$\frac{Du}{Dt} = -\frac{\partial p}{\partial m} + g$$

(43.45)
$$\Pi^{ij} = \int v^i p^j f(\underline{p})\, d\underline{p} = \int v^i p^j f_0(\underline{p}_0) \frac{E}{E_0}\, d\underline{p}_0$$

~~$E/E_0 = [1 + v p_0/E_0]$~~

$$= \int \frac{p^j}{E_0} (v^i E)\, f_0(p_0)\, d\underline{p}_0 = c^2 \int \frac{p^i p^j}{E_0} f_0 d\underline{p}_0$$

$$= c^2 \int \frac{f_0\, dp_0}{E_0} \left[p_0^i + \frac{(\gamma-1)}{v^2} (\underset{\sim}{v} \cdot p_0) v^i + \left(\frac{\gamma E_0}{c^2}\right) v^i \right] \times$$
$$\left[p_0^j + \frac{(\gamma-1)}{v^2} (\underset{\sim}{v} \cdot p_0) v^j + (\gamma E_0 / c^2) v^j \right]$$

$$= c^2 \int \frac{f_0\, dp_0}{E_0} \left\{ p_0^i p_0^j + \frac{(\gamma-1)}{v^2} (\underset{\sim}{v} \cdot p_0) \left[p_0^j v^i + p_0^i v^j \right] + \left(\frac{\gamma E_0}{c^2}\right) \left[p_0^i v^j + p_0^j v^i \right] \right.$$
$$\left. + \frac{(\gamma-1)^2}{v^4} (\underset{\sim}{v} \cdot p_0)^2 v^i v^j + \left(\frac{\gamma E_0}{c^2}\right) \left(\frac{(\gamma-1)}{v^2} (\underset{\sim}{v} \cdot p_0)\right) \left[v^i v^j + v^j v^i \right] \right\}$$

$$= c^2 \int \frac{f_0\, dp_0}{E_0} \left\{ p_0^i p_0^j + \left[\frac{(\gamma-1)}{v^2} (\underset{\sim}{v} \cdot p_0) + \frac{\gamma E_0}{c^2} \right] \left[p_0^j v^i + p_0^i v^j \right] \right.$$
$$\left. + v^i v^j \left[\frac{(\gamma-1)^2}{v^4} (\underset{\sim}{v} \cdot p_0)^2 + 2 \frac{\gamma E_0}{c^2} \frac{(\gamma-1)}{v^2} (\underset{\sim}{v} \cdot p_0) \right] + \frac{\gamma^2 E_0^2}{c^4} v^i v^j \right\}$$

$$\Pi^{ij} = \int \frac{c^2}{E_0}(f_0\, d\underset{\sim}{p}_0)\Big\{ p_0^i p_0^j + \frac{(\gamma-1)(\underset{\sim}{v}\cdot\underset{\sim}{p}_0)}{v^2}(p_0^j v^i + p_0^i v^j) + \frac{(\gamma-1)^2}{v^4}(\underset{\sim}{v}\cdot\underset{\sim}{p}_0)^2 v^i v^j + \frac{\gamma^2 E_0^2}{c^4} v^i v^j \Big\}$$

$c^2 \underset{\sim}{p}_0 = E_0 \underset{\sim}{U}_0$ hence $\frac{c^2}{E_0}\underset{\sim}{p}_0 = \underset{\sim}{U}_0$

and

$$\Pi^{ij} = \int \underset{\sim}{U}_0^i p_0^j f_0\, d\underset{\sim}{p}_0 + \int \frac{(\gamma-1)(\underset{\sim}{v}\cdot\underset{\sim}{U}_0)}{v^2}(p_0^i v^j + p_0^j v^i) f_0\, d\underset{\sim}{p}_0$$
$$+ \int \frac{(\gamma-1)^2(\underset{\sim}{v}\cdot\underset{\sim}{p}_0)(\underset{\sim}{v}\cdot\underset{\sim}{U}_0)}{v^4} v^i v^j f_0\, d\underset{\sim}{p}_0 + \int \frac{\gamma^2 E_0}{c^2} v^i v^j f_0\, d\underset{\sim}{p}_0$$

$$= \int U_0^i p_0^j f_0\, d\underset{\sim}{p}_0 + (\gamma-1)\frac{v^j}{v}\int (\underset{\sim}{U}_0\cdot\underset{\sim}{l})\, p_0^i f_0\, dp_0 + (\gamma-1)\frac{v^i}{v}\int(\underset{\sim}{U}_0\cdot\underset{\sim}{l})\, p_0^j f_0\, dp_0$$
$$+ v^i v^j \frac{(\gamma-1)^2}{v^2}\int (\underset{\sim}{U}_0\cdot\underset{\sim}{l})(\underset{\sim}{p}_0\cdot\underset{\sim}{l}) f_0\, d\underset{\sim}{p}_0 + \frac{\gamma^2}{c^2} v^i v^j \int E_0 f_0\, d\underset{\sim}{p}_0$$

$$= -T^{ij} + \gamma^2 \rho_{00} v^i v^j + (\gamma-1)^2 \frac{p_m v^i v^j}{v^2}$$
$$+ (\gamma-1)\frac{v^j}{v}\int (\underset{\sim}{U}_0\cdot\underset{\sim}{l})\, p_0^i f_0\, d\underset{\sim}{p}_0 + (\gamma-1)\frac{v^i}{v}\int (\underset{\sim}{U}_0\cdot\underset{\sim}{l})\, p_0^j f_0\, d\underset{\sim}{p}_0$$

(43.46)
$$= +p_m \delta^{ij} + \gamma^2 \rho_{00} v^i v^j + (\gamma-1)^2 p_m v^i v^j / v^2$$
$$+ (\gamma-1)\frac{v^j}{v}\int (\underset{\sim}{U}_0\cdot\underset{\sim}{l})(\underset{\sim}{p}_0\cdot\underset{\sim}{l})\frac{v^i}{v} f_0\, d\underset{\sim}{p}_0 + (\gamma-1)\frac{v^i}{v}\int (\underset{\sim}{U}_0\cdot\underset{\sim}{l})(\underset{\sim}{p}_0\cdot\underset{\sim}{l})\frac{v^j}{v} f_0\, d\underset{\sim}{p}_0$$

$$= p_m \delta^{ij} + \gamma^2 \rho_{00} v^i v^j + \left[(\gamma-1)^2 + 2(\gamma-1)\right] p_m \frac{v^i v^j}{v^2}$$

$$(\gamma-1)^2 + 2(\gamma-1) = \gamma^2 - \cancel{2\gamma} + 1 + \cancel{2\gamma} - 2$$
$$= \gamma^2 - 1 = \frac{1}{1-\beta^2} - 1 = \frac{1-(1-\beta^2)}{1-\beta^2} = \beta^2\gamma^2 = \gamma^2\frac{v^2}{c^2}$$

(43.47)
$$\Pi^{ij} = p_m \delta^{ij} + \gamma^2 \rho_{00} v^i v^j + \gamma^2 \frac{\cancel{v^2}}{c^2} p_m \frac{v^i v^j}{\cancel{v^2}} = p_m \delta^{ij} + \gamma^2 v^i v^j \left[\rho_{00} + p_m/c^2\right]$$

Finally !!

(43.47) $$\Pi^{ij} = p_m \delta^{ij} + \gamma^2 v^i v^j \rho_{000}$$

$$\rho_{00} c^2 = \int E_0 f_0 \, dp_0 = m_0 \int \left(c^2 + \tfrac{1}{2} v_0^2\right) f_0 \, dp_0$$

$$= m_0 c^2 n_0 + \frac{m_0}{2} \int v_0^2 f_0 \, dp_0$$

but for f_0 = Maxwellian, $\langle v_0^2 \rangle = 3kT$

$$\gamma = \frac{1}{(1 - v^2/c^2)^{1/2}}$$

$$\underline{E_0 = \gamma m_0 c^2}$$

$$p_0 = \gamma m_0 c$$

$$E_0^2 = p_0^2 c^2 + m_0^2 c^4$$

p.41

$$g_{ij} g^{jk} = \delta_i^{\,k}$$

[illegible]

$$P^{\alpha}_{\ \beta} P^{\beta}_{\ \gamma} = (\delta^{\alpha}_{\beta} + c^{-2} V^{\alpha} V_{\beta})(\delta^{\beta}_{\gamma} + c^{-2} V^{\beta} V_{\gamma})$$

$$= \delta^{\alpha}_{\beta}\delta^{\beta}_{\gamma} + c^{-2}[\delta^{\alpha}_{\beta} V^{\beta} V_{\gamma} + \delta^{\beta}_{\gamma} V^{\alpha} V_{\beta}] \underbrace{+ c^{-4} V^{\alpha} V_{\beta} V^{\beta} V_{\gamma}}_{}$$

$$= \delta^{\alpha}_{\gamma} + c^{-2}[V^{\alpha} V_{\gamma} + V^{\alpha} V_{\gamma}] - c^{-2} V^{\alpha} V_{\gamma} \qquad V_{\beta} V^{\beta} = -c^{2}$$

$$\underbrace{P^{\alpha}_{\ \beta} P^{\beta}_{\ \gamma}}_{} = \delta^{\alpha}_{\gamma} + c^{-2} V^{\alpha} V_{\gamma} = P^{\alpha}_{\ \gamma}$$

$$P^{\alpha\beta} = P^{\alpha\gamma} P^{\beta}_{\ \gamma} = (g^{\alpha\gamma} + c^{-2} V^{\alpha} V^{\gamma})(\delta^{\beta}_{\gamma} + c^{-2} V^{\beta} V_{\gamma})$$

$$= g^{\alpha\gamma}\delta^{\beta}_{\gamma} + c^{-2}(\delta^{\beta}_{\gamma} V^{\alpha} V^{\gamma} + g^{\alpha\gamma} V^{\beta} V_{\gamma})$$

$$+ c^{-4} V^{\alpha} V^{\beta} V^{\gamma} V_{\gamma}$$

$$= g^{\alpha\beta} + c^{-2}[V^{\alpha} V^{\beta} + V^{\beta} V^{\alpha}] - C^{2} V^{\alpha} V^{\beta}$$

$$= g^{\alpha\beta} + c^{-2} V^{\alpha} V^{\beta}$$

$$P^{\alpha\beta} P_{\alpha\beta} = (g^{\alpha\beta} + c^{-2} V^{\alpha} V^{\beta})(g_{\alpha\beta} + c^{-2} V_{\alpha} V_{\beta})$$

$$= (g^{\alpha\beta} g_{\alpha\beta}) + c^{-2}(g_{\alpha\beta} V^{\alpha} V^{\beta} + g^{\alpha\beta} V_{\alpha} V_{\beta}) + c^{-4} V^{\alpha} V^{\beta} V_{\alpha} V_{\beta}$$

$$= (g^{\alpha\beta} g_{\alpha\beta}) + c^{-2}(V^{\beta} V_{\beta} + \cancel{g^{\alpha\beta}} V^{\beta} V_{\beta}) + c^{-4}(V^{\alpha} V_{\alpha} V^{\beta} V_{\beta})$$

$$P^{\alpha\beta} P_{\alpha\beta} = \delta^\alpha_\alpha + c^{-2}(-c^2 - c^2) + c^{-4}(c^4)$$

$$= 4 - 2 + 1 = 3$$

$$V_\alpha P^\alpha_\beta = V_\alpha \left[\delta^\alpha_\beta + c^{-2} V^\alpha V_\beta \right] = V_\beta + c^{-2}(-c^2 V_\beta) \equiv 0$$

$$A^\alpha = (-\cancel{c^{-2} V_\alpha A^\alpha) V^\alpha + P^\alpha_\beta} A^\beta$$

$$= \cancel{(-c^{-2} V^\alpha V_\alpha A^\alpha) + P^\alpha_\beta A^\beta}$$

$$= \cancel{A^\alpha}$$

$$A^\alpha = (-c^{-2} V_\beta A^\beta) V^\alpha + P^\alpha_\beta A^\beta$$

$$= (-c^{-2} V^\alpha V_\beta A^\beta) + \left[\delta^\alpha_\beta + c^{-2} V^\alpha V_\beta \right] A^\beta$$

$$= \delta^\alpha_\beta A^\beta + c^{-2} V^\alpha V_\beta A^\beta - c^{-2} V^\alpha V_\beta A^\beta$$

$$= \delta^\alpha_\beta A^\beta = A^\alpha$$

$$W^{\alpha\beta} = w V^\alpha V^\beta + w^\alpha V^\beta + w^\beta V^\alpha + w^{\alpha\beta}$$

$$= (c^{-4} V_\delta V_\gamma W^{\delta\gamma}) V^\alpha V^\beta + (-c^{-2} P^\alpha_\delta W^{\delta\gamma} V_\gamma) V^\beta + (-c^{-2} P^\beta_\delta W^{\delta\gamma} V_\gamma) V^\alpha$$

$$+ P^\alpha_\delta P^\beta_\gamma W^{\delta\gamma}$$

$$P^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} + c^{-2} V^{\alpha} V_{\beta}$$

$$\begin{array}{r} 216 \\ \underline{195} \\ 41 \end{array}$$

$$P^{\gamma}_{\beta} = \delta^{\gamma}_{\beta} + c^{-2} V^{\gamma} V_{\beta}$$

$$V_{\alpha;\gamma}\left[\delta^{\gamma}_{\beta} + c^{-2} V^{\alpha} V_{\beta}\right] = V_{\alpha;\beta} + c^{-2} V^{\gamma} V_{\beta} V_{\alpha;\gamma}$$

$$= \left(c^{-4} V_{\delta} V_{\gamma} W^{\delta\gamma}\right) V^{\alpha} V^{\beta} - c^{-2}\left(\delta^{\alpha}_{\delta} + c^{-2} V^{\alpha} V_{\delta}\right) W^{\delta\gamma} V_{\gamma} V^{\beta}$$

$$- c^{-2}\left(\delta^{\beta}_{\delta} + c^{-2} V^{\beta} V_{\delta}\right) W^{\delta\gamma} V_{\gamma} V^{\alpha} + \left(\delta^{\alpha}_{\delta} + c^{-2} V^{\alpha} V_{\delta}\right)\left(\delta^{\beta}_{\gamma} + c^{-2} V^{\beta} V_{\gamma}\right) W^{\delta\gamma}$$

$$= c^{-4}\left(V^{\alpha} \cancel{V_{\delta})(V^{\beta} V_{\gamma}}\right) W^{\delta\gamma} - c^{-2}\left(W^{\alpha\gamma} V_{\delta} V^{\beta} + c^{-2} V^{\alpha} \cancel{V_{\delta} V^{\beta} V_{\gamma}} W^{\delta\gamma}\right)$$

$$- c^{-2}\left[W^{\beta\gamma} V_{\gamma} V^{\alpha} + c^{-2} W^{\delta\gamma} \cancel{V^{\beta} V_{\gamma}} V^{\alpha} V_{\delta}\right) + \delta^{\alpha}_{\delta} \delta^{\beta}_{\gamma} W^{\delta\gamma} + c^{-2} V^{\alpha} V_{\delta} \delta^{\beta}_{\gamma} W^{\delta\gamma}$$

$$+ \delta^{\alpha}_{\delta} c^{-2} V^{\beta} V_{\gamma} W^{\delta\gamma} + c^{-4} V^{\alpha} \cancel{V_{\delta} V^{\beta} V_{\gamma}} W^{\delta\gamma}$$

$$= - c^{-2} \cancel{V_{\gamma} V^{\beta}} W^{\alpha\gamma} - c^{-2} \cancel{V_{\gamma} V^{\alpha}} W^{\beta\gamma} + W^{\alpha\beta} + c^{-2} V^{\alpha} \cancel{V_{\delta}} W^{\beta\delta}$$

$$+ c^{-2} V^{\beta} \cancel{V_{\gamma}} W^{\alpha\gamma}$$

$$= W^{\alpha\beta}$$

$$V_{\alpha;\beta} V^{\beta} = \left(\delta V_{\alpha} / \delta\tau\right) = A_{\alpha}$$

$$V_{\alpha;\gamma} P^{\gamma}_{\beta} = V_{\alpha;\gamma}\left[\delta^{\gamma}_{\beta} + c^{-2} V^{\gamma} V_{\beta}\right]$$

$$= V_{\alpha;\beta} + c^{-2} V_{\beta} V^{\gamma} V_{\alpha;\gamma} = V_{\alpha;\beta} + c^{-2} V_{\beta} A_{\alpha}$$

$$\Rightarrow V_{\alpha;\beta} = V_{\alpha;\gamma} P^{\gamma}_{\beta} - c^{-2} V_{\beta} A_{\alpha}$$

$$\omega_{\alpha\beta} = \tfrac{1}{2}\left[V_{\alpha;\gamma}P^{\gamma}_{\beta} - V_{\beta;\gamma}P^{\gamma}_{\alpha}\right]$$

$$e'_{\alpha\beta} = \tfrac{1}{2}\left[V_{\alpha;\gamma}P^{\gamma}_{\beta} + V_{\beta;\gamma}P^{\gamma}_{\alpha}\right]$$

$$V_{\alpha;\gamma}P^{\gamma}_{\beta} = \omega_{\alpha\beta} + e'_{\alpha\beta} = V_{\alpha;\beta} + c^{-2}A_{\alpha}V_{\beta}$$

or $$V_{\alpha;\beta} = e'_{\alpha\beta} + \omega_{\alpha\beta} - c^{-2}A_{\alpha}V_{\beta}$$

Then if $e_{\alpha\beta} = e'_{\alpha\beta} - \tfrac{1}{3}\Theta P_{\alpha\beta}$

we have

$$V_{\alpha;\beta} = e_{\alpha\beta} + \omega_{\alpha\beta} + \tfrac{1}{3}\Theta P_{\alpha\beta} - c^{-2}A_{\alpha}V_{\beta}$$

$$V^{\alpha}e_{\alpha\beta} = \frac{V^{\alpha}}{2}\left(V_{\alpha;\gamma}P^{\gamma}_{\beta} + V_{\beta;\gamma}P^{\gamma}_{\alpha}\right) - \cancel{+}\frac{V^{\alpha}}{3}V^{\gamma}_{;\gamma}P_{\alpha\beta}$$

$$= \tfrac{1}{2}\left\{V^{\alpha}V_{\alpha;\gamma}P^{\gamma}_{\beta} + V^{\alpha}V_{\beta;\gamma}P^{\gamma}_{\alpha}\right\} - \tfrac{1}{3}V^{\alpha}V^{\gamma}_{;\gamma}P_{\alpha\beta}$$

$$= \tfrac{1}{2}\left\{V^{\alpha}V_{\alpha;\gamma}\left(\delta^{\gamma}_{\beta} + c^{-2}V^{\gamma}V_{\beta}\right) + V^{\alpha}V_{\beta;\gamma}\left(\delta^{\gamma}_{\alpha} + c^{-2}V^{\gamma}V_{\alpha}\right)\right\}$$

$$- \tfrac{1}{3}V^{\alpha}V^{\gamma}_{;\gamma}\left[g_{\alpha\beta} + c^{-2}V_{\alpha}V_{\beta}\right]$$

$$= \tfrac{1}{2}\left\{V^{\alpha}V_{\alpha;\beta} + c^{-2}V^{\alpha}V^{\gamma}V_{\alpha;\gamma}V_{\beta} + V^{\alpha}V_{\beta;\alpha} + c^{-2}V^{\alpha}V_{\alpha}V^{\gamma}V_{\beta;\gamma}\right\}$$

$$- \tfrac{1}{3}V_{\beta}V^{\gamma}_{;\gamma} - \tfrac{1}{3}c^{-2}V^{\alpha}V_{\alpha}V_{\beta}V^{\gamma}_{;\gamma}$$

$$= \tfrac{1}{2}\left\{V^{\alpha}V_{\alpha;\beta} + c^{-2}A_{\alpha}V^{\alpha}V_{\beta} + A_{\beta} - V^{\gamma}V_{\beta;\gamma}\right\} - \tfrac{1}{3}\cancel{V_{\beta}V^{\gamma}_{;\gamma}} + \tfrac{1}{3}\cancel{V_{\beta}V^{\gamma}_{;\gamma}}$$

$$V_\alpha P^\alpha_\beta = V^\beta P^\alpha_\beta = 0$$

$$V_\alpha \left[\delta^\alpha_\beta + c^{-2} V^\alpha V_\beta\right] = V_\beta + c^{-2} V_\beta (V^\alpha V_\alpha) = 0$$

Now $V^\gamma V_{\beta;\gamma} = A_\beta$, hence the ~~second~~ 3rd + fourth terms cancel, leaving

$$V^\alpha e_{\alpha\beta} = \frac{1}{2}\left\{V^\alpha V_{\alpha;\beta} + c^{-2} A_\alpha V^\alpha V_\beta\right\}$$

$$= \frac{1}{2}\left\{V^\beta V_{\beta;\alpha} + c^{-2} V_\beta V^\alpha A_\alpha\right\} = \frac{V_\beta}{2}\left[V_{\beta;\alpha} + c^{-2} V^\alpha A_\alpha\right]$$

$V_\alpha P^\alpha_\beta = 0$

$V_\alpha A^\alpha \equiv 0$

~~$V_{\beta;\gamma} P^\gamma_\beta - c^{-2} A_\beta V_\alpha$~~

Now $A_\alpha = a V_\alpha + a_\alpha$

~~$V^\alpha e_{\alpha\beta} = \frac{V_\beta^\beta}{2}\left[V_{\beta;\alpha} + c^{-2} V^\alpha (a V_\alpha + g_{\alpha\gamma} P^\gamma_\beta A^\beta)\right]$~~

~~$= \frac{V_\beta}{2}\left[V_{\beta;\alpha} + c^{2} a V^\alpha V_\alpha + c^{2} V^\alpha P_{\alpha\beta} A^\beta\right]$~~

~~$= \frac{V_\beta}{2}\left[V_{\beta;\alpha} - a + c^{-2} V^\alpha A^\beta (g_{\alpha\beta} + c^{-2} V_\alpha V_\beta)\right]$~~

~~$= \frac{V_\beta}{2}\left[V_{\beta;\alpha} + c^{-2} V_\alpha A^\alpha + c^{-2} V_\beta A^\beta + c^{-4} V^\alpha V_\alpha V_\beta A^\beta\right]$~~

~~$= \frac{V_\beta}{2}\left\{V_{\beta;\alpha} + c^{-2}(V_\alpha A^\alpha + V_\beta A^\beta - V_\beta A^\beta)\right]$~~

~~$= \frac{V_\beta}{2}\left[V_{\beta;\alpha} + c^{-2} V_\alpha A^\alpha\right]$~~

$$A^{\alpha} = V^{\beta} V_{\alpha;\beta}$$

~~find problem~~

$$\frac{d}{d\tau}(V_{\alpha} V^{\alpha}) = 2V_{\alpha} A^{\alpha} = 0 = V$$

$$\text{Thus } 2V_{\alpha}(V^{\beta} V_{\alpha;\beta}) = 2V^{\beta} \cdot V_{\alpha} V_{\alpha;\beta}$$

$$= 2V^{\beta}$$

~~$V_{\beta} V_{\beta;\alpha} = V_{\beta}\left[V_{\beta;\gamma} P^{\gamma}_{\beta} - \bar{c}^{2} A_{\beta} V_{\alpha}\right]$~~

~~$= -\bar{c}^{2}(V_{\beta} A_{\beta} V_{\alpha}) + V_{\beta;\gamma}(V_{\beta} P^{\gamma}_{\beta})$~~

$$= -\bar{c}^{2} V_{\alpha} g_{\beta}$$

$$V^{\alpha} V_{\alpha;\beta} = V^{\alpha}\left[V_{\alpha;\gamma} P^{\gamma}_{\beta} - \bar{c}^{2} A_{\alpha} V_{\beta}\right]$$

$$= -\bar{c}^{2} \overset{0}{\cancel{V^{\alpha} A_{\alpha}}} V_{\beta} + V^{\alpha} V_{\alpha;\gamma} P^{\gamma}_{\beta}$$

$$= (V^{\alpha} V_{\alpha})_{;\beta} - V_{\alpha} V^{\alpha}{}_{;\beta} = (V^{\alpha} V_{\alpha})_{;\beta} - g_{\alpha\gamma} V^{\gamma} (g^{\alpha\delta} V_{\delta})_{;\beta}$$

$$= 0 - g_{\alpha\gamma} g^{\alpha\delta} V^{\gamma} V_{\delta;\beta} + g_{\alpha\gamma} V^{\gamma} V_{\delta} (g^{\alpha\delta})_{;\beta}$$

$$V^{\alpha} V_{\alpha;\beta} = -\delta_{\gamma}^{\delta} V^{\gamma} V_{\delta;\beta} + V^{\alpha} V_{\delta} g^{\alpha\delta}{}_{;\beta}$$

$$= -V^{\delta} V_{\delta;\beta} + V^{\alpha} V_{\delta} g^{\alpha\delta}{}_{;\beta}$$

$$2V_{\alpha} V^{\beta} V_{\alpha;\beta} = 2 g_{\alpha\delta} V^{\delta} g^{\beta\gamma} V_{\gamma} V_{\alpha;\beta}$$

$$= 2 g_{\alpha\delta} g^{\beta\gamma} V_{\gamma} V^{\delta} V_{\alpha;\beta}$$

Energy eqn non-ideal fluid

$$M^{\alpha\beta}{}_{;\beta} = F^\alpha$$

$$M_{\alpha\beta} = \rho_{00} V_\alpha V_\beta + p P_{\alpha\beta} - 2\mu e_{\alpha\beta} + \zeta \Theta P_{\alpha\beta} + c^{-2}(Q_\alpha V_\beta + Q_\beta V_\alpha)$$

$$(\rho_{00} V^\alpha V^\beta)_{;\beta} = V_\beta (\rho_{00})_{;\beta} V^\alpha + \rho_{00} V^\alpha V^\beta{}_{;\beta} + \rho_{00} V^\beta V^\alpha{}_{;\beta}$$

$$= V^\alpha (D\rho_{00}/D\tau) + \rho_{00} V^\alpha \Theta + \rho_{00} A^\alpha$$

$$= V^\alpha [D\rho_{00}/D\tau + \rho_{00}\Theta] + \rho_{00} A^\alpha$$

~~$(V^\alpha Q^\beta)_{;\beta} = V^\alpha{}_{;\beta} Q^\beta + V^\alpha Q^\beta{}_{;\beta}$~~

$$(V^\beta Q^\alpha)_{;\beta} = V^\beta Q^\alpha{}_{;\beta} + V^\beta{}_{;\beta} Q^\alpha = DQ^\alpha/D\tau + \Theta Q^\alpha$$

$$(V^\alpha Q^\beta)_{;\beta} = V^\alpha{}_{;\beta} Q^\beta + V^\alpha Q^\beta{}_{;\beta}$$

$$= V^\alpha Q^\beta{}_{;\beta} +$$

$$M^{\alpha\beta}{}_{;\beta} = V^\alpha \overset{\checkmark}{D\rho_{00}/D\tau} + \overset{\checkmark}{\rho_{00}}\Theta V^\alpha + \rho_{00} \overset{\checkmark}{A^\alpha}$$

$+$

$$+ p_{,\beta} P^{\alpha\beta} + p\, c^{-2}(A^\alpha + V^\alpha\Theta)$$

$$- \zeta\Theta_{,\beta} P^{\alpha\beta} - \zeta\Theta\, c^{-2}(A^\alpha + V^\alpha\Theta)$$

$$+ c^{-2}\Theta Q^\alpha + (DQ^\alpha/D\tau) c^{-2}$$

$$+ V^\alpha Q^\beta{}_{;\beta} \overset{\checkmark}{c^{-2}}$$

$$+ Q_\beta (e^{\alpha\beta} + \omega^{\alpha\beta}) c^{-2}$$

$$+ \tfrac{1}{3}\Theta Q_\beta P^{\alpha\beta} c^{-2}$$

$$-2\mu e^{\alpha\beta}{}_{;\beta}$$

$$\left(P^{\alpha\beta}{}_{;\beta}\right) = \left(g^{\alpha\beta}{}_{;\beta}\right) + \left(c^{-2} V^{\alpha}_{\alpha} V^{\beta}_{\beta}\right)_{;\beta} = c^{-2} V^{\alpha}{}_{;\beta} V^{\beta}_{\beta} + V_{\alpha} V_{\beta;\beta}$$

$$= c^{-2}\left(V^{\beta} V^{\alpha}{}_{;\beta} + V_{\alpha} V^{\beta}{}_{;\beta}\right)$$

$$= c^{-2}\left(A^{\alpha} + V^{\alpha}\Theta\right)$$

$$m^{\alpha\beta}{}_{;\beta} = V^{\alpha}\left\{ D\rho_{00}/D\tau + \rho_{00}\Theta + c^{-2} Q^{\beta}{}_{;\beta} + c^{-2}\Theta p - \zeta\Theta^{2} c^{-2}\right\}$$

$$+ \rho_{00} A^{\alpha} + A^{\alpha}\left(c^{-2} p - c^{-2}\zeta\Theta\right) - 2\mu e^{\alpha\beta}{}_{;\beta} + P^{\alpha\beta}\left(p_{,\beta} - \zeta\theta_{,\beta}\right)$$

$$+ c^{-2}\left[\Theta Q^{\alpha} + Q_{\beta}\left(e^{\alpha\beta} + \omega^{\alpha\beta}\right) + \tfrac{1}{3}\Theta Q_{\beta} P^{\alpha\beta} + DQ^{\alpha}/D\tau\right]$$

$$= V^{\alpha}\left\{D\rho_{00}/D\tau + \Theta\left[\rho_{00} + c^{-2}(p - \zeta\theta)\right\} + A^{\alpha}\left[\rho_{00} + c^{-2}(p - \zeta\theta)\right]$$

$$- 2\mu e^{\alpha\beta}{}_{;\beta} + c^{-2}\left[\Theta Q^{\alpha} + V^{\alpha} Q^{\beta}{}_{;\beta} + Q_{\beta}\left(e^{\alpha\beta} + \omega^{\alpha\beta} + \tfrac{1}{3}\Theta P^{\alpha\beta}\right) + DQ^{\alpha}/D\tau\right]$$

$$+ P^{\alpha\beta}(p - \zeta\theta)_{,\beta}$$

$$\Theta Q^{\alpha} + \tfrac{1}{3}\Theta Q_{\beta} P^{\alpha\beta} = \Theta Q^{\alpha} + \tfrac{1}{3}\Theta Q_{\beta}\left[g^{\alpha\beta} + c^{-2} V^{\alpha} V^{\beta}\right]$$

$$= \Theta Q^{\alpha} + \tfrac{1}{3}\Theta Q_{\beta} g^{\alpha\beta} + \tfrac{c^{-2}}{3}\cancelto{0}{\left(V^{\beta} Q_{\beta}\right)} V^{\alpha}$$

$$= \tfrac{4}{3}\Theta Q^{\alpha}$$

$$m^{\alpha\beta}{}_{;\beta} = V^{\alpha}\left\{D\rho_{00}/D\tau + \Theta\left[\rho_{00} + c^{-2}(p - \zeta\theta)\right]\right\} + A^{\alpha}\left[\rho_{00} + c^{-2}(p - \zeta\theta)\right]$$

$$- 2\mu e^{\alpha\beta}{}_{;\beta} + c^{-2}\left[\tfrac{4}{3}\Theta Q^{\alpha} + V^{\alpha} Q^{\beta}{}_{;\beta} + Q_{\beta}\left(e^{\alpha\beta} + \omega^{\alpha\beta}\right) + DQ^{\alpha}/D\tau\right]$$

$$+ P^{\alpha\beta}(p - \zeta\theta)_{,\beta}$$

$$0 = V_{\alpha} m^{\alpha\beta}{}_{;\beta} = V_{\alpha} V^{\alpha}\left\{D\rho_{00}/D\tau + \Theta\left[\rho_{00} + c^{-2}(p - \zeta\theta)\right]\right\} + 0 \cancel{+} - 2\mu N_{\alpha} e^{\alpha\beta}{}_{;\beta}$$

$$+ c^{-2}\left[0 + V_{\alpha} V^{\alpha} Q^{\beta}{}_{;\beta} + 0 + V_{\alpha} DQ^{\alpha}/D\tau\right] + 0$$

$$= -c^{2}\left\{D\rho_{00}/D\tau + \Theta\left[\rho_{00} + c^{2}(p - \zeta\theta)\right]\right\} - 2\mu V_{\alpha} e^{\alpha\beta}{}_{;\beta} - Q^{\beta}{}_{;\beta} + c^{-2} V_{\alpha} DQ^{\alpha}/D\tau$$

$$V^\alpha \rho_{00;\alpha} = V^\alpha(\rho_{000} - p/c^2)_{;\alpha}$$

$$= V^\alpha \rho_{000,\alpha} - V^\alpha c^{-2} p_{,\alpha} + \rho_{000} V^\alpha{}_{;\alpha} - \rho_{000} V^\alpha{}_{;\alpha}$$

$$= (\rho_{000} V^\alpha)_{;\alpha} - c^{-2} V^\alpha p_{,\alpha} - \rho_{000} V^\alpha{}_{;\alpha}$$

$$c^2 (\rho_{000} V^\alpha)_{;\alpha} - V^\alpha p_{,\alpha} = -2\mu V_\alpha e^{\alpha\beta}{}_{;\beta} + \zeta\Theta^2 - [Q^\beta{}_{;\beta} + c^{-2} V_\alpha DQ^\alpha/D\tau]$$

$$V_\alpha DQ^\alpha/D\tau = D(V_\alpha Q^\alpha)/D\tau - Q^\alpha DV_\alpha/D\tau = -A_\alpha Q^\alpha$$

$$V_\alpha e^{\alpha\beta}{}_{;\beta} = (V_\alpha e^{\alpha\beta})_{;\beta} - e^{\alpha\beta} V_{\alpha;\beta} = -e^{\alpha\beta}(e_{\alpha\beta} + \omega_{\alpha\beta} + \tfrac{1}{3}\Theta P_{\alpha\beta} - c^{-2} A_\alpha V_\beta)$$

,19,20,22,27 $$= -e^{\alpha\beta} e_{\alpha\beta} \cancel{4} - \cancelto{0}{e^{\alpha\beta}\omega_{\alpha\beta}} + \cancelto{0}{\frac{e^{\alpha\beta}}{3}\Theta P_{\alpha\beta}} + \cancelto{0}{e^{\alpha\beta} c^{-2} A_\alpha} V_\beta$$

44.27 44.25

$$= -e^{\alpha\beta} e_{\alpha\beta} - \frac{e^{\alpha\beta}}{2}(V_{\alpha;\gamma} P^\gamma{}_\beta - V_{\beta;\gamma} P^\gamma{}_\alpha)$$

$$= -e^{\alpha\beta} e_{\alpha\beta} - \tfrac{1}{2}\{[\cancelto{0}{(V_\alpha e^{\alpha\beta})_{;\gamma}} - V_\alpha e^{\alpha\beta}{}_{;\gamma}] P^\gamma{}_\beta - [\cancelto{0}{(V_\beta e^{\alpha\beta})_{;\gamma}} - V_\beta e^{\alpha\beta}{}_{;\gamma}] P^\gamma{}_\alpha\}$$

$$= -e^{\alpha\beta} e_{\alpha\beta} + \tfrac{1}{2}\{V_\alpha e^{\alpha\beta}{}_{;\gamma} P^\gamma{}_\beta + V_\beta e^{\alpha\beta}{}_{;\gamma} P^\gamma{}_\alpha\}$$

$$= -e^{\alpha\beta} e_{\alpha\beta} + \tfrac{1}{2}\{V_\alpha e^{\alpha\beta}{}_{;\gamma}[\delta^\gamma_\beta + c^{-2} V^\gamma V^\beta] + V_\beta e^{\alpha\beta}{}_{;\gamma}[\delta^\gamma_\alpha + c^{-2} V^\gamma V^\alpha]\}$$

$$= -e^{\alpha\beta} e_{\alpha\beta} + \tfrac{1}{2}\{V_\alpha e^{\alpha\beta}{}_{;\beta} + c^{-2} V_\alpha V^\gamma V^\beta e^{\alpha\beta}{}_{;\gamma} + V_\beta e^{\alpha\beta}{}_{;\alpha} + c^{-2} e^{\alpha\beta}{}_{;\gamma} V_\beta V^\gamma V^\alpha\}$$

$$c^2(\rho_{000}V^\alpha)_{;\alpha} - V^\alpha p_{,\alpha} = -2\mu V_\alpha e^{\alpha\beta}{}_{;\beta} + \zeta\theta^2 - [Q^\beta{}_{;\beta} - c^{-2}V_\alpha DQ^\alpha/D\tau]$$

$$= -2\mu(-e^{\alpha\beta}V_{\alpha;\beta}) + \zeta\theta^2 - (Q^\beta{}_{;\beta} + c^{-2}A_\alpha Q^\alpha)$$
$$+2\mu(+e^{\alpha\beta}e_{\alpha\beta})$$

$$(\rho_{000}c^2V^\alpha)_{;\alpha} - V^\alpha p_{,\alpha} = 2\mu e^{\alpha\beta}e_{\alpha\beta} + \zeta\theta^2 - (Q^\beta{}_{;\beta} + c^{-2}A_\alpha Q^\alpha)$$

$$(\rho_0 s V^\alpha + Q^\alpha/T)_{;\alpha} = \rho_0 Ds/D\tau + s(\rho_0 V^\alpha)_{;\alpha}$$
$$+ (Q^\alpha/T)_{;\alpha} \geqslant 0$$

$\cancel{V_\alpha V^\alpha}$ $\boxed{c^{-2} \,\big|\, V_\alpha V^\beta Q^\alpha{}_{;\beta}}$ $c^{-2} V_\alpha V^\alpha Q^\beta$

$$-(\rho_{oo} c^2 V^\alpha)_{;\alpha} + V^\alpha p_{,\alpha} + \zeta\theta^2 + 2\mu e^{\alpha\beta} e_{\alpha\beta} - [Q^\beta{}_{;\beta} + c^{-2} A_\alpha Q^\alpha]$$

$$(\rho_{oo} c^2 V^\alpha)_{;\alpha} - V^\alpha p_{,\alpha} = 2\mu e^{\alpha\beta} e_{\alpha\beta} + \zeta\theta^2 - (Q^\beta{}_{;\beta} + c^{-2} A_\alpha Q^\alpha)$$

$$\rho_o T \frac{Ds}{D\tau} = \rho_o \left[\frac{De}{D\tau} + p\frac{D(1/\rho_o)}{D\tau}\right] = 2\mu e^{\alpha\beta} e_{\alpha\beta} + \zeta\theta^2 - (Q^\beta{}_{;\beta} + c^{-2} Q^\alpha A_\alpha)$$

$$\rho_o \frac{Ds}{D\tau} + \left(\frac{Q^\alpha}{T}\right)_{;\alpha} \geq 0$$

$$2\mu \frac{e^{\alpha\beta} e_{\alpha\beta}}{T} + \frac{\zeta\theta^2}{T} - (Q^\beta{}_{;\beta} + c^{-2} Q^\alpha A_\alpha) T^{-1} + \left(\frac{Q^\alpha}{T}\right)_{;\alpha} \geq 0$$

$$-\frac{Q^\beta{}_{;\beta}}{T} + \left(\frac{Q^\beta}{T}\right)_{;\beta} - c^{-2}\frac{Q^\alpha A_\alpha}{T}$$

$$= -\frac{Q^\beta{}_{;\beta}}{T} + \frac{Q^\beta{}_{;\beta}}{T} + Q^\beta\left(\frac{-1}{T^2}\right) T_{,\beta} - c^{-2}\frac{Q^\alpha A_\alpha}{T}$$

$$= -\frac{Q^\alpha}{T^2} T_{,\alpha} - c^{-2}\frac{Q^\alpha A_\alpha}{T} = -\frac{Q^\alpha}{T^2}\left(\frac{T_{,\alpha}}{\cancel{\alpha}} + c^{-2} A_\alpha T\right)$$

$$\left\{\frac{D\rho_{oo}}{D\tau} + \theta\left[\rho_{oo} + c^{-2}(p-\rho\theta)\right]\right\} V^\alpha + \left[\rho_{oo} + c^{-2}(p-\rho\theta)\right] A^\alpha$$

$$+ P^{\alpha\beta}(p-\rho\theta)_{,\beta} - 2\mu\, e^{\alpha\beta}{}_{;\beta} + c^{-2}\Big[DQ^\alpha/D\tau + V^\alpha Q^\beta{}_{;\beta} + \tfrac{4}{3}\theta Q^\alpha$$

$$+ Q_\beta\left(e^{\alpha\beta} + \omega^{\alpha\beta}\right) \Big] = F^\alpha$$

$$V_\alpha V^\alpha \left\{ \frac{D\rho_{oo}}{D\tau} + \theta\left[\rho_{oo} + c^{-2}(p-\rho\theta)\right\} + \cancelto{0}{\left[\rho_{oo} + c^{-2}(p-\rho\theta)\right] V_\alpha A^\alpha}$$

$$+ (p-\rho\theta)_{,\beta}\cancelto{0}{V_\alpha P^{\alpha\beta}} - 2\mu V_\alpha e^{\alpha\beta}{}_{;\beta} + c^{-2}\Big[\cancel{V}_\alpha DQ^\alpha/D\tau + V_\alpha V^\alpha Q^\beta{}_{;\beta} + \tfrac{4}{3}\theta V_\alpha Q^\alpha$$

$$+ V_\alpha Q^\beta\left(e^{\alpha\beta} + \omega^{\alpha\beta}\right)\Big] = V_\alpha F^\alpha = 0$$

$$(-c^2)\left\{ \frac{D\rho_{oo}}{D\tau} + \theta\left[\rho_{oo} + c^{-2}(p-\rho\theta)\right\} + 0 + 0 - 2\mu V_\alpha e^{\alpha\beta}{}_{;\beta}$$

$$+ c^{-2}\Big[V_\alpha DQ^\alpha/D\tau + \underbrace{V_\alpha V^\alpha}_{-c^2} Q^\beta{}_{;\beta} + 0 + Q^\beta\big(\cancelto{0}{V_\alpha e^{\alpha\beta}} + \cancelto{0}{V_\alpha \omega^{\alpha\beta}}\big)$$

$$0 = -c^2\left\{ \frac{D\rho_{oo}}{D\tau} + \theta\left[\rho_{oo} + c^{-2}(p-\rho\theta)\right\} - 2\mu V_\alpha e^{\alpha\beta}{}_{;\beta} + c^{-2} V_\alpha DQ^\alpha/D\tau + c^{-2}(-c^2) Q^\beta{}_{;\beta}$$

~~[illegible]~~

$$= -c^2\left\{ \frac{D\rho_{oo}}{D\tau} + \theta\rho_{ooo} - c^{-2}\rho\theta^2 \right\} - 2\mu V_\alpha e^{\alpha\beta}{}_{;\beta} + c^{-2} V_\alpha DQ^\alpha/D\tau - Q^\beta{}_{;\beta}$$

$$= -c^2\left(\frac{D\rho_{oo}}{D\tau} + \rho_{ooo} V^\alpha{}_{;\alpha}\right) + \rho\theta^2 - 2\mu\left(-e^{\alpha\beta}e_{\alpha\beta}\right) - c^{-2} Q^\alpha A_\alpha - Q^\beta{}_{;\beta}$$

$$= -c^2\left[\rho_{oo,\alpha} + \rho_{ooo} V^\alpha{}_{;\alpha}\right] + \rho\theta^2 + 2\mu e^{\alpha\beta}e_{\alpha\beta} - c^{-2} Q^\alpha A_\alpha - Q^\beta{}_{;\beta}$$

$$-c^2\left[\left(\rho_{ooo}V^\alpha\right)_{;\alpha} - c^{-2} V^\alpha p_{,\alpha}\right] + \rho\theta^2 + 2\mu e^{\alpha\beta}e_{\alpha\beta} - c^{-2} A_\alpha Q^\alpha - Q^\beta{}_{;\beta}$$

$$P_{\alpha\gamma}V^{\gamma} = \left[g_{\alpha\gamma} + c^{-2}V_{\alpha}V_{\gamma}\right]V^{\gamma}$$

$$= \left[V_{\alpha} + c^{-2}V_{\alpha}(-c^{2})\right] = 0$$

$$P_{\alpha\gamma}M^{\gamma\beta}{}_{;\beta} = \cancelto{0}{P_{\alpha\gamma}V^{\gamma}}\left\{\frac{D\rho_{00}}{D\tau} + \theta\left[\rho_{00} + c^{-2}(p-\rho\theta)\right\}\right.$$

$$+ P_{\alpha\gamma}A^{\gamma}\left[\rho_{00} + c^{-2}(p-\rho\theta)\right] + \overset{P_{\alpha}{}^{\beta}}{\boxed{P_{\alpha\gamma}P^{\gamma\beta}}}(p-\rho\theta)_{;\beta}$$

$$- 2\mu P_{\alpha\gamma}e^{\gamma\beta}{}_{;\beta} + c^{-2}P_{\alpha\gamma}\left[(DQ^{\gamma}/D\tau) + \tfrac{4}{3}\theta Q^{\gamma} \cancel{+ V^{\gamma}Q^{\beta}{}_{;\beta}}\right.$$
$$\left. + Q_{\beta}(e^{\gamma\beta} + \omega^{\gamma\beta})\right]$$

$$(P_{\alpha\gamma}P^{\gamma\beta}) = (g_{\alpha\gamma} + c^{-2}V_{\alpha}V_{\gamma})(g^{\gamma\beta} + c^{-2}V^{\gamma}V^{\beta})$$

$$= g_{\alpha\gamma}g^{\gamma\beta} + c^{-2}(g_{\alpha\gamma}V^{\gamma}V^{\beta} + g^{\gamma\beta}V_{\alpha}V_{\gamma}) + c^{-4}V_{\alpha}V_{\gamma}V^{\gamma}V^{\beta}$$

$$= \delta_{\alpha}{}^{\beta} + c^{-2}(V_{\alpha}V^{\beta} + V^{\beta}V_{\alpha}) + c^{-4}V_{\alpha}V^{\beta}(-c^{2})$$

$$= \delta_{\alpha}{}^{\beta} + c^{-2}(V_{\alpha}V^{\beta}) = P_{\alpha}{}^{\beta}$$

$$P_{\alpha\gamma}M^{\alpha\beta}{}_{;\beta} = A_{\alpha}\left[\rho_{00} + c^{-2}(p-\rho\theta)\right] + P_{\alpha}{}^{\beta}(p-\rho\theta)_{;\beta} - 2\mu P_{\alpha\gamma}e^{\gamma\beta}{}_{;\beta}$$

$$+ c^{-2}P_{\alpha\gamma}\left[DQ^{\alpha}/D\tau + \tfrac{4}{3}\theta Q^{\gamma} + Q_{\beta}(e^{\gamma\beta} + \omega^{\gamma\beta})\right]$$

$$P_{\alpha\gamma}Q^{\gamma} = \left[g_{\alpha\gamma} + c^{-2}V_{\alpha}V_{\gamma}\right]Q^{\gamma} = Q_{\alpha}$$

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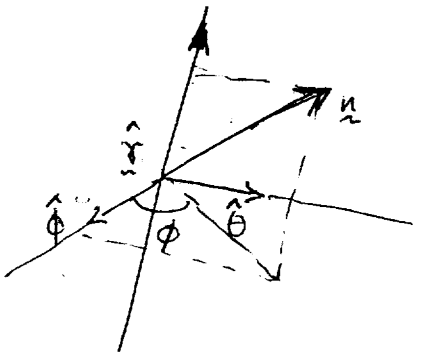
Chap 7

Section 89: The Photon Four Momentum

89.4 $$ds^2 = (h_1 dx^1)^2 + (h_2 dx^2)^2 + (h_3 dx^3)^2$$
$$= dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$



$$n_r = \mu$$
$$n_\theta = (1-\mu^2)^{1/2} \cos\phi$$
$$n_\phi = (1-\mu^2)^{1/2} \sin\phi$$

hence

$$n^{(r)} = \mu$$
$$n^{(\theta)} = (1-\mu^2)^{1/2} \cos\phi / r$$
$$n^{(\phi)} = (1-\mu^2)^{1/2} \sin\phi / r\sin\theta$$

and $M^1 = (h\nu/c)\, n^{(r)}$; $M^2 = (h\nu/c)\, n^{(\theta)}$; $M^{(3)} = (h\nu/c)\, n^{(\phi)}$

1. (89.5) + (89.6) [next page] to obtain (89.10) and (89.11)

$$\begin{pmatrix} M_0^0 \\ M_0^1 \\ M_0^2 \\ M_0^3 \end{pmatrix} = \frac{h\nu_0}{c} \begin{pmatrix} 1 \\ \mu_0 \\ (1-\mu_0^2)^{1/2}\cos\phi_0 \\ (1-\mu_0^2)^{1/2}\sin\phi_0 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v_z/c & 0 & 0 \\ -\gamma v_z/c & [\underline{\underline{I}}+(\gamma-1)\beta^{-2}\underset{\sim}{\beta}\underset{\sim}{\beta}] & 0 & 0 \\ 0 & 0 & [\underline{\underline{I}}+(\gamma-1)\underset{\sim}{\beta}\underset{\sim}{\beta}\beta^{-2} & 0 \\ 0 & 0 & 0 & [\underline{\underline{I}}+(\gamma-1)\underset{\sim}{\beta}\underset{\sim}{\beta}\beta^{-2}] \end{pmatrix} \begin{pmatrix} 1 \\ \mu \\ (1-\mu^2)^{1/2}\cos\phi \\ (1-\mu^2)^{1/2}\sin\phi \end{pmatrix} \frac{h\nu}{c}$$

(a) $$\nu_0 = \gamma\nu - \nu\gamma \frac{v_z}{c}\mu = \nu\gamma(1-\beta\mu)$$

$$\frac{h\nu_0}{c}\mu_0 = -\gamma\beta\frac{h\nu}{c} + [\underline{\underline{I}} + (\gamma-1)\beta^{-2}\underset{\sim}{\beta}\underset{\sim}{\beta}]\,\mu\,\frac{h\nu}{c}$$

$$\Rightarrow \nu_0\mu_0 = -\gamma\beta\nu + \mu\nu(\gamma\cancel{-1}) \cancel{+\mu\nu} = \gamma\nu[\mu-\beta]$$

hence $\mu_0 \cdot \cancel{\gamma\nu}(1-\beta\mu) = \cancel{\gamma\nu}[\mu-\beta]$

(b) $$\mu_0 = (\mu-\beta)/(1-\beta\mu)$$

$$(1-\mu_0^2)^{1/2}\cos\phi_0 \frac{h\nu_0}{c} = [\underline{\underline{I}} + (\gamma-1)\beta^{-2}\underset{\sim}{\beta}\underset{\sim}{\beta}](1-\mu^2)^{1/2}\cos\phi\, h\nu/c$$

$$(1-\mu_0^2)^{1/2}\cos\phi_0\, \nu_0 = (1-\mu^2)^{1/2}\cos\phi\, \nu$$

$$\gamma\nu(1-\beta\mu)(1-\mu_0^2)^{1/2}\cos\phi_0 = \nu(1-\mu^2)^{1/2}\cos\phi$$

$$(1-\mu_0^2)^{1/2}\cos\phi_0 = (1-\mu^2)^{1/2}\cos\phi / [\gamma(1-\beta\mu)]$$

Similarly $(1-\mu_0^2)^{1/2}\sin\phi_0 = (1-\mu^2)^{1/2}\sin\phi / [\gamma(1-\beta\mu)]$

(c) + (d) From which: $\phi_0 = \phi$ $\quad (1-\mu_0^2)^{1/2} = (1-\mu^2)^{1/2}/[\gamma(1-\beta\mu)]$

$$\begin{pmatrix}\nu_0\\ \nu_0\underset{\sim}{n}_0\end{pmatrix} = \begin{pmatrix}\gamma & -\gamma\underset{\sim}{\beta}\\ -\gamma\underset{\sim}{\beta} & \textcircled{I}+(\gamma-1)\beta^{-2}\underset{\sim}{\beta}\underset{\sim}{\beta}\end{pmatrix}\begin{pmatrix}\nu\\ \nu\underset{\sim}{n}\end{pmatrix}$$

(89.5) $$\nu_0 = \gamma\nu - \gamma\beta\nu\underset{\sim}{n} = \gamma\nu(1-\underset{\sim}{\beta}\cdot\underset{\sim}{n}) = \gamma\nu(1-\underset{\sim}{n}\cdot\underset{\sim}{v}/c)$$

$$\nu_0\underset{\sim}{n}_0 = -\gamma\underset{\sim}{\beta}\,\nu + [I+(\gamma-1)\beta^{-2}\underset{\sim}{\beta}\underset{\sim}{\beta}]\cdot\nu\underset{\sim}{n} = \nu\underset{\sim}{n} - \gamma\nu\underset{\sim}{v}/c + \nu(\gamma-1)(\underset{\sim}{v}\cdot\underset{\sim}{n})\underset{\sim}{v}/v2$$

$$= \nu\underset{\sim}{n} - \gamma\nu\underset{\sim}{v}/c\,[1-(\underset{\sim}{n}\cdot\underset{\sim}{v}c/v2)(\gamma-1)/\gamma]$$

now $$(\gamma-1)/\gamma = (\gamma-1)(\gamma+1)/\gamma(\gamma+1) = (\gamma^2-1)/[\gamma(\gamma+1)] = \left[\frac{1}{1-v^2/c^2}-1\right]\frac{1}{\gamma(\gamma+1)}$$

$$= (v^2/c2)\frac{1}{(1-v^2/c^2)}\frac{1}{\gamma(\gamma+1)} = \frac{\gamma 2}{\gamma(\gamma+1)}\frac{v2}{c2} = \frac{\gamma}{\gamma+1}\frac{v2}{c2}$$

hence

(89.6) $$\nu_0 n_0 = \nu\underset{\sim}{n} - \gamma\nu\underset{\sim}{v}/c\left[1-\frac{\underset{\sim}{n}\cdot\underset{\sim}{v}c}{v2}\frac{\gamma}{(\gamma+1)}\frac{v^2}{c2}\right] = \nu\underset{\sim}{n} - \gamma\nu\underset{\sim}{v}/c\left[1-\frac{\underset{\sim}{n}\cdot\underset{\sim}{v}}{c}\frac{\gamma}{(\gamma+1)}\right.$$

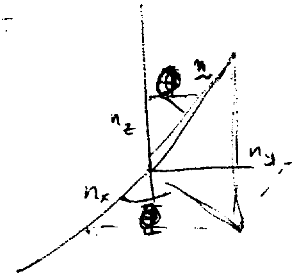
Inverse:

$$\begin{pmatrix}\nu\\ \nu\underset{\sim}{n}\end{pmatrix} = \begin{pmatrix}\gamma & \gamma\beta\\ \gamma\beta & \underset{=}{I}+(\gamma-1)\beta^{-2}\underset{\sim}{\beta}\underset{\sim}{\beta}\end{pmatrix}\begin{pmatrix}\nu_0\\ \nu_0\underset{\sim}{n}\end{pmatrix}$$

$$\nu = \gamma\nu_0 + \gamma\underset{\sim}{\beta}\nu_0\cdot\underset{\sim}{n} = \gamma\nu_0[1+\underset{\sim}{n}\cdot\underset{\sim}{v}/c]$$

$$\nu\underset{\sim}{n} = \gamma\underset{\sim}{\beta}\nu_0 + [\underset{=}{I}+(\gamma-1)\beta^{-2}\underset{\sim}{\beta}\underset{\sim}{\beta}]\,\nu_0\underset{\sim}{n} = \gamma\underset{\sim}{\beta}\,\nu_0 + \nu_0\underset{\sim}{n} + (\gamma-1)(\underset{\sim}{v}\cdot\underset{\sim}{n}/v2)\underset{\sim}{v}$$

$$= \nu_0\underset{\sim}{n} + \gamma\left[1+\left(\frac{\gamma-1}{\gamma}\right)c\frac{\underset{\sim}{v}\cdot\underset{\sim}{n}}{v2}\right]\frac{\underset{\sim}{v}}{c} = \nu_0\underset{\sim}{n} + \gamma\frac{\underset{\sim}{v}}{c}\left[1+\underset{\sim}{v}\cdot\underset{\sim}{n}\left(\frac{\gamma}{\gamma+1}\right)/c\right]$$



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Section 90: Transformation Laws for the Specific Intensity, Opacity + Emissivity

$$N = \left(\frac{I}{h\nu}\right)(d\omega\, d\nu)(dS \cos\theta\, dt) = \left(\frac{I}{h\nu}\right)\left(\frac{\nu d\nu d\omega}{\nu}\right)(dS dt)\,\mu$$

$$N_0 = \left(\frac{I_0}{h\nu_0}\right)(d\omega_0 d\nu_0)(dS \cos\theta_0 dt_0 + \beta\, dS\, dt_0)$$

$$= \left(\frac{I_0}{h\nu_0}\right)\left(\frac{\nu_0 d\nu_0 d\omega_0}{\nu_0}\right)(dS\, dt_0)(\mu_0+\beta)$$

Now since $N = N_0$:

$$\left(\frac{I_0}{\cancel{h}\nu_0}\right)\frac{dt_0}{\nu_0}(\mu_0+\beta)\left[dS\, \cancel{\nu_0 d\nu_0 d\omega_0}\right] = \left(\frac{I}{\cancel{h}\nu}\right)\frac{dt}{\nu}\mu\left[dS\, \cancel{\nu d\nu d\omega}\right]$$

$dt_0 = \gamma\, dt$

$\mu = (\mu_0+\beta)/(1+\beta\mu_0)$ but $(1+\beta\mu_0) = \nu/\gamma\nu_0$ hence

$$\mu = (\mu_0+\beta)(\gamma\nu_0/\nu)$$

Therefore

$$\frac{I_0}{\nu_0^2}\cancel{\gamma dt}\,(\cancel{\mu_0+\beta}) = \frac{I}{\nu^2}\cancel{dt}\,(\cancel{\mu_0+\beta})\left(\frac{\cancel{\gamma}\nu_0}{\nu}\right)$$

(90.3) $\Rightarrow \quad I(\mu,\nu) = (\nu/\nu_0)^3\, I_0(\mu_0,\nu_0)$

or $\quad (I/\nu^3) = (I_0/\nu_0^3)$ = Lorentz invariant

$$\eta_0\, d\omega_0\, d\nu_0\, dV_0\, dt_0 / h\nu_0 = \eta\, d\omega\, d\nu\, dV\, dt / h\nu$$

$dV dt$ is an invariant; and $\nu\, d\nu\, d\omega$ is invariant, hence

$$\eta_0\left(\frac{\nu_0 d\nu_0 d\omega_0}{\nu_0}\right)\frac{1}{\nu_0} = \eta\left(\frac{\nu d\nu d\omega}{\nu}\right)\frac{1}{\nu}$$

(90.6) $\Rightarrow \eta_0/\nu_0^2 = \eta/\nu^2$

$$\chi_0 I_0 (d\nu_0 d\omega_0 \cancel{dV_0 dt_0})/\cancel{h}\nu_0 = \chi I (d\nu d\omega \cancel{dV dt})/\cancel{h}\nu$$

$$\Rightarrow \chi_0 I_0 (\nu_0 d\nu_0 d\omega_0/\nu_0)(1/\nu_0) = \chi I(\nu d\nu d\omega/\nu)(1/\nu)$$

$$\chi_0 I_0/\nu_0^2 = \chi I/\nu^2$$

But $I = (\nu^3/\nu_0^3) I_0$ hence

$$\chi_0 I_0/\nu_0^2 = \chi I_0(\nu^3/\nu_0^3)/\nu^2$$

$$\chi_0\nu_0 = \chi\nu$$

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I}{\partial \mu} = \eta - \chi I$$

89.7: $\nu = \gamma\nu_0(1 + \underset{\sim}{n}_0 \cdot \underset{\sim}{v}/c)$

90.3: $I = (\nu/\nu_0)^3 I_0$

90.6: $\eta = (\nu/\nu_0)^2 \eta_0$

90.8: $\chi = (\nu_0/\nu)\,\chi_0$

$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\nu^3}{\nu_0^3}I_0\right) + \mu\frac{\partial}{\partial r}\left(\frac{\nu^3}{\nu_0^3}I_0\right) + \frac{(1-\mu^2)}{r}\frac{\partial}{\partial \mu}\left(\frac{\nu^3}{\nu_0^3}I_0\right) = \frac{\nu^2}{\nu_0^2}\eta_0 - \left(\frac{\nu_0}{\nu}\right)\chi_0\left(\frac{\nu^3}{\nu_0^3}\right)I_0$$

$$= \frac{\nu^3}{\nu_0^3}\left\{\frac{1}{c}\frac{\partial I_0}{\partial t} + \mu\frac{\partial I_0}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I_0}{\partial \mu}\right\} - 3\frac{\nu^3}{\nu_0^4}\left\{\frac{I_0}{c}\frac{\partial \nu_0}{\partial t} + \mu I_0\frac{\partial \nu_0}{\partial r} + \frac{(1-\mu^2)I_0}{r}\frac{\partial \nu_0}{\partial \mu}\right\}$$

$$= \frac{\nu^2}{\nu_0^2}\left[\eta_0 - \chi_0 I_0\right]$$

$$\Rightarrow \frac{\nu}{\nu_0}\left\{\frac{1}{c}\frac{\partial I_0}{\partial t} + \mu\frac{\partial I_0}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I_0}{\partial \mu}\right\} - 3\frac{\nu}{\nu_0^2}I_0\left\{\frac{1}{c}\frac{\partial \nu_0}{\partial t} + \mu\frac{\partial \nu_0}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial \nu_0}{\partial \mu}\right\} = \eta_0 - \chi_0 I_0$$

now

$$\frac{\partial}{\partial t} = \left.\frac{\partial}{\partial t}\right|_{r\mu_0\nu_0} + \left.\frac{\partial \mu_0}{\partial t}\right|_{r\mu\nu}\frac{\partial}{\partial \mu_0} + \left.\frac{\partial \nu_0}{\partial t}\right|_{r\mu\nu}\frac{\partial}{\partial \nu_0}$$

hence

$$\frac{\nu}{\nu_0}\frac{1}{c}\frac{\partial I_0}{\partial t} \to \frac{\nu}{\nu_0}\frac{1}{c}\left[\frac{\partial I_0}{\partial t} + \frac{\partial \mu_0}{\partial t}\frac{\partial I_0}{\partial \mu_0} + \frac{\partial \nu_0}{\partial t}\frac{\partial I_0}{\partial \nu_0}\right\} \; ; \; \frac{1}{c}\frac{\partial \nu_0}{\partial t} \to \frac{1}{c}\left\{\frac{\partial \nu_0}{\partial t} + \right.$$

Section 91: The Radiation Stress-Energy Tensor and Four-Force Vector

$$R^{\alpha\beta} = c^{-1}\int d\nu \int d\omega \; I\, n^\alpha n^\beta \qquad n^0 = 1$$

$$\int d\nu \int d\omega \; (I/\nu^3)\left(\frac{h\nu}{c} n^\alpha\right)\left(\frac{h\nu}{c} n^\beta\right) = \frac{h^2}{c^2}\int d\nu \int d\omega \; I\, n^\alpha n^\beta$$

= product of $m^\alpha m^\beta$; (I/ν^3) ; $\nu d\nu d\omega$

$= (h^2/c)\, R^{\alpha\beta}$

$$R^{\alpha\beta} = c^2 \int f_R \, M^\alpha m^\beta d^3M / \tilde{e} \qquad \tilde{e} = h\nu \quad m^\alpha = \frac{h\nu}{c} n^\alpha \text{ etc.}$$

From (63.4) $f_R = \frac{c^2}{h^4 \nu^3} I$

Also $d^3M = p^2 dp\, d\omega$, $p = \frac{h\nu}{c} \Rightarrow d^3M = \frac{h^3}{c^3}\nu^2 d\nu\, d\omega$

hence

$$R^{\alpha\beta} = c^2 \int \left[\frac{c^2}{h^4\nu^3} I\right]\left(\frac{h\nu}{c} n^\alpha\right)\left(\frac{h\nu}{c} n^\beta\right)\frac{h^3}{c^3}\nu^2 d\nu\, d\omega \,/ h\nu$$

$$= \frac{c^4 h^5}{c^5 h^5}\int \frac{I}{\nu^3}\nu^2 n^\alpha n^\beta \cdot \frac{\nu^2 d\nu\, d\omega}{\nu}$$

(91.6) $$\boxed{R^{\alpha\beta} = c^{-1}\int I n^\alpha n^\beta d\nu\, d\omega}$$

(91.7) From (64.4) $E_\nu = c^{-1}\int I\, d\omega$

(64.5) $E = \int d\nu\, E_\nu = c^{-1}\int d\nu \int I\, d\omega = R^{00}$

(65.2) $\underset{\sim}{F}_\nu = \int I\, \underset{\sim}{n}\, d\omega \qquad F^i_\nu = \int I n^i\, d\omega$

(65.4) $F^i = \int d\nu\, F_\nu = \int d\nu \int d\omega\, I n^i \qquad i = 1, 2, 3$

hence

$$R^{0i} = c^{-1} F^i$$

$$R^{i0} = c^{-1} F^i$$

Finally $P^{ij}_\nu = c^{-1}\int I n^i n^j d\omega$ (66.3)

$$P^{ij} = c^{-1}\int d\nu \int I n^i n^j d\omega = R^{ij} \qquad i, j = 1, 2, 3$$

Relate E to E° :

$$E = c^{-1}\int I \, d\nu \, d\omega = c^{-1}\int \frac{I}{\nu^3} \underline{(\nu \, d\nu \, d\omega)} \left(\frac{h\nu}{c}\right)\left(\frac{h\nu}{c}\right)\frac{c^2}{h^2}$$

$$= c^{-1}\int\left[\frac{I^{\circ}}{\nu_0^{3}}(\nu_0 \, d\nu_0 \, d\omega_0)\right] \nu_0^{2}\gamma^2[1+\underset{\sim}{\beta}\cdot\underset{\sim}{n}]^2$$

$$= c^{-1}\int I_0 \, d\nu_0 \, d\omega_0 \cdot \gamma^2[1+\underset{\sim}{\beta}\cdot\underset{\sim}{n}]^2$$

$$E = c^{-1}\gamma^2\left\{\int I_0 \, d\nu_0 \, d\omega_0 + 2\underset{\sim}{\beta}\cdot\int I_0 \, d\nu_0 \, d\omega_0 \underset{\sim}{n}_0 + \int I_0 \, d\nu_0 \, d\omega_0 \underset{\sim}{\beta}\cdot\underset{\sim}{n}_0 \cdot \beta\cdot\underset{\sim}{n}_0\right.$$

(91.10) $$E = \gamma^2\left\{E_0 + 2c^{-2}v_i F_0^{i} + c^{-2}v_i v_j P_0^{ij}\right\}$$

Lorentz transform of F:

$$F^{i} = \int I n^{i} \, d\nu \, d\omega = \int\left[\frac{I}{\nu^3}(\nu \, d\nu \, d\omega)\right]\nu^2 n^{i}$$

$$= \int\left[\frac{I_0}{\nu_0^{3}}\nu_0 \, d\nu_0 \, d\omega_0\right]\nu_0 \, \gamma \, [1+\underset{\sim}{\beta}\cdot\underset{\sim}{n}] \, \nu_0\left\{n_0^{i} + \gamma v^{i}/c\left[1+\frac{\gamma \underset{\sim}{n}_0\cdot\underset{\sim}{v}}{c(\gamma+1)}\right]\right.$$

$$= \gamma\int I_0 \, d\nu_0 \, d\omega_0 \, [1+\underset{\sim}{\beta}\cdot\underset{\sim}{n}_0][n_0^{i} + \gamma v^{i}/c + \gamma^2 n_0^{j} v_j v^{i}/c^2(\gamma+1)]$$

$$= \gamma\int I_0 \, d\nu_0 \, d\omega_0 \, (1+v_k n_0^{k}/c)[n_0^{i} + \gamma v^{i}/c + \gamma^2 n_0^{j} v_j v^{i}/c^2(\gamma+1)]$$

$$= \gamma\int I_0 \, d\nu_0 \, d\omega_0 \{n_0^{i} + \gamma v^{i}/c + \gamma^2 n_0^{j} v_j v^{i}/c^2(\gamma+1) + v_k n_0^{i} n_0^{k} c^{-1} + \gamma v^{i} v_k n_0^{k}/c$$
$$+ \gamma^2 n_0^{j} n_0^{k} v^{i} v_j v_k/c^3(\gamma+1)\}$$

$$= (\gamma^2 v^{i}/c)\int I_0 \, d\nu_0 \, d\omega_0 + \gamma\int I_0 \, d\nu_0 \, d\omega_0 \, n_0^{i} + (\gamma^3 v^{i} v_j/c^2(\gamma+1))\int I_0 n_0^{j} \, d\nu_0 \, d\omega_0$$
$$+ \gamma v_k\int I_0 n_0^{i} n_0^{k} \, d\nu_0 \, d\omega_0 + \gamma^2 v^{i} v_k/c^2\int I_0 n_0^{k} \, d\nu_0 \, d\omega_0 + [\gamma^3 v^{i} v_j v_k/c^3(\gamma+1)]\int I_0 n_0^{j} n_0^{k} \, d\nu$$

$$= \gamma^2 v^{i} E_0 + \gamma F_0^{i} + [\gamma^3/(\gamma+1)]c^{-2} v^{i} v_j F_0^{j} + \gamma v_k P_0^{ik} + \gamma^2 c^{-2} v^{i} v_k F_0^{k} + \frac{\gamma^3}{(\gamma+1)} v^{i} c^{-2} v_j v_k P_0^{jk}$$

$$\gamma\left\{F_0^{i} + \gamma v^{i} E_0 + v_k P_0^{ik} + c^{-2} v^{i}\gamma\left[v_k F_0^{k} + \frac{\gamma}{(\gamma+1)} v_j F_0^{j} + \frac{\gamma}{(\gamma+1)} v_j v_k P_0^{jk}\right]\right\}$$

$$\frac{\gamma}{\gamma+1} v_k F_0^{k} + \frac{(\gamma+1)}{(\gamma+1)} v_k F_0^{k} = \frac{2\gamma+1}{\gamma+1} v_k F_0^{k}$$

(91.11) Hence $F^i = \gamma\{F_0^i + \gamma v^i E_0 + v_k P_0^{ik} + \frac{\gamma v^i}{c^2(\gamma+1)}[(2\gamma+1)v_k F_0^k + \gamma v_j v_k P_0^{ik}]\}$

Lorentz transform of P^{ij}

$$P^{ij} = \int I_i \, d\nu \, d\omega \, n^i n^j = \int \frac{I_0}{\nu_0^3}(\nu_0 d\nu_0 d\omega_0)\cdot(\nu n^i)(\nu n^j)$$

$$\nu n^i = \nu_0\{n_0^i + (\gamma v^i/c)[1 + \gamma n_0^k v_k^i / c(\gamma+1)]\}$$

$$\nu n^j = \nu_0\{n_0^j + \gamma v^j/c[1 + \gamma n_0^l v_l / c(\gamma+1)]\}$$

hence

$$P^{ij} = c^{-1}\int I_0 d\nu_0 d\omega_0 \left\{n_0^i + \frac{\gamma v^i}{c}\left[1 + \frac{\gamma n_0^k v_k}{c(\gamma+1)}\right]\right\}\left\{n_0^j + \frac{\gamma v^j}{c}\left[1 + \frac{\gamma n_0^l v_l}{c(\gamma+1)}\right]\right\}$$

$$= c^{-1}\int I_0 d\nu_0 d\omega_0 \left\{n_0^i + \frac{\gamma v^i}{c} + \frac{\gamma^2 n_0^k v^i v_k}{c^2(\gamma+1)}\right\}\left\{n_0^j + \frac{\gamma v^j}{c} + \frac{\gamma^2}{c^2}\frac{n_0^l v^j v_l}{(\gamma+1)}\right\}$$

$$= c^{-1}\int I_0 d\nu_0 d\omega_0 \{n_0^i n_0^j + n_0^i \gamma v^j c^{-1} + n_0^j \gamma v^i c^{-1} + \frac{\gamma^2}{c^2}\frac{n_0^i n_0^l v^j v_l}{(\gamma+1)} + \frac{\gamma^2 v^i v^j}{c^2}$$
$$+ \frac{\gamma^3}{c^3}\frac{n_0^l v^i v^j v_l}{(\gamma+1)} + \frac{\gamma^2}{c^2}\frac{n_0^k n_0^j v^i v_k}{(\gamma+1)} + \frac{\gamma^3}{c^3}\frac{n_0^k v^i v^i v_k}{(\gamma+1)} + \frac{\gamma^4}{c^4}\frac{n_0^k n_0^l v^i v_k v^j v_l}{(\gamma+1)^2}\}$$

$$= P_0^{ij} + \gamma c^{-2} v^j F_0^i + \gamma c^{-2} v^i F_0^j + \frac{\gamma^2}{c^2(\gamma+1)} v^j v_l P_0^{il} + \frac{\gamma^2}{c^2} v^i v^j E_0$$
$$+ \frac{\gamma^3}{c^4(\gamma+1)} v^i v^j v_l F_0^l + \frac{\gamma^2}{c^2(\gamma+1)} v^i v_k P^{ik} + \frac{\gamma^3}{c^4(\gamma+1)} v_k v^i v^j F_0^k + \frac{\gamma^4}{c^4(\gamma+1)^2} v^i v^j v_k v_l P_0^{kl}$$

$$= P_0^{ij} + \frac{\gamma}{c^2}[v^j F_0^i + v^i F_0^j + \gamma v^i v^j E_0]$$
$$+ \frac{\gamma^2}{c^2(\gamma+1)}[v^j v_l P_0^{il} + \frac{\gamma}{c^2} v^i v^j v_l F_0^l + v^i v_k P^{kj} + \frac{\gamma}{c^2} v^i v^j v_k F_0^k + \frac{\gamma^2}{c^2(\gamma+1)} v^i v^j$$

(91. $$P^{ij} = P_0^{ij} + \gamma c^{-2}[v^i F_0^j + v^j F_0^i + \gamma v^i v^j E_0]$$
$$+ \frac{\gamma^2}{c^2(\gamma+1)}[v^j v_k P_0^{ik} + v^i v_k P_0^{kj} + 2\gamma c^{-2} v^i v^j v_k F_0^k] + \frac{\gamma^4}{c^4(\gamma+1)^2} v^i v^j v_k v_l P_0$$

One dimensional flow:

E:
$$E = \gamma^2 [E_0 + 2c^{-1} v_i F_0^i + c^{-2} v_i v_j P_0^{ij}]$$

only one surviving velocity component, hence sums over i+j become: $v_i F_0^i = v F_0$ $\quad v_i v_j P_0^{ij} = v^2 P_0$ and $v^2 c^{-2} P_0 = \beta^2 P_0$

(91.13) $$E = \gamma^2 [E_0 + 2\beta F_0 + \beta^2 P_0]$$

F:
$$F^i = \gamma [F_0^i + \gamma v^i E_0 + v_k P_0^{ik} + (\gamma v^i / c^2(\gamma+1)) [(2\gamma+1) v_k F_0^k + \gamma v_j v_k P_0^{ik}]$$
$$= \{\gamma^2 v E_0 + F_0 [\gamma + \gamma^2\beta^2 (2\gamma+1)/(\gamma+1)] + P_0 v \gamma [1 + \gamma^2\beta^2/((\gamma+1)]$$

Now $\beta^2 = (\gamma^2 - 1)/\gamma^2$, hence $\gamma^2\beta^2/(\gamma+1) = \gamma^2(\gamma+1)(\gamma-1)/\gamma^2(\gamma+1)$
$= (\gamma-1)$

Therefore the coeff of F_0 is:
$$\gamma + \gamma^2\beta^2(2\gamma+1)/(\gamma+1) = \gamma + (2\gamma+1)(\gamma-1) = \gamma + 2\gamma^2 - \gamma - 1 = 2\gamma^2 - 1 = \frac{2}{1-\beta^2} - 1$$
$$= [2 - (1-\beta^2)]/(1-\beta^2) = (1+\beta^2)/(1-\beta^2) = \gamma^2(1+\beta^2)$$

and the coeff of $\gamma P_0 v$:
$$1 + \gamma^2\beta^2/(\gamma+1) = 1 + (\gamma-1) = \gamma$$

hence
$$F = \gamma^2 v E_0 + \gamma^2(1+\beta^2) F_0 + \gamma^2 v P_0$$

(91.14) $$\boxed{F = \gamma^2 [v E_0 + (1+\beta^2) F_0 + v P_0]}$$

P:
$$P^{ij} = P_0^{ij} + \gamma c^{-2} [v^i F_0^j + v^j F_0^i + \gamma v^i v^j E_0].$$
$$+ \frac{\gamma^2}{c^2(\gamma+1)} [v^j v_k P_0^{ik} + v^i v_k P_0^{kj} + 2\gamma c^{-2} v^i v^j v_k F_0^k] + \left(\frac{\gamma^2}{c^2(\gamma+1)}\right)^2 v^i v^j v_k v_l P_0^{kl}$$

$$= P_0 + 2\gamma c^{-1}\beta F_0 + \gamma^2\beta^2 E_0 + \frac{\gamma^2}{(\gamma+1)} [2\beta^2 P_0 + 2\gamma c^{-1}\beta^3 F_0] + \left[\frac{\gamma^2}{(\gamma+1)}\right]^2 \beta^4 P_0$$
$$= \gamma^2\beta^2 E_0 + P_0 \left\{1 + \frac{2\beta^2\gamma^2}{(\gamma+1)} + \left[\frac{\beta^2\gamma^2}{\gamma+1}\right]^2\right\} + F_0 \cdot 2\gamma c^{-1}\beta \left[1 + \frac{\gamma^2\beta^2}{(\gamma+1)}\right]$$

$$= \gamma^2\beta^2 E_0 + P_0 \{1 + 2(\gamma-1) + (\gamma-1)^2\} + 2\gamma c^{-1}\beta F_0 \{1 + (\gamma-1)\}$$
$$= \gamma^2\beta^2 E_0 + P_0 \{\cancel{1} + \cancel{2\gamma} - \cancel{2} + \gamma^2 - \cancel{2\gamma} + \cancel{1}\} + 2\gamma^2 c^{-1}\beta F_0$$
91.15) $$= \gamma^2 \{P_0 + \beta^2 E_0 + 2v F_0\}$$

To first order in β this becomes:

(91.16a) $E = E_0 + 2\beta c^{-1} F_0 = E_0 + 2 v c^{-2} F_0$

(91.17a) $F = F_0 + V\,(E_0 + P_0)$

(91.18a) $P = P_0 + 2\beta c^{-1} F_0 = P_0 + 2 v c^{-2} F_0$

Then using $E_0 = 4\pi c^{-1} J_0$; $E = 4\pi c^{-1} J$

$F_0 = 4\pi H_0 \qquad F = 4\pi H$

$P_0 = 4\pi c^{-1} K_0 \qquad P = 4\pi c^{-1} R$

we have

(91.16b) $J = J_0 + 2\beta F_0$

(91.17b) $H = H_0 + \beta(J_0 + K_0)$

(91.18b) $K = K_0 + 2\beta H_0$

$$
\begin{aligned}
\mathcal{G}^0 &= c^{-1}\int d\nu\, d\omega\, [\chi I - \eta] \\
&= c^{-1}\int (\nu\, d\nu\, d\omega)\left[\nu^{-2}(\chi I - \eta)\right]\nu \\
&= c^{-1}\int (\nu_0\, d\nu_0\, d\omega_0)\,\nu_0^{-2}(\chi_0 I_0 - \eta_0)\,\gamma\nu_0\,(1 + \underset{\sim}{\beta}\cdot\underset{\sim}{n}) \\
&= c^{-1}\int d\nu_0\, d\omega_0\,(\chi_0 I_0 - \eta_0) + \gamma\underset{\sim}{\beta}\cdot\left[c^{-1}\int d\nu_0\, d\omega_0\,[I\chi - \eta]\,\underset{\sim}{n}\right]
\end{aligned}
$$

(91.22a) $\mathcal{G}^0 = \mathcal{G}_0^0 + \gamma\underset{\sim}{\beta}\cdot\underset{\sim}{\mathcal{G}}_0$

$$
\begin{aligned}
\mathcal{G}^1 &= c^{-1}\int d\nu\, d\omega\,[\chi I - \eta]\,\underset{\sim}{n} \\
&= c^{-1}\int (\nu\, d\nu\, d\omega)\,\nu^{-2}[\chi I - \eta]\,(\nu\underset{\sim}{n}) \\
&= c^{-1}\int (\nu_0\, d\nu_0\, d\omega_0)\,\nu_0^{-2}[\chi_0 I_0 - \eta_0]\,\nu_0\left[\underset{\sim}{n}_0 + \gamma\underset{\sim}{\beta}\left(1 + \frac{\gamma\underset{\sim}{n}_0\cdot\underset{\sim}{\beta}}{(\gamma+1)}\right)\right] \\
&= c^{-1}\int d\nu_0\, d\omega_0\,[\chi_0 I_0 - \eta_0]\left[\underset{\sim}{n}_0 + \gamma\underset{\sim}{\beta} + \gamma^2(\underset{\sim}{n}_0\cdot\underset{\sim}{\beta})\underset{\sim}{\beta}/(\gamma+1)\right]
\end{aligned}
$$

For one-dimensional flow,

$$
\begin{aligned}
\underset{\sim}{n}_0 + \gamma\underset{\sim}{\beta} + \gamma^2(\underset{\sim}{n}_0\cdot\underset{\sim}{\beta})\underset{\sim}{\beta}/(\gamma+1) &= n_0 + \gamma\beta + [\gamma^2\beta^2/(\gamma+1)]\,n_0 \\
&= n_0 + (\gamma - 1)n_0 + \gamma\beta = \gamma(\beta + n_0)
\end{aligned}
$$

$\mathcal{G}^1 = c^{-1}\int d\nu_0\, d\omega_0\,[\chi_0 I_0 - \eta_0]\,[\gamma\beta + \gamma n_0]$

(91.22b) $\quad = \gamma\beta\,\mathcal{G}_0^0 + \gamma\,\mathcal{G}_0^1$

The reverse procedure, with $\nu_0 = \gamma\nu(1 - \underset{\sim}{n}\cdot\underset{\sim}{\beta})$

and $\nu_0 \underline{n}_0 = \nu[\underset{\sim}{n} - \gamma\underset{\sim}{\beta}(1 - \gamma\underset{\sim}{n}\cdot\underset{\sim}{\beta}/(\gamma+1))]$

yields for 1-D flow:

(91,23a) $$G_0^0 = \gamma(G^0 - \beta G^1)$$

(91.23b) $$G_0^1 = \gamma(G^1 - \beta G^0)$$

Section 92 : Covariant Form of the Transfer Equation

Generalize the Boltzmann equation

$$\partial f/\partial t + v^i(\partial f/\partial x_i) + \dot{p}^i(\partial f/\partial p_i) = (Df/Dt)_{coll}$$

to

(92.4) $$\left(\frac{dx^\alpha}{d\lambda}\right)\frac{\partial f}{\partial x^\alpha} + \left(\frac{dp^\alpha}{d\lambda}\right)\left(\frac{\partial f}{\partial p^\alpha}\right) = \left(\frac{\delta f}{\partial\lambda}\right)_{coll} = p^\alpha\left(\frac{\partial f}{\partial x^\alpha}\right) + \dot{p}^\alpha\left(\frac{\partial f}{\partial p^\alpha}\right)$$

(3) where we define $p^\alpha \equiv (dx^\alpha/d\lambda)$

(.12.5) $$\dot{p}^\alpha \equiv (dp^\alpha/d\lambda)$$

λ is used in (92.4) instead of proper time τ because photon trajectories lie on null cone.

If $M^\alpha = p^\alpha$ and ϵ = source, $-\alpha f_R$ = sink then:

$$M^\alpha \frac{\partial f_R}{\partial x^\alpha} + \dot{M}^\alpha \frac{\partial f_R}{\partial M^\alpha} = \epsilon - \alpha f_R \equiv \left(\frac{\delta f_R}{\delta\lambda}\right)$$

From (63,4): $I = (h^4\nu^3/c^2) f_R \Rightarrow f_R = (I/\nu^3)(c^2/h^4) = \mathcal{I}\, c^2/h^4$

$$M^\alpha \frac{c^2}{h^4}\frac{\partial\mathcal{I}}{\partial x^\alpha} + \dot{M}^\alpha \frac{c^2}{h^4}\frac{\partial\mathcal{I}}{\partial M^\alpha} = \epsilon - \alpha\frac{c^2}{h^4}\mathcal{I}$$

or

(92.7) $$M^\alpha\frac{\partial\mathcal{I}}{\partial x^\alpha} + \dot{M}^\alpha\frac{\partial\mathcal{I}}{\partial M^\alpha} = \epsilon\left(\frac{h^4}{c^2}\right) - \alpha\mathcal{I} \equiv e - a\mathcal{I}$$

In an inertial frame, $\dot{M}^\alpha \equiv 0$ for photons, and ν = constant, hence

$$M^\alpha \frac{\partial}{\partial x^\alpha}(I/\nu^3) = \frac{h\nu}{c}\frac{n^\alpha}{\nu^3}\frac{\partial I}{\partial x^\alpha}$$

$$= \frac{h}{c}\frac{1}{\nu^2}\left[\frac{1}{c}\frac{\partial I}{\partial t} + \underset{\sim}{n}\cdot\nabla I\right] = e - a\frac{I}{\nu^3}$$

$$\frac{1}{\nu^2}\left[\frac{1}{c}\frac{\partial I}{\partial t} + \underset{\sim}{n}\cdot\nabla I\right] = \frac{c}{h}e - \frac{c}{h}a\frac{I}{\nu^3}$$

Right-hand-side must scale as ν^{-2} like lhs, and we identify

$$(c/h)e = \eta_\nu/\nu^2$$
$$(c/h)a = \nu\chi_\nu$$

Note: in § 92 $c=h=1$ units are used, and then $e = \eta_\nu/\nu^2$ $a = \nu\chi_\nu$

So now we have:

$$\nu^{-2}[\eta - \chi I] = \nu^{-2}\left[\frac{1}{c}\frac{\partial I}{\partial t} + (\underset{\sim}{n}\cdot\nabla)I\right] = \frac{c}{h}M^\alpha \mathcal{I}_{;\alpha}$$

But ~~$\mathcal{I}$~~ = world scalar; $\mathcal{I}_{;\alpha}$ = covariant and $M^\alpha\mathcal{I}_{;\alpha}$ = invariant.

hence $\frac{c}{h}M^\alpha\mathcal{I}_{;\alpha} = \frac{c}{h}M^{\alpha'}\mathcal{I}'_{;\alpha'} = \nu'^{-2}\left[\frac{1}{c}\frac{\partial I'}{\partial t'} + (\underset{\sim}{n}'\cdot\nabla')I'\right]$

and since $\nu^{-2}[\eta - \chi I]$ is also invariant:

(92.11) $$[\eta' - \chi' I'] = \frac{1}{c}\frac{\partial I'}{\partial t'} + (\underset{\sim}{n}'\cdot\nabla')I'$$

Alternate derivation: boost matrix is: $\begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} = L^{\alpha'}{}_{\beta}$

(35.39): $B'_{\alpha} = (\partial x^\beta/\partial x^{\alpha'})B_\beta = L^\beta{}_{\alpha'}B_\beta$

(92.12) $$\begin{pmatrix} \frac{1}{c}\partial/\partial t \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} c^{-1}\partial/\partial t' \\ \partial/\partial x' \\ \partial/\partial y' \\ \partial/\partial z' \end{pmatrix} = \begin{pmatrix} \gamma c^{-1}\partial/\partial t' - \beta\gamma\,\partial/\partial z' \\ \partial/\partial x' \\ \partial/\partial y' \\ \gamma\,\partial/\partial z' - \beta\gamma c^{-1}\partial/\partial t' \end{pmatrix}$$

Then

$$\frac{1}{c}\frac{\partial}{\partial t} + \underset{\sim}{n}\cdot\nabla = \frac{1}{c}\frac{\partial}{\partial t} + n_x\frac{\partial}{\partial x} + n_y\frac{\partial}{\partial y} + n_z\frac{\partial}{\partial z}.$$

$$= \left(\frac{\gamma}{c}\frac{\partial}{\partial t'} - \gamma\beta\frac{\partial}{\partial z'}\right) + n_x\frac{\partial}{\partial x'} + n_y\frac{\partial}{\partial y'} + n_z\gamma\left(\frac{\partial}{\partial z'} - \frac{\beta}{c}\frac{\partial}{\partial t'}\right)$$

$$= \left(\frac{\gamma}{c}\frac{\partial}{\partial t'} - \frac{\gamma n_z\beta}{c}\frac{\partial}{\partial t'}\right) + n_x\frac{\partial}{\partial x'} + n_y\frac{\partial}{\partial y'} + \left(\gamma n_z\frac{\partial}{\partial z'} - \beta\gamma\frac{\partial}{\partial z'}\right)$$

$$= \frac{\gamma}{c}(1 - n_z\beta)\frac{\partial}{\partial t'} + n_x\frac{\partial}{\partial x'} + n_y\frac{\partial}{\partial y'} + \gamma(n_z - \beta)\frac{\partial}{\partial z'}$$

From (89.9) $\gamma(1 - n_z\beta) = \nu'/\nu$; $n_x = n_x'\nu'/\nu$; $n_y = n_y'\nu'/\nu$; $\gamma(n_z - \beta) = n_z'\nu'/\nu$

hence

(92.13) $$\frac{1}{c}\frac{\partial}{\partial t} + \underset{\sim}{n}\cdot\nabla = \frac{1}{c}\frac{\nu'}{\nu}\frac{\partial}{\partial t'} + \frac{\nu'}{\nu}n_x'\frac{\partial}{\partial x'} + \frac{\nu'}{\nu}n_y'\frac{\partial}{\partial y'} + \frac{\nu'}{\nu}n_z'\frac{\partial}{\partial z'}$$

$$= \frac{\nu'}{\nu}\left[\frac{1}{c}\frac{\partial}{\partial t'} + \underset{\sim}{n}'\cdot\nabla'\right]$$

Now $I_\nu = (\nu^3/\nu'^3) I_\nu'$; $\eta_\nu = (\nu^2/\nu'^2)\eta_\nu'$; $\chi_\nu = (\nu'/\nu)\chi_\nu'$

hence

$$\eta_\nu - \chi_\nu I_\nu = (\nu^2/\nu'^2)\,[\eta_\nu' - \chi_\nu' I_\nu']$$

(92.15) and $\left\{\frac{\nu'}{\nu}\left[\frac{1}{c}\frac{\partial}{\partial t'} + \underset{\sim}{n}'\cdot\nabla'\right]\right\}\left\{(\nu^3/\nu'^3) I_\nu'\right\} = (\nu^2/\nu'^2)\,[\eta_\nu' - \chi_\nu' I_\nu']$;

When relative motion is constant $\frac{\partial}{\partial x^\alpha}[(\nu^3/\nu'^3) I_\nu] = (\nu^3/\nu'^3)\,\partial I_\nu/\partial x^\alpha$

and

$$\frac{\nu^2}{\nu'^2}\left\{\left[\frac{1}{c}\frac{\partial}{\partial t'} + \underset{\sim}{n}'\cdot\nabla'\right] I_\nu'\right\} = \frac{\nu^2}{\nu'^2}\left\{\eta_\nu' - \chi_\nu' I_\nu'\right\}$$

which yields (92.11) by cancelling (ν^2/ν'^2).

Section 7.2 : The Dynamical Equations for a Radiating Fluid

Section 93 : The Inertial-Frame Transfer Equation for a Moving Fluid

$$\chi(\nu, \underset{\sim}{n}) = \frac{\nu_0}{\nu}\chi_0(\nu_0) = \gamma(1 - \underset{\sim}{\beta}\cdot\underset{\sim}{n})\,\chi_0[\gamma\nu(1 - \underset{\sim}{\beta}\cdot\underset{\sim}{n})]$$

To order v/c, $\gamma \to 1$. Then $\chi_0(\nu - \nu\underset{\sim}{\beta}\cdot\underset{\sim}{n}) \simeq \chi_0(\nu) - \nu\underset{\sim}{\beta}\cdot\underset{\sim}{n}\,\frac{\partial\chi_0(\nu)}{\partial\nu}$

and

$$\chi(\nu, \underset{\sim}{n}) \cong (1 - \underset{\sim}{\beta}\cdot\underset{\sim}{n})\left[\chi_0(\nu) - \nu\underset{\sim}{\beta}\cdot\underset{\sim}{n}\,\frac{\partial\chi_0(\nu)}{\partial\nu}\right]$$

$$\simeq \chi_0(\nu) - \underset{\sim}{\beta}\cdot\underset{\sim}{n}\,\chi_0(\nu) - \nu\underset{\sim}{\beta}\cdot\underset{\sim}{n}\,\frac{\partial\chi_0(\nu)}{\partial\nu}$$

(93.2) $$\chi(\nu, \underset{\sim}{n}) = \chi_0(\nu) - \underset{\sim}{\beta}\cdot\underset{\sim}{n}\left[\chi_0(\nu) + \nu\frac{\partial\chi_0(\nu)}{\partial\nu}\right]$$

$$\eta(\nu, \underset{\sim}{n}) = \frac{\nu^2}{\nu_0^2}\eta_0(\nu_0) \cong [1 + \underset{\sim}{n}\cdot\underset{\sim}{\beta}]^2\left[\eta_0(\nu) - \nu\underset{\sim}{n}\cdot\underset{\sim}{\beta}\,\frac{\partial\eta_0(\nu)}{\partial\nu}\right]$$

$$\cong \eta_0(\nu) + 2\underset{\sim}{n}\cdot\underset{\sim}{\beta}\ \eta_0(\nu) - \nu\underset{\sim}{n}\cdot\underset{\sim}{\beta}\,\partial\eta_0(\nu)/\partial\nu$$

(93.3) $$\eta(\nu, \underset{\sim}{n}) = \eta_0(\nu) + \underset{\sim}{n}\cdot\underset{\sim}{\beta}\,[2\eta_0(\nu) - \nu\,\partial\eta_0(\nu)/\partial\nu]$$

Then

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \underset{\sim}{n}\cdot\underset{\sim}{\nabla}\right) I(\nu, \underset{\sim}{n}) = \eta(\nu, \underset{\sim}{n}) - \chi(\nu, \underset{\sim}{n})\, I(\nu, \underset{\sim}{n})$$

$$= \eta_0 + \underset{\sim}{n}\cdot\underset{\sim}{\beta}\,[2\eta_0(\nu) - \nu\,\partial\eta_0/\partial\nu] - \{\chi_0 - \underset{\sim}{\beta}\cdot\underset{\sim}{n}\,(\chi_0 + \nu\,\partial\chi_0/\partial\nu)\} I$$

(93.4) $$\left(\frac{1}{c}\frac{\partial}{\partial t} + \underset{\sim}{n}\cdot\underset{\sim}{\nabla}\right) I = \eta_0 - \chi_0 I + \underset{\sim}{n}\cdot\underset{\sim}{\beta}\,\{2\eta_0 - \nu\,\partial\eta_0/\partial\nu + (\chi_0 + \nu\,\partial\chi_0/\partial\nu) I\}$$

moment equation:

$$\frac{1}{4\pi}\int d\omega\left\{\frac{1}{c}\frac{\partial I}{\partial t}+\underset{\sim}{n}\cdot\nabla I\right\}=\frac{1}{4\pi}\int d\omega\left\{\eta_0-\chi_0 I+\underset{\sim}{\beta}\cdot\underset{\sim}{n}\left[2\eta_0-\nu\frac{\partial\eta_0}{\partial\nu}+\left[\chi_0+\nu\frac{\partial\chi_0}{\partial\nu}\right]I\right]\right.$$

$$=\frac{1}{c}\frac{\partial}{\partial t}\left[\frac{1}{4\pi}\int I\,d\omega\right]+\frac{1}{4\pi}\int n\,\frac{\partial I}{\partial x}\,d\omega=\eta_0\frac{1}{4\pi}\int d\omega-\chi_0\frac{1}{4\pi}\int I\,d\omega$$

$$+\,2\underset{\sim}{\beta}\cdot\left[\frac{1}{4\pi}\int\underset{\sim}{n}\,d\omega\right]-\nu\frac{\partial\eta_0}{\partial\nu}\,\underset{\sim}{\beta}\cdot\frac{1}{4\pi}\int\underset{\sim}{n}\,d\omega+\left[\chi_0+\nu\frac{\partial\chi_0}{\partial\nu}\right]\underset{\sim}{\beta}\cdot\frac{1}{4\pi}\int\underset{\sim}{n}\,I\,d\omega$$

Now $\int n^i\,d\omega=0$ $\quad\frac{1}{4\pi}\int I\,d\omega=J_\nu\quad\frac{1}{4\pi}\int n^i I\,d\omega=H^i$, hence

$\int d\omega=4\pi$

$$\frac{1}{c}\frac{\partial J_\nu}{\partial t}+\frac{1}{4\pi}\int n^i\frac{\partial I}{\partial x^i}\,d\omega=\eta_0-\chi_0 J_\nu+\cancelto{0}{\frac{2\beta_i}{4\pi}\int n^i\,d\omega}-\nu\frac{\partial\eta_0}{\partial\nu}\cancelto{0}{\beta_i/4\pi\int n^i\,d\omega}$$

$$+\,\chi_0\beta_i H^i+\nu\frac{\partial\chi_0}{\partial\nu}\beta_i H^i$$

To evaluate $\frac{1}{4\pi}\int n^i\frac{\partial I}{\partial x^i}\,d\omega$, we assume (?) $\frac{\partial n^i}{\partial x^i}\equiv 0$, then it become

$$\frac{1}{4\pi}\frac{\partial}{\partial x^i}\int n^i I\,d\omega=\quad\partial H^i_\nu/\partial x^i$$

hence

(93.5) $$\frac{1}{c}\frac{\partial J_\nu}{\partial t}+\frac{\partial H^i_\nu}{\partial x^i}=\eta_0-\chi_0 J_\nu+\left[\chi_0+\nu\frac{\partial\chi_0}{\partial\nu}\right]\beta_i H^i$$

First Moment Equation:

$$\frac{1}{4\pi}\int\underset{\sim}{n}\,d\omega\,\frac{1}{c}\frac{\partial I}{\partial t}+\frac{1}{4\pi}\int\underset{\sim}{n}\,d\omega\,(\underset{\sim}{n}\cdot\nabla I)=\frac{\eta_0}{4\pi}\int\underset{\sim}{n}\,d\omega-\frac{\chi_0}{4\pi}\int\underset{\sim}{n}\,I\,d\omega+2\eta_0\underset{\sim}{\beta}\cdot\frac{1}{4\pi}\int\underset{\sim}{n}\,\underset{\sim}{n}\,d\omega$$

$$-\,\nu\frac{\partial\eta_0}{\partial\nu}\frac{\underset{\sim}{\beta}}{4\pi}\cdot\int\underset{\sim}{n}\,\underset{\sim}{n}\,d\omega+\chi_0\frac{\underset{\sim}{\beta}}{4\pi}\cdot\int\underset{\sim}{n}\,\underset{\sim}{n}\,I\,d\omega+\nu\frac{\partial\chi_0}{\partial\nu}\frac{\underset{\sim}{\beta}}{4\pi}\int\underset{\sim}{n}\,\underset{\sim}{n}\,I\,d\omega$$

or $$\frac{1}{4\pi}\frac{1}{c}\frac{\partial}{\partial t}\int n^i I\,d\omega+\frac{1}{4\pi}\int n^i\left(n^j\frac{\partial I}{\partial x^j}\right)d\omega=\cancel{\frac{\eta_0}{4\pi}\int n_i\,d\omega}-\frac{\chi_0}{4\pi}\int n_i I\,d\omega+\frac{2\eta_0}{4\pi}\beta_j\int n^i n^j\,d\omega$$

$$-\,\nu\frac{\partial\eta_0}{\partial\nu}\frac{\beta_j}{4\pi}\int n^i n^j\,d\omega+\left[\chi_0+\nu\frac{\partial\chi_0}{\partial\nu}\right]\cdot\frac{\beta_j}{4\pi}\int n^i n^j I\,d\omega$$

$$\frac{1}{c}\frac{\partial H^i_\nu}{\partial t}+\frac{\partial K^{ij}_\nu}{\partial x^j}=-\chi_0 H^i_\nu+\frac{2\eta_0}{3}\beta_i-\nu\frac{\partial\eta_0}{\partial\nu}\frac{\beta^i}{3}+\left[\chi_0+\nu\frac{\partial\chi_0}{\partial\nu}\right]\beta_j K^{ij}_\nu$$

(93.6) $$\frac{1}{c}\frac{\partial H^i_\nu}{\partial t}+\frac{\partial K^{ij}_\nu}{\partial x^j}=-\chi_0 H^i_\nu+\frac{\beta^i}{3}\left[2\eta_0-\nu\frac{\partial\eta_0}{\partial\nu}\right]+\left[\chi_0+\nu\frac{\partial\chi_0}{\partial\nu}\right]\beta_j K^{ij}_\nu$$

In $\int n^i n^j\,d\omega$, the integral $=0$ when $i\neq j$. Pick μ and ϕ so that

$n^i = \mu$. By definition $d\omega = \mu d\phi$. Hence for $i=j$ the

(93.7) integral is $\int_0^{2\pi} d\phi \int_{-1}^{1} \mu^2 d\mu = 2\pi \int \mu^2 d\mu = 2\pi \cdot \frac{\mu^3}{3}\Big]_{-1}^{+1} = 2\pi \cdot \frac{2}{3} = \frac{4\pi}{3}$

To write (93.5) and (93.6) in terms of E_ν, F_ν^i and P_ν^{ij}, use:

$$J_\nu = \frac{c}{4\pi} E_\nu \qquad H_\nu^i = \frac{1}{4\pi} F_\nu^i \qquad K_\nu^{ij} = \frac{c}{4\pi} P_\nu^{ij}$$

(93.5) becomes

$$\frac{\partial E_\nu}{\partial t} + \frac{\partial F_\nu^i}{\partial x^i} = 4\pi \eta_0 - c\chi_0 E_\nu + \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu}\right] \beta_i F_\nu^i$$

or

(93.5a) $$\frac{\partial E_\nu}{\partial t} + \frac{\partial F_\nu^i}{\partial x^i} = 4\pi\left[\eta_0 - \chi_0 J_\nu\right] + \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu}\right] \beta_i F_\nu^i$$

and (93.6) becomes:

(93.5b) $$\frac{1}{c} \frac{\partial F_\nu^i}{\partial t} + c \frac{\partial P_\nu^{ij}}{\partial x^j} = -\chi_0 F_\nu^i + \frac{4\pi\beta_i}{3}\left[2\eta_0 - \nu \frac{\partial \eta_0}{\partial \nu}\right] + \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu}\right] v_j P^{ij}$$

note: $2\eta_0 - \nu \frac{\partial \eta_0}{\partial \nu} = 3\eta_0 - \partial(\nu\eta_0)/\partial\nu$

Integrating over frequency, (93.5a) becomes the energy equation:

(93.8) $$\frac{\partial E}{\partial t} + \frac{\partial F^i}{\partial x^i} = 4\pi \int (\eta_0 - \chi_0 J_\nu)\, d\nu + \beta_i \int \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu}\right] F_\nu^i \, d\nu$$

$$= c G^0$$

and (93.6a) becomes the momentum equation:

(93.9) $$\frac{1}{c^2} \frac{\partial F^i}{\partial t} + \frac{\partial P^{ij}}{\partial x^j} = -\frac{1}{c} \int \chi_0 F_\nu^i \, d\nu + \frac{4\pi v_i}{3 c^2} \int \left[2\eta_0 - \nu \frac{\partial \eta_0}{\partial \nu}\right] d\nu$$

$$+ \frac{v_j}{c} \int \left[\chi_0 + \nu \frac{\partial \chi_0}{\partial \nu}\right] P_\nu^{ij} \, d\nu = -G^i$$

where $\int[2\eta_0 - \nu\,\partial\eta_0/\partial\nu]\,d\nu = \int[3\eta_0\, d\nu] - \int[\partial(\nu\eta_0)/\partial\nu]\, d\nu = 3\int \eta_0 d\nu - 0$

For grey material, χ_0 is not a fn. of ν and $\eta_0(\nu)/\chi_0(\nu) = B_\nu$, the Planck function

(93.8) becomes

$$\frac{\partial E}{\partial t} + \frac{\partial F^i}{\partial x^i} = 4\pi\chi_0 \int (B_\nu - J_\nu)\, d\nu + \chi_0 \beta_i \int F_\nu^i \, d\nu \, ;$$

using κ for χ_0 and integrating:

$$\frac{\partial E}{\partial t} + \frac{\partial F^i}{\partial x^i} = 4\pi\kappa\,(B - cE/4\pi) + \kappa\, \beta_i F^i$$

(93.10) $$E_{,t} + F^i_{\ ,i} = \kappa\,(4\pi B - cE) + \kappa\, \beta_i F^i$$

(93.9) $c^{-2}F^{i}_{,t} + P^{ij}_{,j} = -c^{-1}\int \chi_0 F^{i}_{\nu}\,d\nu + 4\pi v^{i} c^{-2}\int \eta_0\, d\nu + v^{j} c^{-1}\int\left[\chi_0 + \nu\frac{\partial \chi_0}{\partial \nu}\right] P^{ij}_{\nu}\,d\nu$

for grey material:

(93.11) $c^{-2}F^{i}_{,t} + P^{ij}_{,j} = -c^{-1}\kappa F^{i} + 4\pi v^{i} c^{-2}\kappa B + v_j c^{-1}\kappa P^{ij}$

(91.16a): $E \approx E_0 + 2vc^{-2}F_0$

if $E_0 \approx 4\pi B/c$ (diffusion regime) then

$E = 4\pi B/c + 2vc^{-2}F_0$

or $4\pi B - cE = -2\utilde{\beta}\cdot\utilde{F}_0$

and $E_{,t} + F^{i}_{,i} = \chi(4\pi B - cE) + \chi\,\utilde{\beta}\cdot\utilde{F}$

$= \chi\utilde{\beta}\cdot(\utilde{F} - 2\utilde{F}_0)$

From (91.17a), $F = F_0 + v(E_0 + P_0)$

thus:

$E_{,t} + F^{i}_{,t} = \chi\utilde{\beta}\cdot[\utilde{F}_0 + \utilde{v}(E_0+P_0) - 2\utilde{F}_0]$

$= \chi\utilde{\beta}\cdot[-\utilde{F}_0 + \utilde{v}(E_0 + P_0)]$

(93.12) $E_{,t} + F^{i}_{,i} \approx \chi\utilde{\beta}\cdot(-\utilde{F}_0)$ in the <u>static</u> diffusion regime.

Order of magnitude estimate of $(\chi\utilde{v}\cdot\utilde{F}/c)/\utilde{\nabla}\cdot\utilde{F}$:

$\chi \sim 1/\lambda_p$; $\utilde{\nabla}\cdot\utilde{F} \sim F/l$, hence

$(\chi\utilde{v}\cdot\utilde{F}/c)/\utilde{\nabla}\cdot\utilde{F} \approx (vF/c\lambda_p)(F/l) \sim lv/c\lambda_p$

O magnitude estimate of $(c^{-2}F^{i}_{,t})/(\chi F^{i}/c)$

$\approx (c^{-2}F/t_f)(\lambda_p^{-1}F/c)$

and $l/t_f \sim v$, hence $t_f^{-1} \sim vl^{-1}$ and

ratio $\approx (v/lc)(\ \lambda_p) = v\lambda_p/lc$

Thus (93.11) becomes

$P^{ij}_{,j} \approx -\chi c^{-1}F^{i} + \chi v^{i}c^{-1}(4\pi B/c) + \chi c^{-1}v_j P^{ij}$

$\chi c^{-1}F^{i} = \chi v^{i} c^{-2}4\pi B + \chi c^{-1}v_j P^{ij} \quad -P^{ij}_{;j}$

(93.13) $F^{i} = 4\pi v^{i}c^{-1}B - (c/\chi)P^{ij}_{;j} + v_j P^{ij}$

$\eta - \chi I \rightarrow \eta - \chi J = \chi\left(\frac{\eta}{\chi} - J\right)$ Diffusion limit $\eta/\chi = B$, hence

$\eta - \chi J = \chi(B - J) = (\chi/4\pi)(4\pi B - cE)$ as $J = cE/4\pi$

Assume (show in §97) that diffusion limit gives: $E_0 \Rightarrow 4\pi B/c$;

$P^{ij} = P_0^{ij} + O(\lambda_p v/\ell c) = \frac{1}{3}E_0 \delta^{ij} + O(\lambda_p v/\ell c)$; $F_0 \approx -c\chi^{-1}\underline{\nabla}\cdot P_0$

then

$$F^i \approx v^i E_0 + v_j\left[P_0^{ij} + O(\lambda_p v/\ell c)\right] - (c/\chi)\left(P_0^{ij} + O(\lambda_p v/\ell c)\right)_{;j}$$

(93.14) $$F^i \approx v^i E_0 + v_j P_0^{ij} - F_0 \approx F_0^i + \frac{4}{3}E_0 v^i$$

Streaming limit: $\ell/\lambda_p \lesssim 1$; $E \approx P$; $F \approx cE$

Diffusion limit: $E \approx 3P$

Static diffusion: $t_f \gg t_d$, $(v/c) \ll (\lambda_p/\ell)$; $(F/cE) = O(\lambda_p/\ell)$

Dynamic diffusion: $t_f \lesssim t_d$, $(v/c) \gtrsim (\lambda_p/\ell)$; $F \sim (cE)(v/c) \equiv (vE)$

Equation (91.17a): $F \approx F_0 + v(E_0 + P_0)$

Dynamic diffusion $v(E_0 + P_0) \gg F_0$; $E_0 \approx 3P_0$

$F \approx v \cdot \frac{4}{3}E_0$ and $(F/cE) \sim O(v/c)$

For estimate of $S = \chi[4\pi B - cE)$ 80.20:

$$c^{-1}\partial E_\nu/\partial t = \frac{1}{3}\underline{\nabla}\cdot(\chi_\nu^{-1}\nabla E_\nu) + (4\pi/c)\eta_\nu - \chi_\nu E_\nu$$

or $$c^{-1}\partial E/\partial t = 1/3\, \nabla\cdot(\chi^{-1}\nabla E) + c^{-1}\chi[4\pi B - cE]$$

Scaling: $c^{-1}E/t_f \qquad \frac{1}{3}\frac{1}{\ell}\lambda_p \frac{E}{\ell} \qquad -c^{-1}S$

$c^{-1}Ev\ell^{-1} \qquad \frac{1}{3}E\lambda_p/\ell^2 \qquad c^{-1}S$.

for dynamic diffusion, where time derivative dominates over ∇^2 term

$$c^{-1}S \sim c^{-1}Ev/\ell \Rightarrow S \sim E(v/\ell)$$

for Static diffusion where $\partial/\partial t$ term is ignored:

$$c^{-1}S \sim \frac{1}{3}E\lambda_p/\ell^2 \Rightarrow S \sim E(\lambda_p c/\ell^2)$$

Transfer equation, Streaming limit (93.4): on a fluid-flow timescale $t_f = \ell/v$

(a) $$c^{-1}\frac{\partial I}{\partial t} + \underset{\sim}{n}\cdot\nabla I = \eta_0 - \chi_0 I + \left(\frac{n\cdot v}{c}\right)\left\{2\eta_0 - \nu\frac{\partial \eta_0}{\partial \nu} + \left[\chi_0 + \nu\frac{\partial \chi_0}{\partial \nu}\right] I\right\}$$

$$\frac{I}{ct_f} \qquad \frac{I}{\ell} \qquad \frac{I}{\lambda_p} \qquad \frac{I}{\lambda_p} \qquad \frac{v}{c}\,\frac{I}{\lambda_p}$$

multiply thru by ℓ/I

(a) $\frac{\ell}{ct_f} = \frac{\ell v}{c\ell} = \frac{v}{c}$; 1 ; $\frac{\ell}{\lambda_p}$; $\frac{\ell}{\lambda_p}$; $\frac{v}{c}\frac{\ell}{\lambda_p}$

on a radiation-flow timescale t_R,

(b) $\frac{1}{c}\frac{\partial}{\partial t} = \frac{1}{c}\frac{1}{t_R} = \frac{1}{c}\frac{c}{\ell} = \frac{1}{\ell}$ and $\frac{I}{\ell}\cdot\left(\frac{\ell}{I}\right) = 1$

Transfer equation: diffusion limit

$$c^{-1}\frac{\partial I}{\partial t} + \underline{n}\cdot\nabla I = \chi_0[\eta_0/\chi_0 - I] + \underline{n}\cdot\underline{\beta}\left\{2\eta_0 - \nu\frac{\partial\eta_0}{\partial\nu} + (\chi_0 + \nu\frac{\partial\chi_0}{\partial\nu}]I\right\}$$

$$\sim \chi_0[\eta_0/\chi_0 - \tfrac{c}{4\pi}E]$$

(I/ct_f) (I/ℓ) $\frac{1}{4\pi\lambda_p}[\sim 4\pi B - cE]$ $\frac{v}{c}\frac{1}{\lambda_p}[O(4\pi B - cE)]$

Static diffusion:

$(I/\ell)(v/c)$ (I/ℓ) $E(c\lambda_p/\ell^2)$, $(v/c)(E\lambda_p c/\ell^2 + 3I/\lambda_p)$

$I/\ell\{$ $\frac{v}{c}$ 1 $\frac{E}{I}(c\lambda_p/\ell)$ $(v\lambda_p/\ell)E/I + (v/c)(\ell/$

But $E/I \sim 4\pi/c$ hence term order is:

(c) $\frac{v}{c}$ 1 (λ_p/ℓ) $(v/c)(\lambda_p/\ell) + (v/c)(\ell/\lambda$

Dynamic diffusion:

$(I/\ell)(v/c)$ (I/ℓ) $E(v/\ell)$ $E(v/c)(v/\ell)$; $(v/c)I/\lambda_p$

$E \sim (4\pi/c)I \Rightarrow$

$(I/\ell)(v/c)$ (I/ℓ) $(I/\ell)(v/c)$. $(I/\ell)(v/c)(v/c)$; $(v/c)(I/\ell)($

(d) or v/c 1 v/c v^2/c^2; $(v/c)(\ell/\lambda_p)$

Radiative momentum equation (93.)

$$c^{-2}F^i_{,t} + P^{ij}_{;j} = -c^{-1}\int\chi_0 F^i d\nu + 4\pi c^{-2}v^i\int\eta_0 d\nu + c^{-1}v_j\int[\chi_0 + \nu\partial\chi_0/\partial\nu]P^{ij}d\nu$$

(a) Streaming limit; fluid-flow timescale

$c^{-2}F/t_f$ P/ℓ $c^{-1}F/\lambda_p$ $c^{-2}v\lambda_p^{-1}I$ $c^{-1}v\lambda_p^{-1}P$

now $F \simeq cE$, $E \gtrsim P$, $t_f^{-1} \sim v/\ell$

$c^{-2}(cE)v/\ell$ E/ℓ $c^{-1}(cE)/\lambda_p$ $c^{-2}v\lambda_p^{-1}(cE)$ $c^{-1}v\lambda_p^{-1}E$

(a) $(E/\ell)\{\ v/c \quad 1 \quad \ell/\lambda_p \quad (v/c)(\ell/\lambda_p) \quad (v/c)(\ell/\lambda_p)\}$

(b) Radiation time scale (streaming) $c^{-1}\partial/\partial t \to c^{-1}t_\ell^{-1} = \ell^{-1}$ and time-der term is $O(1)$

$$\{\ 1 \quad 1 \quad \ell/\lambda_p \quad (v/c)(\ell/\lambda_p) \quad (v/c)(\ell/\lambda_p)\}$$

(c) Static diffusion:

$$c^{-2}F/t_f \quad P/\ell \quad c^{-1}F/\lambda_p \quad c^{-2}vI/\lambda_p \quad c^{-1}v\lambda_p^{-1}P$$

$$c^{-2}F(v/\ell) \quad E/\ell \quad c^{-1}F/\lambda_p \quad c^{-2}v\lambda_p^{-1}(cE) \quad c^{-1}v\lambda_p^{-1}E$$

$F \sim cE\lambda_p/\ell$

$$c^{-2}(cE\lambda_p v/\ell^2) \quad E/\ell \quad c^{-1}(cE\lambda_p/\ell\lambda_p) \quad c^{-1}v\lambda_p^{-1}E \quad c^{-1}v\lambda_p^{-1}E$$

$$E(c^{-1}\lambda_p v/\ell^2) \quad E/\ell \quad (E/\ell) \quad (v/c)E/\lambda_p \quad (v/c)(E/\lambda_p)$$

$$(E/\ell)\{(v/c)(\lambda_p/\ell) \quad 1 \quad 1 \quad (v/c)(\ell/\lambda_p) \quad (v/c)(\ell/\lambda_p)\}$$

(d) Dynamic diffusion: $F \sim cE(v/c)$

$$c^{-2}(cEv/c)(v/\ell) \quad (E/\ell) \quad c^{-1}\lambda_p^{-1}(vE) \quad c^{-2}vcE/\lambda_p \quad c^{-1}v\lambda_p^{-1}E$$

$$(E/\ell)(v^2/c^2) \quad (E/\ell) \quad (E/\ell)(v/c)(\ell/\lambda_p) \quad (E/\ell)(v/c)(\ell/\lambda_p) \quad (E/\ell)(v/c)(\ell/\lambda_p)$$

$$v^2/c^2 \quad 1 \quad (v/c)(\ell/\lambda_p) \quad (v/c)(\ell/\lambda_p) \quad (v/c)(\ell/\lambda_p)$$

Radiative Energy Equation (93.8)

$$E_{,t} + F^{i}_{,t} = 4\pi\int\eta_0 d\nu - 4\pi\int\chi_0 J d\nu + (v/c)\int[\chi_0 + \nu\,\partial\chi_0/\partial\nu]F\,d\nu$$

(a) $E/t_f \quad F/\ell \quad J/\lambda_p \quad J/\lambda_p \quad (v/c)F/\lambda_p$

in streaming limit $cE \simeq F$; also $J \sim cE$

Fluid-flow timescale

$$E(v/c) \quad cE/\ell \quad cE/\lambda_p \quad cE/\lambda_p \quad (v/c)(cE/\lambda_p)$$

$$cE/\ell\{\ v/c \quad 1 \quad \ell/\lambda_p \quad \ell/\lambda_p \quad (v/c)(\ell/\lambda_p)$$

on radiation timescale, streaming limit:

$$1 \quad 1 \quad \ell/\lambda_p \quad \ell/\lambda_p \quad (v/c)(\ell/\lambda_p)$$

(b) Static Diffusion

$$E_{,t} + F^{i}_{\ ,i} = 4\pi \int (\eta_0 - \chi_0 J)\, d\nu + (v/c) \int [\chi_0 + \nu \partial\chi_0/\partial\nu] F\, d\nu$$

$$E/t_f \qquad F/\ell \qquad E(\lambda_p c/\ell^2) \qquad (v/c)\lambda_p^{-1} F$$

$$E v/\ell \qquad (cE\lambda_p/\ell^2) \qquad E(\lambda_p c/\ell^2) \qquad (v/c)\lambda_p^{-1}(cE\lambda_p/\ell)$$

$$(cE/\ell)(\lambda_p/\ell)\{(v/c)(\ell/\lambda_p) \qquad 1 \qquad 1 \qquad (v/c)(\ell/\lambda_p)$$

(c) Dynamic Diffusion

$$E v/\ell \qquad \ell^{-1}(Ev) \qquad E(v/\ell) \qquad (v/c)\lambda_p^{-1}(vE)$$

$$(Ev/\ell)\{ \quad 1 \qquad 1 \qquad 1 \qquad (v/c)\ell/\lambda_p \quad \}$$

§94 Inertial-Frame Equations of Radiation Hydrodynamics

(91.8)
rewritten with
r the 1st spatial coord.

$$\mathcal{R} = \textcircled{R} = \begin{pmatrix} E & c^{-1}F & 0 & 0 \\ c^{-1}F & P & 0 & 0 \\ 0 & 0 & \frac{1}{2}(E-P) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(E-P) \end{pmatrix}$$

physical components
(t, r, θ, ϕ)

(91.9)
rewritten
in E, F, P

$$R^{\alpha\beta} = \begin{pmatrix} E & c^{-1}F & 0 & 0 \\ c^{-1}F & P & 0 & 0 \\ 0 & 0 & \frac{1}{2}(E-P)/r^2 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(E-P)/r^2\sin^2\theta \end{pmatrix}$$

contravariant components
in (t, r, θ, ϕ)

Use (A3.91) on (91.8) with $\partial/\partial\theta = \partial/\partial\phi = 0$

$$\mathcal{R}^{1j}{}_{;j} = r^{-2}\partial(r^2\mathcal{R}_{11})/\partial r - r^{-1}(\mathcal{R}_{22} + \mathcal{R}_{33})$$

$$= r^{-2}\partial(r^2P)/\partial r - r^{-1}[\tfrac{1}{2}(E-P) + \tfrac{1}{2}(E-P)]$$

$$= \qquad /\partial r + 2rP) - r^{-1}(E-P)$$

$$\mathcal{R}^{1j}{}_{;j} = \partial P/\partial r + r^{-1}[2P - E + P] = \partial P/\partial r + r^{-1}[3P-E]$$

$$-G^1 = \mathcal{R}^{10}{}_{,0} + \mathcal{R}^{1\beta}{}_{;\beta}$$

$$= c^{-1}\frac{\partial}{\partial t}(c^{-1}F) + \partial P/\partial r + [3P-E)/r$$

(94.5) $$-G^1 = c^{-2}\partial F/\partial t + \partial P/\partial r + (3P-E)/r$$

$$-G^0 = \mathcal{R}^{00}{}_{,t} + \mathcal{R}^{0}{}_{;} = c^{-1}\frac{\partial}{\partial t}(E) + \mathcal{R}^{01}{}_{;1}$$

$$= c^{-1}\partial E/\partial t + r^{-2}\partial(r^2c^{-1}F)/\partial r$$

(94.4) or $$\underline{-cG^0 = \partial E/\partial t + r^{-2}\partial(r^2F)/\partial r}$$

To write (94.6) $M^{\alpha\beta}{}_{;\beta} = F^\alpha + G^\alpha$ and (94.8) $(M^{\alpha\beta} + R^{\alpha\beta})_{;\beta} = F^\alpha$ in cartesian coordinates, we need $M^{\alpha\beta}{}_{;\beta}$ from §42

$$(\rho_1 - p/c^2)_{,t} + (\rho_1 v^j)_{,j} = v_j f^j/c^2 \qquad (42.3)$$

$$(\rho_1 v^i)_{,t} + (\rho_1 v^i v^j + p\delta^{ij})_{,j} = f^i \qquad (42.4)$$

Then (94.6) becomes:

$$c^{-1}M^{0\beta}{}_{;\beta} = (\rho_1 - p/c^2)_{,t} + (\rho_1 v^j)_{,j} = v_j f^j/c^2 + c^{-1}G^0$$

(94.9a) $$\Rightarrow \quad (\rho_1 c^2 - p)_{,t} + (\rho_1 c^2 v^j)_{,j} = v_j f^j + cG^0$$

and

$$M^{i\beta}{}_{;\beta} = (M^{i0})_{,0} + (M^{ij})_{,j}$$
$$= c^{-1}(c\rho v^i)_{,t} + (\rho v^i v^j + p\delta^{ij})_{,j} = f^i + G^i$$
$$M^{i\beta}{}_{;\beta} = (\rho v^i)_{,t} + (\rho v^i v^j + p\delta^{ij})_{,} = f^i + G^i$$

(94.10a) $$\text{or} \quad (\rho v^i)_{,t} + (\rho v^i v^j)_{,j} = f^i - p_{,i} + G^i$$

And (94.8) becomes:

$$c\,M^{0\beta}{}_{;\beta} + c\,R^{0\beta}{}_{;\beta} = (\rho_1 c^2 - p)_{,t} + (c^2\rho_1 v^j)_{,j} + E_{,t} + F^j{}_{,j}$$

$$\Rightarrow$$

(94.9a) $$c(M^{0\beta} + R^{0\beta})_{;\beta} = (\rho_1 c^2 - p + E)_{,t} + (c^2\rho_1 v^j + F^j)_{,j} = v_j f^j$$

and

$$(M^{i\beta} + R^{i\beta})_{;\beta} = c^{-1}(\rho_1 c v^i)_{,t} + (\rho_1 v^i v^j + p\delta^{ij})_{,j} + c^{-1}(c^{-1}F^i)_{,t} + P^{ij}{}_{,j}$$
$$= (\rho_1 v^i)_{,t} + (\rho_1 v^i v^j + p\delta^{ij})_{,j} + c^{-2}F^i{}_{,t} + P^{ij}{}_{,j} = f^i$$

or

(94.10b) $$(M^{i\beta} + R^{i\beta})_{;\beta} = (\rho_1 v^i + c^{-2}F^i)_{,t} + (\rho_1 v^i v^j + P^{ij})_{,j} = f^i - p_{,i}$$

Subtract $c^2 \times$ continuity equation ($(\gamma\rho_0)_{,t} + (\gamma\rho_0 v^j)_{,j} = 0$) from 94.9a and 94.9b:

First write both in terms of ρ and e: $\rho_1 = \gamma^2\rho_0(1 + h/c^2)$.

$$= \gamma\rho(1 + e/c^2 + \gamma p/\rho c^2)$$

$$\boxed{\rho_1 \equiv \gamma^2\rho_{000} \equiv \gamma^2\rho_0(1 + h/c^2)}$$

$\gamma\rho_0 \equiv \rho$

$h = e + \gamma p/\rho$

(94.9a) becomes:

$$(\gamma\rho c^2 + \gamma\rho h - p]_{,t} + [\gamma\rho c^2 v^j + \gamma\rho h v^j]_{,j} = v_j f^j + G^0_c$$
$$\Rightarrow [\gamma\rho c^2 + \gamma\rho(e + \gamma p/\rho) - p]_{,t} + [\gamma\rho c^2 v^j + \gamma\rho(e + \gamma p/\rho)v^j]_{,j} = v_j f^j + cG^0$$
$$\Rightarrow [\gamma\rho c^2 + \gamma\rho e + (\gamma^2 - 1)p]_{,t} + [\gamma\rho c^2 v^j + \gamma\rho e v^j + \gamma^2 p v^j]_{,j} = v_j f^j + cG^0$$
$$-(c^2\rho)_{,t} \qquad - \qquad (c^2\rho v^j)_{,j} \qquad = 0$$

(94.11a) $$[(\gamma - 1)\rho c^2 + \gamma\rho e + (\gamma^2 - 1)p]_{,t} + \{[(\gamma - 1)\rho c^2 + \gamma\rho e + \gamma^2 p]v^j\}_{,j} = v_j f^j + cG^0$$

(94.9b) becomes:

$$\begin{array}{l} [\gamma\rho c^2(1+e/c^2+\gamma p/\rho c^2)-p+E]_{,t}+[\gamma\rho c^2v^j(1+e/c^2+\gamma p/\rho c^2)+F^i]_{,j}=v_jf^j \\ \qquad -(c^2\rho)_{,t} \qquad\qquad\qquad\qquad\qquad\qquad\qquad -(c^2\rho v^j)_{,j} \qquad\qquad = 0 \\ \hline \end{array}$$

(94.11b) $$[(\gamma-1)\rho c^2+\gamma\rho e+(\gamma^2-1)p]_{,t}+\{(\gamma-1)\rho c^2+\gamma\rho e+\gamma^2 D]v^j+F^j]_{,j}=v_jf^j$$

Momentum Eqns (94.10)

$$\begin{array}{l} (\gamma^2\rho_{000}v_i)_{,t}+(\gamma^2\rho_{000}v_iv^j)_{,j} = f_i - p_{,i} + G \\ v^i/c^2(\gamma^2\rho_{000}c^2-p)_{,t}+v^i/c^2(\gamma^2\rho_{000}c^2v^j)_{,j} = (v^i/c^2)v_jf^j+(v^i/c^2)cG^0 \\ \hline \end{array}$$

$$[\gamma^2\rho_{000}v^i]_{,t}-v^i(\gamma^2\rho_{000}-p/c^2)_{,t}+(\gamma^2\rho_{000}v^iv^j)_{,j}-v^i(\gamma^2\rho_{000}v^j)_{,j}$$
$$= f_i-(v^iv_j/c^2)f^j-p_{,i}+G^i-(v^i/c)G^0$$

$$= v^i(\cancel{\gamma^2\rho_{000}})_{,t}+\gamma^2\rho_{000}v^i_{,t}+v^i(p/c^2)_{,t}+v^i(\gamma^2\cancel{\rho_{000}}v^j)_{,j}+\gamma^2\rho_{000}v^jv^i_{,j}-v^i(\gamma^2\cancel{\rho_{000}}v^j)_{,j}$$
$$-v^i(\gamma^2\cancel{\rho_{000}})_{,t} \qquad = f_i-f_jv^iv^j/c^2-p_{,i}+G^i-(v^i/c)G^0$$

$\Rightarrow$

$$\gamma^2\rho_{000}v^i_{,t}+(v^i/c^2)p_{,t}+\gamma^2\rho_{000}v^jv^i_{,j}=f_i-f_jv^iv^j/c^2)-p_{,i}+G^i-(v^i/c)G^0$$
$$\gamma^2\rho_{000}[v^i_{,t}+v^jv^i_{,j}]=f_i-f_jv^iv^j/c^2-p_{,i}-(v^i/c^2)p_{,t}+G^i-(v^i/c)G^0$$

$(94.12a)^?$ $$\Rightarrow \rho_1[D\underset{\sim}{v}/Dt]=\underset{\sim}{f}-\underset{\sim}{\nabla}p-(\underset{\sim}{v}/c^2)(\underset{\sim}{v}\cdot\underset{\sim}{f}+\partial p/\partial t)+\underset{\sim}{G}-\underset{\sim}{v}c^{-1}G^0$$

$$\begin{array}{l} (\rho_1v^i+c^{-2}F^i)_{,t}+(\rho_1v^iv^j+P^{ij})_{,j}=f^i-p_{,i} \\ (v^i/c^2)(\rho_1c^2-p+E)_{,t}+(v^i/c^2)(\rho_1c^2v^j+F^j)_{,j}=v_jf^j(v^i/c^2) \\ \hline \end{array}$$

$$\rho_1v^i_{,t}+\cancel{v^i_{,t}\rho_1}+c^{-2}F^i_{,t}-\cancel{v_i\rho_{1,t}}+(pv^i/c^2-Ev_i/c^2)_{,t}$$
$$+\rho_1v^jv^i_{,j}+v^i\cancel{(\rho_1v^j)}_{,j}-v^i\cancel{(\rho_1v^j)}_{,j}+P^{ij}_{,j}-c^{-2}v^iF^j{}_{,j}=f^i-p_{,i}-v^iv_jf^j/c^2$$
$$\rho_1v^i_{,t}+\rho_1v^jv^i_{,j}+F^i_{,t}+p_{,t}v^ic^{-2}-E_{,t}v^ic^{-2}+(P^{ij}-c^{-2}v^iF^j)_{,j}=f^i-p_{,i}-v^iv_jf^j/c^2$$

(94.12b) $$\rho_1(D\underset{\sim}{v}/Dt)+c^{-2}[\underset{\sim}{E}_{,t}+(p_{,t}-E_{,t})\underset{\sim}{v}]+[\underset{\sim}{\nabla}\cdot\underset{\sim}{P}-\underset{\sim}{v}(\underset{\sim}{\nabla}\cdot\underset{\sim}{F})c^{-2}]=\underset{\sim}{f}-\nabla\underset{\sim}{p}-\underset{\sim}{v}(\underset{\sim}{v}\cdot\underset{\sim}{f})$$

$$\text{or } \rho_1 (D\underset{\sim}{v}/Dt) = \underset{\sim}{f} - \nabla p - \underset{\sim}{v} c^{-2}[p_{,t} + \underset{\sim}{v}\cdot\underset{\sim}{f}] - (\nabla\cdot \mathcal{P} + c^{-2}\underset{\sim}{E}_{,t}) + \underset{\sim}{v} c^{-2}(E_{,t} + \nabla\cdot\underset{\sim}{E})$$

Writing $dt = \gamma d\tau$ and $\rho_* = \gamma^{-1}\rho_1$ or $\rho_1 = \gamma \rho_*$, we have

$$(94.12a) \quad \rho_* (D\underset{\sim}{v}/D\tau) = \underset{\sim}{f} - \nabla p - c^{-2}\underset{\sim}{v}(p_{,t} + \underset{\sim}{f}\cdot\underset{\sim}{v}) + \underset{\sim}{G} - c^{-1}\underset{\sim}{v}G^0$$

and

$$(94.12b) \quad \rho_* (D\underset{\sim}{v}/D\tau) = \underset{\sim}{f} - \nabla p - \underset{\sim}{v}c^{-2}[\partial p/\partial t + \underset{\sim}{v}\cdot\underset{\sim}{f}] - (\nabla\cdot\underset{\sim}{\mathcal{P}} + c^{-2}\partial\underset{\sim}{E}/\partial t) + \underset{\sim}{v}c^{-2}(\partial E/\partial t + \underset{\sim}{\nabla}\cdot\underset{\sim}{E})$$

Dropping terms of higher order than (v/c):

$$(94.13a) \quad \rho\ (D\underset{\sim}{v}/Dt) = \underset{\sim}{f} - \underset{\sim}{\nabla} p + \underset{\sim}{G} - vc^{-1}G^0$$

$$(94.13b) \quad \rho\ (D\underset{\sim}{v}/Dt) = \underset{\sim}{f} - \nabla p - (\nabla\cdot\underset{\sim}{\mathcal{P}} + c^{-2}\partial\underset{\sim}{E}/\partial t) + \underset{\sim}{v}c^{-2}(\partial E/\partial t + \underset{\sim}{\nabla}\cdot$$

The Total Energy Equation

Take (94.11) and let $\gamma \to 1$, $(\gamma - 1) \to \frac{1}{2}v^2/c^2$, then

$$[\rho v^2/2 + \rho e + \cancel{pv^2/c^2}]_{,t} + \{[\tfrac{1}{2}\rho v^2 + \rho e + p]v^i\}_{,i} = v_i f^i + cG^0$$

$$(94.15a) \quad (\rho v^2/2 + \rho e)_{,t} + \{[\rho v^2/2 + \rho e + p]v^i\}_{,i}$$

$$(94.15b) \quad [\rho v^2/2 + \rho e + E]_{,t} + \{[\rho v^2/2 + \rho e + p]v^i + F^i\}_{,i} = v_i f^i$$

$$[\rho(e + v^2/2)]_{,t} + \{[\rho(e+v^2/2) + p]v^i\}_{,t} = (e+v^2/2)\partial\rho/\partial t + \rho\,(e+v^2/2)_{,t}$$
$$+ (pv^i)_{,i} + (e+v^2/2)_{,i}\,\rho v^i + (e+v^2/2)(\rho v_i)_{,i}$$
$$= (e+v^2/2)\underbrace{[\partial\rho/\partial t + \nabla\cdot(\underset{\sim}{v}\rho)]}_{\equiv 0 \text{ by continuity eqn.}} + \rho\,\partial(e+v^2/2)/\partial t + \rho\underset{\sim}{v}\cdot\nabla(e+v^2/2) + (pv^i)_{,i}$$

(94.16a) hence $\rho D(e+v^2/2)/Dt + \underset{\sim}{\nabla}\cdot(p\underset{\sim}{v}) = \underset{\sim}{v}\cdot\underset{\sim}{f} + cG^0$

Similarly,

$$[\rho(e+v^2/2)+E]_{,t} + \{[\rho(e+v^2/2)+p]v^i + F^i\}_{,i} = v_i f^i$$

becomes

$$(94.16b) \quad \rho\, D(e+v^2/2)/Dt + \underset{\sim}{\nabla}\cdot(p\underset{\sim}{v}) + \partial E/\partial t + \underset{\sim}{\nabla}\cdot\underset{\sim}{F} = \underset{\sim}{v}\cdot\underset{\sim}{f}$$

Mechanical Energy Equation

$$\underset{\sim}{v}\cdot\{\rho(D\underline{v}/Dt) = \underset{\sim}{f} - \underset{\sim}{\nabla}p + \underset{\sim}{G} - \underset{\sim}{v}c^{-1}G^0\}$$

$$\rho\underset{\sim}{v}\cdot\partial\underline{v}/\partial t + \rho\underset{\sim}{v}(\underset{\sim}{v}\cdot\nabla)\underset{\sim}{v} = \underset{\sim}{v}\cdot\underset{\sim}{f} - \underset{\sim}{v}\cdot\nabla p + \underset{\sim}{v}\cdot G - v^2c^{-1}G^0$$

$$\rho\, D(v^2/2)/Dt = -\underset{\sim}{v}\cdot\underline{\nabla}p + \underset{\sim}{v}\cdot(\underset{\sim}{f}+\underset{\sim}{G})$$

$$\underset{\sim}{v}\cdot\rho\, D\underline{v}/Dt = \underset{\sim}{v}\cdot\underset{\sim}{f} - \underset{\sim}{v}\cdot\underline{\nabla p} - c^{-2}\underset{\sim}{v}\cdot\partial\underset{\sim}{E}/\partial t - \underset{\sim}{v}\cdot\underline{\nabla}\cdot\underline{\underline{P}}$$
$$+ c^{-2}v^2(\cancel{\partial\underline{E}/\partial t + \underset{\sim}{\nabla}}\cdot\underset{\sim}{E})$$

$$\Rightarrow \rho\, D(v^2/2)/Dt = -\underset{\sim}{v}\cdot\nabla p - \underline{v}\cdot\underline{\nabla}\cdot\underline{\underline{P}} + \underset{\sim}{v}\cdot\underset{\sim}{f} - (\underline{v}/c^2)\cdot\partial\underline{E}/\partial t$$
$$= -\underset{\sim}{v}\cdot\underline{\nabla}p + \underset{\sim}{v}\cdot\underline{f} - \underset{\sim}{v}\cdot(c^{-2}\partial\underline{E}/\partial t + \underline{\nabla}\cdot\underline{\underline{P}})$$

Gas Energy Equation

Start with: $(94.6) \quad m^{\alpha\beta}{}_{;\beta} = F^\alpha + G^\alpha$

$(94.9a) \quad (\rho_1 c^2 - p)_{,t} + (\rho_1 c^2 v^j)_{;j} = v_j f^j + cG^0 \quad = c\, m^{0\beta}{}_{;\beta}$

$\therefore (94.10a) \quad (\rho_1 v_i)_{,t} + (\rho_1 v_i v^j)_{;j} = f_i - P_{;i} + G_i = m^{i\beta}{}_{;\beta}$

$\rho_1 = \gamma\rho + \cancel{\gamma}\rho e c^{-2} + \gamma^2 p c^{-2}$

Thus $m^{0\beta}{}_{;\beta} = c^{-1}[\gamma\rho c^2 + \gamma\rho e + (\gamma^2-1)p]_{,t} + [(\gamma\rho c^2 v^j + \gamma\rho e v^j + \gamma^2 p v^j)]_{;j}$

and $\gamma\, m^{0\beta}{}_{;\beta} = -\gamma[\gamma\rho c^2 + \gamma\rho e + (\gamma^2-1)p]_{,t} \mp \cancel{\gamma}[\gamma\rho c^2 v^j + \gamma\rho e v^j + \gamma^2 p v^j]_{;j}$

add $\gamma c^2 \times$[cont eqn]: $\quad + \gamma(\rho c^2)_{,t} \quad + \gamma(\rho c^2 v^j)_{;j}$

$$\Rightarrow -\gamma\{(\gamma-1)\rho c^2 + \gamma\rho e + (\gamma^2-1)p\}_{,t} - \gamma\{(\gamma-1)\rho c^2 v^j + \gamma\rho e v^j + \gamma^2 p v^j\}_{;j}$$

Take limit $\gamma\to 1$, $\gamma - 1 \to \frac{1}{2}\frac{v^2}{c^2}$ $\gamma^2 - 1 \to \frac{v^2}{c^2}$, then

$$V_0\, m^{0\beta}{}_{;\beta} \to -\{\tfrac{1}{2}\rho v^2 + \rho e + \tfrac{v^2}{c^2}p\}_{,t} - \{\tfrac{1}{2}\rho v^2 v^j + \cancel{\gamma}\rho e v^j + p v^j\}_{;j}$$

and dropping $O(v^2/c^2)$ terms

$$V_0 M^{0\beta}{}_{;\beta} = -\left(\tfrac{1}{2}\rho v^2\right)_{,t} - (\rho e)_{,t} - \left(\tfrac{1}{2}\rho v^2 v^j\right)_{,j} - (\rho e v^j)_{,j} - (p v^j)_{,j}$$

$$= -\left(\tfrac{1}{2}\rho v^2\right)_{,t} - \rho(De/Dt) - \left(\tfrac{1}{2}\rho v^2 v^j\right)_{,j} - (p v^j)_{,j}$$

we also have

$\rho_1 = \gamma\rho + \gamma\rho e c^{-2} + \gamma p c^{-2}$
$$V_i M^{i\beta}{}_{;\beta} = v_i(\rho_1 v_i)_{,t} + v_i(\rho_1 v_i v^j)_{,j} + v_i p_{,i}$$

$$= v_i\left[(\gamma\rho + \gamma\rho e c^{-2} + \gamma p c^{-2})v_i\right]_{,t} + v_i\left[(\gamma\rho + \gamma\rho e c^{-2} + \gamma p c^{-2})v_i v^j\right]_{,j} + v_i p_{,i}$$

letting $\gamma \to 1$ and dropping $O(v^2/c^2)$ terms, we have:

$$V_i M^{i\beta}{}_{;\beta} \cong v_i(\rho v_i)_{,t} + v_i(\rho v_i v^j)_{,j} + v_i p_{,i}$$

$$= \left(\tfrac{1}{2}\rho v^2\right)_{,t} + \left(\tfrac{1}{2}\rho v^2 v^j\right)_{,j} + v_i p_{,i}$$

Adding $V_0 M^{0\beta}{}_{;\beta} + V_i M^{i\beta}{}_{;\beta}$

$$= -\cancel{\left(\tfrac{1}{2}\rho v^2\right)_{,t}} - \rho(De/Dt) - \cancel{\left(\tfrac{1}{2}\rho v^2 v^j\right)_{,j}} - (p v^j)_{,j}$$
$$+ \cancel{\left(\tfrac{1}{2}\rho v^2\right)_{,t}} + \qquad + \cancel{\left(\tfrac{1}{2}\rho v^2 v^j\right)_{,j}} + v_i p_{,i}$$

$$= -\rho(De/Dt) - (p v^j)_{,j} + v_i p_{,i}$$

$$= -\rho(De/Dt) - \cancel{v^i p_{,i}} - p v^i{}_{,i} + \cancel{v_i p_{,i}}$$

$$= -\rho(De/Dt) - p v^i{}_{,i}$$

But $D\rho/Dt + \rho v^i{}_{,i} = 0 \Rightarrow v^i{}_{,i} = -\rho^{-1}(D\rho/Dt) = \rho(D\rho^{-1}/Dt)$

hence

$$V_0 M^{0\beta}{}_{;\beta} + V_i M^{i\beta}{}_{;\beta} = -\rho(De/Dt) - p\rho(D\rho^{-1}/Dt) = V_\alpha F^\alpha + V_\alpha G^\alpha$$

or, finally

$$\rho(De/Dt) + p\rho(D\rho^{-1}/Dt) = -V_\alpha F^\alpha - V_\alpha G^\alpha$$

or $\rho\left[De/Dt + p\,D(1/\rho)/Dt\right] = -V_\alpha F^\alpha - V_\alpha G^\alpha$

$$\left(\rho_{000} V^\alpha V^\beta + p g^{\alpha\beta}\right)_{;\beta} = F^\alpha + G^\alpha$$

$$V_\alpha\left(\rho_{000} V^\alpha V^\beta + p g^{\alpha\beta}\right)_{;\beta} = N_\alpha F^\alpha + V_\alpha G^\alpha$$

$$= V_\alpha V^\alpha (\rho_{000} V^\beta)_{;\beta} + V_\alpha \rho_{000} V^\beta (V^\alpha)_{;\beta} + V_\alpha (p g^{\alpha\beta})_{;\beta} = V_\alpha F^\alpha + V_\alpha G^\alpha$$

$$= -c^2 (\rho_{000} V^\beta)_{;\beta} + \rho_{000} V^\beta \; V_\alpha V^\alpha{}_{;\beta} + V^\alpha p_{,\alpha}$$

For the heck of it, show by calculation that $V_\alpha V^\alpha{}_{;\beta} \equiv 0$:

$$V_\alpha V^\alpha{}_{;\beta} = [\gamma(-c, \underline{v})]\,[\gamma(c, \underline{v})]_{;\beta}$$

$$= V_0 V^0{}_{;\beta} + V_i V^i{}_{;\beta}$$

$\beta = 0$
$$= -\gamma c (\gamma c)_{,t} + \gamma v_i (\gamma v_i)_{,t}$$

$$= -\gamma c^2 (\gamma)_{,t} + \gamma^2 (\tfrac{1}{2} v^2)_{,t} + \gamma v^2 \gamma_{,t} = \gamma (v^2 - c^2)\gamma_{,t} + \gamma^2 (\tfrac{1}{2} v^2)_{,t}$$

now $\gamma_{,t} = [(1-\beta^2)^{-1/2}]_{,t} = -\tfrac{1}{2}(1-\beta^2)^{-3/2}(-2\beta) c^{-1} v_{,t} = \gamma^3 c^{-2} v v_{,t} = \gamma^3 c^{-2} (\tfrac{1}{2} v^2)_{,t}$

$$= \gamma^2 . (\gamma/c^2)(\tfrac{1}{2} v^2)_{,t}$$

and $V_\alpha V^\alpha{}_{;0} = \gamma (v^2 - c^2) \cdot \gamma^2 (\gamma/c^2)(\tfrac{1}{2} v^2)_{,t} + \gamma^2 (\tfrac{1}{2} v^2)_{,t}$

$$= \gamma^2 (\tfrac{1}{2} v^2)_{,t} \left[(\gamma^2/c^2)(v^2 - c^2) + 1\right] = \gamma^2 (\tfrac{1}{2} v^2)_{,t} \left[\gamma^2 (\beta^2 - 1) + 1\right]$$

$$= \gamma^2 (\tfrac{1}{2} v^2)_{,t} \left[\gamma^2 (-1/\gamma^2) + 1\right] \equiv 0$$

$\beta = j$ $\qquad V_\alpha V^\alpha{}_{;j} = V_0 V^0{}_{;j} + V_i V^i{}_{;j}$

$$= -\gamma c (\gamma c)_{,j} + \gamma v_i (\gamma v_i)_{,j} = -\gamma c^2 \gamma_{,j} + \gamma^2 (\tfrac{1}{2} v^2)_{,j} + \gamma v^2 \gamma_{,j}$$

$$= \gamma (v^2 - c^2) \gamma_{,j} + \gamma^2 (\tfrac{1}{2} v^2)_{,j}$$

again $\gamma_{,j} = [(1-\beta^2)^{-1/2}]_{,j} = -\tfrac{1}{2}\gamma^3 (-2\beta_i) c^{-1} v_{i,j} = \gamma^3 c^{-2} v_i v_{i,j} = \gamma^3 c^{-2} (\tfrac{1}{2} v^2)_{,j}$

$$= \gamma^2 (\gamma/c^2)(\tfrac{1}{2} v^2)_{,j}$$

$$V_\alpha V^\alpha{}_{;j} = \gamma (v^2 - c^2) \cdot \gamma^2 (\gamma/c^2)(\tfrac{1}{2} v^2)_{,j} + \gamma^2 (\tfrac{1}{2} v^2)_{,j}$$

$$= \gamma^2 (\tfrac{1}{2} v^2)_{,j} \left[(\gamma^2/c^2)(v^2 - c^2) + 1\right] \equiv 0$$

Hence we have

$$V_\alpha M^{\alpha\beta}{}_{;\beta} = -c^2 (\rho_{000} V^\beta)_{;\beta} + V^\alpha p_{,\alpha}$$

$$= -\left[\rho_0 (c^2 + e + p/\rho_0) V^\beta\right]_{;\beta} + V^\alpha p_{,\alpha}$$

$$\rho_{000} = \rho_0 + \rho_0 e c^{-2} + p c^{-2}$$
$$= (\rho_0/c^2)(c^2 + e + p/\rho_0)$$
$$c^2 \rho_{000} = \rho_0 (c^2 + e + p/\rho_0)$$

Now take $V_\alpha M^{\alpha\beta}{}_{;\beta} + (c^2+c+p/\rho_0)[(\rho_0 V^\alpha)_{;\alpha}$

$$= -[\rho_0(c^2+e+p/\rho_0)V^\beta]_{;\beta} + (c^2+e+p/\rho_0)(\rho_0 V^\beta)_{;\beta} + V^\alpha p_{,\alpha}$$

$$= -(c^2+e+p/\rho_0)(\rho_0 V^\beta)_{;\beta} - \rho_0 V^\beta(c^2+e+p/\rho_0)_{;\beta}$$

$$+ (c^2+e+p/\rho_0)(\rho_0 V^\beta)_{;\beta} + V^\beta p_{,\beta}$$

$$= -\rho_0 V^\beta (e+p/\rho_0)_{;\beta} + V^\beta p_{,\beta}$$

$$= -\rho_0 V^\beta e_{;\beta} - \rho_0 V^\beta \rho_0^{-1} p_{,\beta} - \rho_0 V^\beta p\,(1/\rho_0)_{;\beta} + V^\beta p_{,\beta}$$

$$= -\rho_0 V^\beta e_{;\beta} - V^\beta p_{,\beta} - \rho_0 p\, V^\beta (\rho_0^{-1})_{;\beta} + V^\beta p_{;\beta}$$

$$= -\rho_0 V^\beta e_{;\beta} - \rho_0 p\, V^\beta (\rho_0^{-1})_{;\beta}$$

But $V^\beta \frac{\partial}{\partial x^\beta} = \frac{D}{D\tau}$, hence

(94.18) $$V_\alpha M^{\alpha\beta}{}_{;\beta} = -\rho_0 [De/D\tau + p\,D(1/\rho_0)/D\tau] = V_\alpha F^\alpha + V_\alpha G^\alpha$$

Now $V_\alpha G^\alpha = \gamma(-c, \underset{\sim}{v})\cdot(G^0, \underline{G}) = -\gamma c G^0 + \gamma \underset{\sim}{v}\cdot\underline{G} = \gamma(-cG^0 + \underset{\sim}{v}\cdot\underline{G})$

hence

$$\rho_0[De/D\tau + p\,D(1/\rho_0)/D\tau] = \gamma(cG^0 + \underset{\sim}{v}\cdot\underset{\sim}{G})$$

and $Dt = \gamma D\tau \Rightarrow$

(94.19a) $$\rho_0[De/Dt + p\,D(1/\rho_0)/Dt] = cG^0 - \underset{\sim}{v}\cdot\underline{G}$$

Finally $G^0 = -c^{-1}(E_{,t} + F^i{}_{,i})$ (94.2)

$$\underline{G} = R^{j\beta}{}_{;\beta} = c^{-1}(c^{-1}F^j)_{,t} + P^{ij}{}_{,j} = c^{-2}F^j{}_{,t} + P^{ij}{}_{,j}$$

hence

(94.19b) $$\rho_0[De/Dt + p\,D(1/\rho_0)/Dt] = -[E_{,t} + F^i{}_{,i}] - \underset{\sim}{v}\cdot[c^{-2}\underset{\sim}{F}_{,t} + \underset{\sim}{\nabla}\cdot\underline{\underline{P}}]$$

(95.1) $\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I}{\partial \mu} = \eta - \chi I$

now $I/\nu^3 = I_0/\nu_0^3$, $\eta/\nu^2 = \eta_0/\nu_0^2$ and $\chi\nu = \chi_0\nu_0$, hence

$$\frac{1}{c}\frac{\partial}{\partial t}\left(I_0\frac{\nu^3}{\nu_0^3}\right) + \mu\frac{\partial}{\partial r}\left(I_0\frac{\nu^3}{\nu_0^3}\right) + \frac{(1-\mu^2)}{r}\frac{\partial}{\partial\mu}\left(I_0\frac{\nu^3}{\nu_0^3}\right) = \frac{\nu^2}{\nu_0^2}(\eta_0 - \chi_0 I_0)$$

and

$$\frac{\partial}{\partial t}\left(I_0\frac{\nu^3}{\nu_0^3}\right) = \frac{\nu^3}{\nu_0^3}\frac{\partial I_0}{\partial t} - \frac{3\nu^3}{\nu_0^4}I_0\frac{\partial\nu_0}{\partial t} = \frac{\nu^3}{\nu_0^3}\left[\frac{\partial I}{\partial t} - \frac{3}{\nu_0}\frac{\partial \nu_0}{\partial t}\right] \quad \text{etc.}$$

$$\Rightarrow \frac{\nu^3}{\nu_0^3}\left[\frac{1}{c}\frac{\partial I_0}{\partial t} + \mu\frac{\partial I_0}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I_0}{\partial\mu}\right] - \frac{3\nu^3}{\nu_0^4}I_0\left[\frac{1}{c}\frac{\partial\nu_0}{\partial t} + \mu\frac{\partial\nu_0}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\nu_0}{\partial\mu}\right] = \frac{\nu^2}{\nu_0^2}(\eta_0 - \chi_0 I_0$$

$\Rightarrow$

(95.2) $$\frac{\nu}{\nu_0}\left[\frac{1}{c}\frac{\partial I_0}{\partial t} + \mu\frac{\partial I_0}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial I_0}{\partial\mu}\right] - \frac{3\nu I_0}{\nu_0^2}\left[\frac{1}{c}\frac{\partial\nu_0}{\partial t} + \mu\frac{\partial\nu_0}{\partial r} + \frac{(1-\mu^2)}{r}\frac{\partial\nu_0}{\partial\mu}\right] = \eta_0 - \chi_0 I_0$$

Use (89.10) and (89.11) to evaluate derivatives $\frac{\partial\mu_0}{\partial t}, \frac{\partial\mu_0}{\partial r}, \frac{\partial\nu_0}{\partial t}, \frac{\partial\nu_0}{\partial r}, \frac{\partial\mu_0}{\partial\mu}, \frac{\partial\nu_0}{\partial\mu}$

(89.10): $\mu_0 = (\mu-\beta)/(1-\beta\mu)$ $\quad \nu_0 = \gamma\nu(1-\beta\mu)$ $\quad (1-\mu_0^2)^{1/2} = (1-\mu^2)^{1/2}/\gamma(1-\beta\mu)$

(89.11): $\mu = (\mu_0+\beta)/(1+\beta\mu_0)$ $\quad \nu = \gamma\nu_0(1+\beta\mu_0)$ $\quad (1-\mu^2)^{1/2} = (1-\mu_0^2)^{1/2}/\gamma(1+\beta\mu_0)$

$$\frac{\partial\mu_0}{\partial t} = \frac{1}{(1-\beta\mu)}\left(-\frac{\partial\beta}{\partial t}\right) + \frac{(\mu-\beta)}{(1-\beta\mu)^2}\left(+\mu\frac{\partial\beta}{\partial t}\right) = \frac{\partial\beta}{\partial t}\left[\frac{-(1-\beta\mu)+\mu(\mu-\beta)}{(1-\beta\mu)^2}\right] = \frac{\partial\beta/\partial t}{(1+\beta\mu)^2}\left[\mu^2-1\right]$$

now

$$(1-\beta\mu) = \frac{\mu-\beta}{\mu_0} = \frac{1}{\mu_0}\left[\frac{\mu_0+\beta}{1+\beta\mu_0} - \beta\right] = \frac{1}{\mu_0}\left[\frac{\mu_0+\beta-\beta-\beta^2\mu_0}{1+\beta\mu_0}\right] = \frac{\mu_0(1-\beta^2)}{\mu_0(1+\beta\mu_0)} = \frac{\gamma^{-2}}{(1+\beta}$$

hence $(\mu^2-1) = (\mu_0^2-1)/\gamma^2(1+\beta\mu_0)^2$

$$(\mu^2-1)/(1-\beta\mu)^2 = (\mu_0^2-1)\gamma^{-2}\cancel{(1+\beta\mu_0)^{-2}}\,\cancel{(1+\beta\mu_0)^2}\gamma^4 = \gamma^2(\mu_0^2-1)$$

(95.6a) $$\frac{\partial\mu_0}{\partial t} = \gamma^2(\mu_0^2-1)\frac{\partial\beta}{\partial t} = -\gamma^2(1-\mu_0^2)\frac{\partial\beta}{\partial t}$$

an identical analysis (almost) yields

(95.7a) $$\frac{\partial\mu_0}{\partial r} = -\gamma^2(1-\mu_0)^2\frac{\partial\beta}{\partial r}$$

$$\frac{\partial\nu_0}{\partial t} = \frac{\partial}{\partial t}\left[\gamma\nu(1-\beta\mu)\right] = \frac{\partial\gamma}{\partial t}\nu(1-\beta\mu) + \gamma\nu\left(-\mu\frac{\partial\beta}{\partial t}\right) = \frac{\nu_0}{\gamma}\frac{\partial\gamma}{\partial t} - \gamma\mu\nu\frac{\partial\beta}{\partial t}$$

now $\frac{\partial\gamma}{\partial t} = \frac{\partial}{\partial t}\left[(1-\beta^2)^{-1/2}\right] = -\frac{1}{2}(1-\beta^2)^{3/2}\left(-2\beta\frac{\partial\beta}{\partial t}\right) = \gamma^3\beta\frac{\partial\beta}{\partial t}$

and $\mu\nu = \gamma\nu_0(1+\beta\mu_0)\frac{(\mu_0+\beta)}{(1+\beta\mu_0)} = \gamma\nu_0(\mu_0+\beta)$

so

$$\frac{\partial\nu_0}{\partial t} = \frac{\nu_0}{\gamma}\left(\gamma^3\beta\frac{\partial\beta}{\partial t}\right) - \gamma^2\nu_0(\mu_0+\beta)\frac{\partial\beta}{\partial t} = \gamma^2\nu_0\frac{\partial\beta}{\partial t}\left[\beta-\mu_0-\beta\right] = -\gamma^2\mu_0\nu_0\frac{\partial}{\partial}$$

(95.6b) $$\frac{\partial\nu_0}{\partial t} = -\gamma^2\mu_0\nu_0\frac{\partial\beta}{\partial t}$$

The same analysis, replacing $\frac{\partial}{\partial t}$ with $\frac{\partial}{\partial r}$ yields

(95.7b) $$\frac{\partial \nu_0}{\partial r} = -\gamma^2 \mu_0 \nu_0 \frac{\partial \beta}{\partial r}$$

$$\frac{\partial \mu}{\partial \mu_0} = \frac{\partial}{\partial \mu_0}\left(\frac{\mu_0+\beta}{1+\beta\mu_0}\right) = \frac{1}{1+\beta\mu_0} - \frac{\mu_0+\beta}{(1+\beta\mu_0)^2}\,\beta = \frac{1}{(1+\beta\mu_0)^2}\left[1+\beta\mu_0-\beta\mu_0-\beta^2\right]$$

and thus $= (1+\beta^2)/(1+\beta\mu_0)^2 = \gamma^{-2}(1+\beta\mu_0)^{-2}$

(95.8) $$\frac{\partial \mu_0}{\partial \mu} = \left(\frac{\partial \mu}{\partial \mu_0}\right)^{-1} = \gamma^2(1+\beta\mu_0)^2$$

$$\left.\frac{\partial \nu_0}{\partial \mu}\right|_{\nu t} = \frac{\partial}{\partial \mu}\left[\gamma\nu(1-\beta\mu)\right] = -\gamma\nu\beta = -\gamma\beta\left[\gamma\nu_0(1+\beta\mu_0)\right]$$

(95.8) $$\frac{\partial \nu_0}{\partial \mu} = -\gamma^2\beta\nu_0(1+\beta\mu_0)$$ Disagrees with D's text.

We use these results in the transfer eqn (95.2) with the derivative expressions:

$$\left.\frac{\partial}{\partial t}\right|_{r\mu\nu} = \left.\frac{\partial}{\partial t}\right|_{r\mu_0\nu_0} + \left.\frac{\partial \mu_0}{\partial t}\right|_{r\mu\nu}\frac{\partial}{\partial \mu_0} + \left.\frac{\partial \nu_0}{\partial t}\right|_{r\mu\nu}\frac{\partial}{\partial \nu_0}$$

$$\left.\frac{\partial}{\partial r}\right|_{t\mu\nu} = \left.\frac{\partial}{\partial r}\right|_{t\mu_0\nu_0} + \left.\frac{\partial \mu_0}{\partial r}\right|_{t\mu\nu}\frac{\partial}{\partial \mu_0} + \left.\frac{\partial \nu_0}{\partial r}\right|_{t\mu\nu}\frac{\partial}{\partial \nu_0}$$

$$\left.\frac{\partial}{\partial \mu}\right|_{tr\nu} = \cancel{\left.\frac{\partial}{\partial \mu}\right|_{tr\nu_0}} + \left.\frac{\partial \mu_0}{\partial \mu}\right|_{tr\nu}\frac{\partial}{\partial \mu_0} + \left.\frac{\partial \nu_0}{\partial \mu}\right|_{tr\nu}\frac{\partial}{\partial \nu_0}$$

q the results just derived, we have:

$$\frac{\partial \mu_0}{\partial t} = -\gamma^2(1-\mu_0^2)\partial\beta/\partial t \qquad \frac{\partial \nu_0}{\partial t} = -\gamma^2\nu_0\mu_0(\partial\beta/\partial t)$$

$$\partial\mu_0/\partial r = -\gamma^2(1-\mu_0^2)\partial\beta/\partial r \qquad \partial\nu_0/\partial r = -\gamma^2\nu_0\mu_0(\partial\beta/\partial r)$$

$$\partial\mu_0/\partial\mu = \gamma^2(1+\beta\mu_0)^2. \qquad \partial\nu_0/\partial\mu = -\beta\gamma^2(1+\beta\mu_0)\nu_0$$

need $\nu = \gamma\nu_0(1+\beta\mu_0)$ or $(\nu/\nu_0) = \gamma(1+\beta\mu_0)$

$$\frac{1}{\nu} = \frac{1}{c}\left.\frac{\partial I_0}{\partial t}\right|_{r\mu_0\nu_0} - \frac{\gamma^2}{c}(1-\mu_0^2)\frac{\partial\beta}{\partial t}\frac{\partial I_0}{\partial \mu_0} - \frac{\gamma^2}{c}\nu_0\mu_0\frac{\partial\beta}{\partial t}\frac{\partial I_0}{\partial \nu_0}$$

$$\nu = \frac{\mu_0+\beta}{(1+\beta\mu_0)}\left\{\left.\frac{\partial I_0}{\partial r}\right|_{r\mu_0\nu_0} - \gamma^2(1-\mu_0^2)\frac{\partial\beta}{\partial r}\frac{\partial I_0}{\partial \mu_0} - \gamma^2\nu_0\mu_0\frac{\partial\beta}{\partial r}\frac{\partial I_0}{\partial \nu_0}\right\}$$

$$\partial\mu|_{r\mu\nu} = \frac{(1-\mu_0^2)}{\gamma^2(1+\beta\mu_0)^2}\left\{\gamma^2\frac{(1+\beta\mu_0)^2}{r}\frac{\partial I_0}{\partial \mu_0} - \beta\gamma^2\frac{(1+\beta\mu_0)\nu_0}{r}\frac{\partial I_0}{\partial \nu_0}\right\}$$

and $\frac{\nu}{\nu_0}\left[\frac{1}{c}\frac{\partial I_0}{\partial t}+\mu\frac{\partial I_0}{\partial r}+\frac{(1-\mu^2)}{r}\frac{\partial I_0}{\partial\mu}\right]$

$$=\gamma(1+\beta\mu_0)\left\{\frac{1}{c}\frac{\partial I_0}{\partial t}+\frac{(\mu_0+\beta)}{(1+\beta\mu_0)}\frac{\partial I_0}{\partial r}\right\}$$

$$+\gamma(1+\beta\mu_0)\left\{-\frac{\gamma^2}{c}(1-\mu_0^2)\frac{\partial\beta}{\partial t}-\frac{(\mu_0+\beta)}{(1+\beta\mu_0)}\gamma^2(1-\mu_0^2)\frac{\partial\beta}{\partial r}+\frac{(1-\mu_0^2)}{\cancel{\gamma^2(1+\beta\mu_0)^2}}\frac{\cancel{\gamma^2(1+\beta\mu_0)^2}}{r}\right\}\frac{\partial I_0}{\partial\mu_0}$$

$$+\gamma(1+\beta\mu_0)\left\{-\frac{\gamma^2}{c}\nu_0\mu_0\frac{\partial\beta}{\partial t}-\frac{(\mu_0+\beta)}{(1+\beta\mu_0)}\gamma^2\nu_0\mu_0\frac{\partial\beta}{\partial r}-\frac{(1-\mu_0^2)}{\cancel{\gamma^2}(1+\beta\mu_0)^{\cancel{2}}}\frac{\beta\cancel{\gamma^2(1+\beta\mu_0)}}{r}\nu_0\right\}\frac{\partial I_0}{\partial\nu_0}$$

$$=\frac{\gamma}{c}(1+\beta\mu_0)\frac{\partial I_0}{\partial t}+\gamma(\mu_0+\beta)\frac{\partial I_0}{\partial r}$$

$$+\gamma(1-\mu_0^2)\left\{-\frac{\gamma^2}{c}(1+\beta\mu_0)\frac{\partial\beta}{\partial t}-\gamma^2(\mu_0+\beta)\frac{\partial\beta}{\partial r}+\frac{(1+\beta\mu_0)}{r}\right\}\frac{\partial I_0}{\partial\mu_0}$$

$$+\gamma\nu_0\left\{-\frac{\gamma^2}{c}(1+\beta\mu_0)\mu_0\frac{\partial\beta}{\partial t}-\gamma^2(\mu_0+\beta)\mu_0\frac{\partial\beta}{\partial r}-\frac{(1-\mu_0^2)}{r}\beta\right\}\frac{\partial I_0}{\partial\nu_0}$$

$$=\frac{\gamma}{c}(1+\beta\mu_0)\frac{\partial I_0}{\partial t}+\gamma(\mu_0+\beta)\frac{\partial I_0}{\partial r}+\gamma(1-\mu_0^2)\left\{\frac{(1+\beta\mu_0)}{r}-\frac{\gamma^2(1+\beta\mu_0)}{c}\frac{\partial\beta}{\partial t}-\gamma^2(\mu_0+\beta)\frac{\partial\beta}{\partial r}\right\}\frac{\partial I_0}{\partial\mu_0}$$
$$-\gamma\nu_0\left\{\frac{(1-\mu_0^2)}{r}\beta+\frac{\gamma^2(1+\beta\mu_0)\mu_0}{c}\frac{\partial\beta}{\partial t}+\gamma^2(\mu_0+\beta)\mu_0\frac{\partial\beta}{\partial r}\right\}\frac{\partial I_0}{\partial\nu_0}$$

Next the term: $-3I_0(\nu/\nu_0^2)\left[\frac{1}{c}\frac{\partial\nu_0}{\partial t}+\mu\frac{\partial\nu_0}{\partial r}+\frac{(1-\mu^2)}{r}\frac{\partial\nu_0}{\partial\mu}\right]$

$$[\;]=\frac{1}{c}\left(-\gamma^2\mu_0\nu_0\frac{\partial\beta}{\partial t}\right)+\frac{(\mu_0+\beta)}{(1+\beta\mu_0)}\left(-\gamma^2\nu_0\mu_0\frac{\partial\beta}{\partial r}\right)+\frac{(1-\mu_0^2)}{r\cancel{\gamma}(\cancel{1+\beta\mu_0})^2}\left(-\beta\gamma^{\cancel{2}}(1+\cancel{\beta\mu_0})\nu_0\right)$$

$$=-\nu_0\left\{\frac{\gamma^2\mu_0}{c}\frac{\partial\beta}{\partial t}+\frac{\gamma^2(\mu_0+\beta)\mu_0}{1+\beta\mu_0}\frac{\partial\beta}{\partial r}+\frac{\beta\cancel{\gamma}(1-\mu_0^2)}{(1+\beta\mu_0)r}\right\}$$

$$-3I_0\frac{\gamma(1+\beta\mu_0)}{\cancel{\nu_0}}(-\cancel{\nu_0})\left\{\qquad\right\}=+3I_0\gamma\left\{\frac{\gamma^2\mu_0}{c}(1+\beta\mu_0)\frac{\partial\beta}{\partial t}+\gamma^2(\mu_0+\beta)\mu_0\frac{\partial\beta}{\partial r}+\frac{\beta}{r}(1-\mu_0^2)\right\}$$

$$=3\gamma I_0\left\{\frac{\beta(1-\mu_0^2)}{r}+\frac{\gamma^2}{c}\mu_0(1+\beta\mu_0)\frac{\partial\beta}{\partial t}+\gamma^2\mu_0(\mu_0+\beta)\frac{\partial\beta}{\partial r}\right\}$$

The whole eqn becomes:

$$\frac{\gamma}{c}(1+\beta\mu_0)\frac{\partial I_0}{\partial t}+\gamma(\mu_0+\beta)\frac{\partial I_0}{\partial r}+\gamma(1-\mu_0^2)\left\{\frac{(1+\beta\mu_0)}{r}-\frac{\gamma^2}{c}(1+\beta\mu_0)\frac{\partial\beta}{\partial t}-\gamma^2(\mu_0+\beta)\frac{\partial\beta}{\partial r}\right\}\frac{\partial I_0}{\partial\mu_0}$$
$$-\gamma\nu_0\left\{\frac{(1-\mu_0^2)}{r}\beta+\frac{\gamma^2}{c}(1+\beta\mu_0)\mu_0\frac{\partial\beta}{\partial t}+\gamma^2\mu_0(\mu_0+\beta)\frac{\partial\beta}{\partial r}\right\}\frac{\partial I_0}{\partial\nu_0}$$
(95.9) $$+3\gamma\left\{\frac{\beta(1-\mu_0^2)}{r}+\frac{\gamma^2}{c}\mu_0(1+\beta\mu_0)\frac{\partial\beta}{\partial t}+\gamma^2\mu_0(\mu_0+\beta)\frac{\partial\beta}{\partial r}\right\}I_0=\eta_0-\chi_0 I_0$$

$$3\gamma\left[\frac{\beta(1-\mu_0^2)}{r}+\frac{\gamma^2}{c}\mu_0(1+\beta\mu_0)\frac{\partial\beta}{\partial t}+\gamma^2\mu_0(\mu_0+\beta)\frac{\partial\beta}{\partial r}\right]I_0$$

$$-I_0\frac{\partial}{\partial\mu_0}\left\{\gamma(1-\mu_0^2)\left[\frac{1+\beta\mu_0}{r}-\gamma^2(\mu_0+\beta)\frac{\partial\beta}{\partial r}-\frac{\gamma^2}{c}(1+\beta\mu_0)\frac{\partial\beta}{\partial t}\right]\right\}$$

sign error $$\oplus\, I_0\frac{\partial}{\partial\gamma_0}\left\{\gamma\gamma_0\left[\frac{\beta(1-\mu_0^2)}{r}+\gamma^2\mu_0(\mu_0+\beta)\frac{\partial\beta}{\partial r}+\frac{\gamma^2}{c}\mu_0(1+\beta\mu_0)\frac{\partial\beta}{\partial t}\right]\right\}$$

$$\frac{1}{I_0}=3\gamma\frac{\beta(1-\mu_0^2)}{r}+\frac{3\gamma\cdot\gamma^2}{c}\mu_0(1+\beta\mu_0)\frac{\partial\beta}{\partial t}\overset{3\gamma}{+}\gamma^2\mu_0(\mu_0+\beta)\frac{\partial\beta}{\partial r}$$

$$+\frac{2\gamma\mu_0(1+\beta\mu_0)}{r}-2\gamma\mu_0\cdot\frac{\gamma^2}{c}(1+\beta\mu_0)\frac{\partial\beta}{\partial t}-2\gamma\mu_0\gamma^2(\mu_0+\beta)\frac{\partial\beta}{\partial r}$$

$$-\gamma(1-\mu_0^2)\frac{\beta}{r}+\gamma(1-\mu_0^2)\frac{\gamma^2}{c}\beta\frac{\partial\beta}{\partial t}+\gamma(1-\mu_0^2)\gamma^2\frac{\partial\beta}{\partial r}$$

$$\oplus\,\gamma\beta\frac{(1-\mu_0^2)}{r}\oplus\gamma\cdot\frac{\gamma^2}{c}\mu_0(1+\beta\mu_0)\frac{\partial\beta}{\partial t}\oplus\gamma\gamma^2\mu_0(\mu_0+\beta)\frac{\partial\beta}{\partial r}$$

$$=\frac{\gamma}{r}\left\{3\beta(1-\mu_0^2)+2\mu_0(1+\beta\mu_0)-(1-\mu_0^2)\beta\oplus\beta(1-\mu_0^2)\right\}$$

$$+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left\{3\mu_0(1+\beta\mu_0)-2\mu_0(1+\beta\mu_0)+(1-\mu_0^2)\beta\oplus\mu_0(1+\beta\mu_0)\right\}$$

$$+\gamma^3\frac{\partial\beta}{\partial r}\left\{3\mu_0(\cancel{\mu_0+\beta})-2\mu_0(\cancel{\mu_0+\beta})+(1-\mu_0^2)\oplus\mu_0(\cancel{\mu_0+\beta})\right\}$$

~~$=\frac{\gamma}{r}\left\{\beta(1-\mu_0^2)+2\mu_0(1+\beta\mu_0)\right\}$~~

~~$+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left\{\mu_0(1+\cancel{\beta\mu_0})+(1-\mu_0^2)\beta-\mu_0(\cancel{1+\beta\mu_0})\right\}$~~

~~$+\gamma^3\frac{\partial\beta}{\partial r}\left\{1-\mu_0^2\right\}$~~

$\beta+2\mu_0+\beta\mu_0^2$

~~$=\frac{\gamma}{r}\left\{\beta-\beta\mu_0^2+2\mu_0+2\beta\mu_0^2\right\}$~~

~~$+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\beta(1-\mu_0^2)+\gamma^3\frac{\partial\beta}{\partial r}(1-\mu_0^2)$~~

line 4 of new eqn (95.9)

$$= \frac{\gamma}{r}\left\{3\beta(1-\mu_0^2)+2\mu_0(1+\beta\mu_0)\right\} + \frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left\{2\mu_0(1+\beta\mu_0)+\beta(1-\mu_0^2)\right\} + \gamma^3\frac{\partial\beta}{\partial r}\left\{2\mu_0(\mu_0+\beta)+(1-\mu_0\right.$$

$$= \frac{\gamma}{r}\left\{3\beta-3\beta\mu_0^2+2\mu_0+2\beta\mu_0^2\right\} + \frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left\{2\mu_0+2\beta\mu_0^2+\beta-\beta\mu_0^2\right\} + \gamma^3\frac{\partial\beta}{\partial r}\left\{2\mu_0^2+2\beta\mu_0+1-\mu_0^2\right\}$$

$$= \frac{\gamma}{r}\left\{\beta(3-\mu_0^2)+2\mu_0\right\} + \frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left\{2\mu_0+\beta(\mu_0^2+1)\right\} + \gamma^3\frac{\partial\beta}{\partial r}\left\{2\beta\mu_0+\mu_0^2+1\right\}$$

(a) $$\frac{\gamma}{c}\int(1+\beta\mu_0)\frac{\partial I_0}{\partial t}\frac{d\omega_0}{4\pi}+\gamma\int(\mu_0+\beta)\frac{\partial I_0}{\partial r}\frac{d\omega_0}{4\pi}+\gamma\int(1-\mu_0^2)\frac{(1+\beta\mu_0)}{r}\frac{\partial I_0}{\partial\mu_0}\frac{d\omega_0}{4\pi}-\frac{\gamma^3}{c}\int(1-\mu_0^2)(1+\beta\mu_0)\frac{\partial\beta}{\partial t}\frac{\partial I_0}{\partial\mu_0}\frac{d\omega_0}{4\pi}$$

$$-\gamma^3\frac{\partial\beta}{\partial r}\int(1-\mu_0^2)(\mu_0+\beta)\frac{\partial I_0}{\partial\mu_0}-\frac{\gamma\nu_0}{r}\beta\int(1-\mu_0^2)\frac{\partial I_0}{\partial\nu_0}\frac{d\omega_0}{4\pi}-\frac{\gamma^3\nu_0}{c}\frac{\partial\beta}{\partial t}\int\mu_0(1+\beta\mu_0)\frac{\partial I}{\partial\nu_0}\frac{d\omega}{4\pi}-\gamma^3\nu_0\frac{\partial\beta}{\partial r}\int\mu_0(\mu_0+\beta)\frac{\partial I_0}{\partial\nu_0}\frac{d\omega}{4}$$

$$+\frac{3\gamma\beta}{r}\int(1-\mu_0^2)I_0\frac{d\omega_0}{4\pi}+\frac{3\gamma^3}{c}\frac{\partial\beta}{\partial t}\int\mu_0(1+\beta\mu_0)I_0\frac{d\omega_0}{4\pi}+3\gamma^3\frac{\partial\beta}{\partial r}\int\mu_0(\mu_0+\beta)I_0\frac{d\omega_0}{4\pi}=\int(\eta_0-\chi_0 I_0)\frac{d\omega_0}{4\pi}$$

(b) $$=\frac{\gamma}{c}\left[\frac{\partial J_0}{\partial t}+\beta\frac{\partial H_0}{\partial t}\right]+\gamma\left[\frac{\partial H_0}{\partial r}+\beta\frac{\partial J_0}{\partial r}\right]+\frac{\gamma}{r}\int(1-\mu_0^2+\beta\mu_0-\beta\mu_0^3)\frac{\partial I_0}{\partial\mu_0}\frac{d\omega_0}{4\pi}-\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\int(1-\mu_0^2+\beta\mu_0-\beta\mu_0^3)\frac{\partial I_0}{\partial\mu_0}\frac{d\omega_0}{4\pi}$$

$$-\gamma^3\frac{\partial\beta}{\partial r}\int(\mu_0-\mu_0^3+\beta-\beta\mu_0^2)\frac{\partial I_0}{\partial\mu_0}\frac{d\omega_0}{4\pi}-\frac{\gamma\nu_0\beta}{r}\left(\frac{\partial J_0}{\partial\nu_0}-\frac{\partial K_0}{\partial\nu_0}\right)-\frac{\gamma^3\nu_0}{c}\frac{\partial\beta}{\partial t}\left(\frac{\partial H_0}{\partial\nu_0}+\beta\frac{\partial K_0}{\partial\nu_0}\right)-\gamma^3\nu_0\frac{\partial\beta}{\partial r}\left(\frac{\partial K_0}{\partial\nu_0}+\beta\frac{\partial H_0}{\partial\nu_0}\right.$$

$$+\frac{3\gamma\beta}{r}(J_0-K_0)+\frac{3\gamma^3}{c}\frac{\partial\beta}{\partial t}(H_0+\beta K_0)+3\gamma^3\frac{\partial\beta}{\partial r}(\beta H_0+K_0)=\eta_0-\chi_0 J_0$$

Evaluate $$\int(1-\mu_0^2+\beta\mu_0-\beta\mu_0^3)\frac{\partial I_0}{\partial\mu_0}d\mu_0=\int\frac{\partial}{\partial\mu_0}\left\{(1-\mu_0^2+\beta\mu_0-\beta\mu_0^3)I_0\right\}d\mu_0$$

$$-\int I_0\frac{\partial}{\partial\mu_0}(1-\mu_0^2+\beta\mu_0-\beta\mu_0^3)\,d\mu_0$$

$$=\left[I_0(1-\mu_0^2+\beta\mu_0-\beta\mu_0^3)\right]_{\mu_0=-1}^{+1}-\int I_0\left[-2\mu_0+\beta-3\beta\mu_0^2\right]d\mu_0$$

$$=I_0(\mu_0=+1)(1-1+\beta\cdot1-\beta\cdot1^3)-I_0(-1)\left[1-1+\beta(-1+1)\right]+\int I_0(2\mu_0-\beta+3\beta\mu_0^2)\,d\mu_0$$

(c) $$=0+0+(2H_0-\beta J_0+3\beta K_0)=-(\beta J_0-2H_0-3\beta K_0)$$

Similarly $$\int(\mu_0-\mu_0^3+\beta-\beta\mu_0^2)\frac{\partial I_0}{\partial\mu_0}d\mu_0=0-\int I_0\frac{\partial}{\partial\mu_0}(\mu_0-\mu_0^3+\beta-\beta\mu_0^2)\,d\mu_0$$

$$=-\int I_0(1-3\mu_0^2-2\beta\mu_0)\,d\mu_0=\int I_0(2\beta\mu_0+3\mu_0^2-1)\,d\mu_0$$

(d) $$=(-J_0+2\beta H_0+3K_0)=-(J_0-2\beta H_0-3K_0)$$

Thus the three terms in $\partial I_0/\partial\mu_0$ integrate to:

(e) $$\left(-\frac{\gamma}{r}+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\right)(\beta J_0-2H_0-3\beta K_0)+\gamma^3\frac{\partial\beta}{\partial r}(J_0-2\beta H_0-3K_0)$$

Add these terms to the 3rd line of (b):

$$-\frac{\gamma}{r}(\beta J_0-2H_0-3\beta K_0)+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}(\beta J_0-2H_0-3\beta K_0)+\gamma^3\frac{\partial\beta}{\partial r}(J_0-2\beta H_0-3K_0)$$

$$+\frac{\gamma}{r}(3\beta J_0-3\beta K_0)\qquad+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}(3H_0+3\beta K_0)\qquad+\gamma^3\frac{\partial\beta}{\partial r}(3\beta H_0+3K_0)$$

$$=\frac{\gamma}{r}(2\beta J_0+2H_0)\qquad+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}(\beta J_0+H_0)+\gamma^3\frac{\partial\beta}{\partial r}(J_0+\beta H_0)$$

Thus the entire zeroth moment equation becomes:

(95.11) $$\frac{\gamma}{c}\left(\frac{\partial J_0}{\partial t}+\beta\frac{\partial H_0}{\partial t}\right)+\gamma\left(\frac{\partial H_0}{\partial r}+\beta\frac{\partial J_0}{\partial r}\right)-\frac{\gamma\nu_0\beta}{r}\left(\frac{\partial J_0}{\partial \nu_0}-\frac{\partial K_0}{\partial \nu_0}\right)-\frac{\gamma^3\nu_0}{c}\frac{\partial\beta}{\partial t}\left(\frac{\partial H_0}{\partial \nu_0}+\beta\frac{\partial K_0}{\partial \nu_0}\right)$$
$$-\gamma^3\nu_0\frac{\partial\beta}{\partial r}\left(\frac{\partial K_0}{\partial \nu_0}+\beta\frac{\partial H_0}{\partial \nu_0}\right)+\frac{2\gamma}{r}(\beta J_0+H_0)+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}(\beta J_0+H_0)+\gamma^3\frac{\partial\beta}{\partial r}(J_0+\beta H_0)=\eta_0-\chi_0 J_0$$

Now start again and integrate against $\mu_0\, d\omega_0/4\pi$:

$$\frac{\gamma}{c}\int(\mu_0+\beta\mu_0^2)\frac{\partial I_0}{\partial t}\frac{d\omega_0}{4\pi}+\gamma\int(\mu_0^2+\beta\mu_0)\frac{\partial I_0}{\partial r}\frac{d\omega_0}{4\pi}+\gamma\int(\mu_0-\mu_0^3)\left\{\frac{(1+\beta\mu_0)}{r}-\frac{\gamma^2}{c}(1+\beta\mu_0)\frac{\partial\beta}{\partial t}-\gamma^2(\mu_0+\beta)\frac{\partial\beta}{\partial r}\right\}\frac{\partial I_0}{\partial\mu_0}\frac{d\omega_0}{4\pi}$$
$$-\gamma\nu_0\int\left\{\frac{(1-\mu_0^2)}{r}\beta+\frac{\gamma^2}{c}(1+\beta\mu_0)\mu_0\frac{\partial\beta}{\partial t}+\gamma^2\mu_0(\mu_0+\beta)\frac{\partial\beta}{\partial r}\right\}\mu_0\frac{\partial I_0}{\partial\nu_0}$$
$$+3\gamma\int\left\{\beta\frac{(1-\mu_0^2)}{r}+\frac{\gamma^2}{c}\mu_0(1+\beta\mu_0)\frac{\partial\beta}{\partial t}+\gamma^2\mu_0(\mu_0+\beta)\frac{\partial\beta}{\partial r}\right\}\mu_0 I_0\frac{d\omega_0}{4\pi}=\int\mu_0(\eta_0-\chi_0 I_0)\frac{d\omega_0}{4\pi}$$

$$=\frac{\gamma}{c}\left(\frac{\partial H_0}{\partial t}+\beta\frac{\partial K_0}{\partial t}\right)+\gamma\left(\beta\frac{\partial H_0}{\partial r}+\frac{\partial K_0}{\partial r}\right)-\gamma\nu_0\left\{\frac{\beta}{r}\int(\mu_0-\mu_0^3)\frac{\partial I_0}{\partial\nu_0}\frac{d\omega_0}{4\pi}+\frac{\gamma}{c^2}\int(\mu_0^2+\beta\mu_0^3)\frac{\partial\beta}{\partial t}\frac{\partial I_0}{\partial\nu_0}\frac{d\omega_0}{4\pi}+\gamma^2\frac{\partial\beta}{\partial r}\int(\mu_0^3+\beta\mu_0^2)\right\}$$

$$+\frac{\gamma}{r}\int(\mu_0-\mu_0^3)(1+\beta\mu_0)\frac{\partial I_0}{\partial\mu_0}\frac{d\omega_0}{4\pi}-\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\int(\mu_0-\mu_0^3)(1+\beta\mu_0)\frac{\partial I_0}{\partial\mu_0}\frac{d\omega_0}{4\pi}-\gamma^3\frac{\partial\beta}{\partial r}\int(\mu_0-\mu_0^3)(\mu_0+\beta)\frac{\partial I_0}{\partial\mu_0}\frac{d\omega_0}{4\pi}$$

$$+3\gamma\frac{\beta}{r}\int(\mu_0-\mu_0^3)I_0\frac{d\omega_0}{4\pi}+\frac{3\gamma^3}{c}\frac{\partial\beta}{\partial t}\int(\mu_0^2+\beta\mu_0^3)I_0\frac{d\omega_0}{4\pi}+3\gamma^3\frac{\partial\beta}{\partial r}\int(\mu_0^3+\beta\mu_0^2)I_0\frac{d\omega_0}{4\pi}=\int(\eta_0-\chi I_0)\mu_0\frac{d\iota}{4}$$

Again we must integrate terms in $\frac{\partial I_0}{\partial\mu_0}$ by parts:

$$\int(\mu_0-\mu_0^3)(1+\beta\mu_0)\frac{\partial I_0}{\partial\mu_0}d\mu_0=\int(\mu_0-\mu_0^3+\beta\mu_0^2-3\mu_0^4)\frac{\partial I_0}{\partial\mu_0}d\mu_0$$
$$=\int\frac{\partial}{\partial\mu_0}\left\{(\mu_0-\mu_0^3+\beta\mu_0^2-\beta\mu_0^4)I_0\right\}d\mu_0-\int I_0\frac{\partial}{\partial\mu_0}(\mu_0+\beta\mu_0^2-\mu_0^3-\beta\mu_0^4)\,d\mu_0$$
$$=\Big[(\mu_0-\mu_0^3+\beta\mu_0^2-\beta\mu_0^4)I_0\Big]_{-1}^{+1}-\int I_0(1+2\beta\mu_0-3\mu_0^2-4\beta\mu_0^3)\,d\mu_0$$
$$=0-(J_0+2\beta H_0-3K_0-4\beta N_0)=(-J_0+3K_0-2\beta H_0+4\beta N_0)$$

and $$\int(\mu_0-\mu_0^3)(\mu_0+\beta)\frac{\partial I_0}{\partial\mu_0}d\mu_0=\int(\mu_0^2-\mu_0^4+\beta\mu_0-\beta\mu_0^3)\frac{\partial I_0}{\partial\mu_0}d\mu_0$$
$$=-\int I_0\frac{\partial}{\partial\mu_0}(\mu_0^2-\mu_0^4+\beta\mu_0-\beta\mu_0^3)d\mu_0=-\int I_0\left[2\mu_0-4\mu_0^3+\beta-3\beta\mu_0^2\right]d\mu_0$$
$$=-(2H_0-4N_0+\beta J_0-3\beta K_0)=(-\beta J_0-2H_0+3\beta K_0+4N_0)$$

The terms in $\partial I_0/\partial\mu_0$ then become:

$$\frac{\gamma}{r}(-J_0+3K_0-2\beta H_0+4\beta N_0)-\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}(-J_0+3K_0-2\beta H_0+4\beta N_0)-\gamma^3\frac{\partial\beta}{\partial r}(-\beta J_0-2H_0+3\beta K_0+4N_0)$$

and the terms in I_0 grow:

$$\frac{\gamma}{r}(3\beta H_0-3\beta N_0)+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}(3K_0+3\beta N_0)+\gamma^3\frac{\partial\beta}{\partial r}(3\beta K_0+3N_0)$$

Adding these two sets of terms we obtain:

$$\frac{\gamma}{r}\left[-J_0+\beta H_0+3K_0+\beta N_0\right)+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left(J_0+2\beta H_0-\beta N_0\right)+\gamma^3\frac{\partial\beta}{\partial r}\left(\beta J_0+2H_0-N_0\right)$$

Thus the 1st moment equation becomes

$$\frac{\gamma}{c}\left(\frac{\partial H_0}{\partial t}+\beta\frac{\partial K_0}{\partial t}\right)+\gamma\left(\beta\frac{\partial H_0}{\partial r}+\frac{\partial K_0}{\partial r}\right)-\gamma\nu_0\left\{\frac{\beta}{r}\left(\frac{\partial H_0}{\partial \nu_0}-\frac{\partial N_0}{\partial \nu_0}\right)+\frac{\gamma^2}{c^2}\frac{\partial\beta}{\partial t}\left(\frac{\partial K_0}{\partial \nu_0}+\beta\frac{\partial N_0}{\partial \nu_0}\right)+\gamma^2\frac{\partial\beta}{\partial r}\left(\beta\frac{\partial K_0}{\partial \nu_0}+\frac{\partial N_0}{\partial \nu_0}\right)\right.$$

$$\tag{95.12} +\frac{\gamma}{r}\left(-J_0+\beta H_0+3K_0+\beta N_0\right)+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left(J_0+2\beta H_0-\beta N_0\right)+\gamma^3\frac{\partial\beta}{\partial r}\left(\beta J_0+2H_0-N_0\right)=-\chi_0 H_0$$

Integrate over $d\nu_0$ to obtain frequency-integrated moment equations:
(95.11) becomes:

$$\frac{\gamma}{c}\int\left(\frac{\partial J_0}{\partial t}+\beta\frac{\partial H_0}{\partial t}\right)d\nu_0+\gamma\int\left(\frac{\partial H_0}{\partial r}+\beta\frac{\partial J_0}{\partial r}\right)d\nu_0-\frac{\gamma\beta}{r}\int\left(\frac{\partial J_0}{\partial \nu_0}-\frac{\partial K_0}{\partial \nu_0}\right)\nu_0\,d\nu_0-\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\int\left(\frac{\partial H_0}{\partial \nu_0}+\beta\frac{\partial K_0}{\partial \nu_0}\right)\nu_0\,d\nu_0$$

$$-\gamma^3\frac{\partial\beta}{\partial r}\int\left(\frac{\partial K_0}{\partial \nu_0}+\beta\frac{\partial H_0}{\partial \nu_0}\right)\nu_0\,d\nu_0+\frac{2\gamma}{r}\int(\beta J_0+H_0)\,d\nu_0+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\int(\beta J_0+H_0)\,d\nu_0+\gamma^3\frac{\partial\beta}{\partial r}\int(J_0+\beta H_0)\,d\nu_0$$

$$=\int(\eta_0-\chi_0 J_0)\,d\nu_0$$

Using now the symbols J_0, K_0, H_0 + N_0 to mean frequency-integrated quantities, we have

$$\frac{\gamma}{c}\left(\frac{\partial J_0}{\partial t}+\beta\frac{\partial H_0}{\partial t}\right)+\gamma\left(\frac{\partial H_0}{\partial r}+\beta\frac{\partial J_0}{\partial r}\right)+\frac{2\gamma}{r}\left(\beta J_0+H_0\right)+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left(\beta J_0+H_0\right)+\gamma^3\frac{\partial\beta}{\partial r}\left(J_0+\beta H_0\right)$$

$$-\frac{\gamma\beta}{r}\int\nu_0\,d\nu_0\left(\frac{\partial J_0(\nu_0)}{\partial \nu_0}-\frac{\partial K_0(\nu_0)}{\partial \nu_0}\right)-\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\int\nu_0\,d\nu_0\left(\frac{\partial H_0(\nu_0)}{\partial \nu_0}+\beta\frac{\partial K_0(\nu_0)}{\partial \nu_0}\right)-\gamma^3\frac{\partial\beta}{\partial r}\int\nu_0\,d\nu_0\left(\frac{\partial K_0(\nu_0)}{\partial \nu_0}+\beta\frac{\partial H_0(\nu_0)}{\partial \nu_0}\right)$$

$$=\int\left[\eta_0(\nu_0)-\chi_0(\nu_0)J_0(\nu_0)\right]d\nu_0$$

Integrating the terms in $\partial/\partial\nu_0$ by parts we obtain:

$$\int\nu_0\,d\nu_0\left(\frac{\partial J_0(\nu_0)}{\partial \nu_0}-\frac{\partial K_0(\nu_0)}{\partial \nu_0}\right)=\int\frac{\partial}{\partial\nu_0}\left[\nu_0 J_0(\nu_0)-\nu_0 K_0(\nu_0)\right]d\nu_0-\int\left(J_0(\nu_0)+K_0(\nu_0)\right)d\nu_0$$

$$=\left[\nu_0 J_0-\nu_0 K_0\right]_{\nu_0=0}^{\infty}-\left(J_0-K_0\right)$$

As $\nu_0\to\infty$, J_0 and $K_0\to 0$ exponentially, while at $\nu_0=0$, J_0+K_0 are finite

hence $\int\ = 0-(J_0+K_0)$

and

$$-\frac{\gamma\beta}{r}\int\nu_0\,d\nu_0\left(\frac{\partial J_0(\nu_0)}{\partial \nu_0}-\frac{\partial K_0(\nu_0)}{\partial \nu_0}\right)=+\frac{\gamma\beta}{r}\left(J_0-K_0\right)$$

A similar analysis gives:

$$-\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\int\left(\frac{\partial H_0(\nu_0)}{\partial\nu_0}+\beta\frac{\partial K_0(\nu_0)}{\partial\nu_0}\right)\nu_0\,d\nu_0 = +\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\int(H_0(\nu_0)+\beta K_0(\nu_0))\,d\nu_0 = \frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}(H_0+\beta K_0)$$

and

$$-\gamma^3\frac{\partial\beta}{\partial r}\int\left(\frac{\partial K_0(\nu_0)}{\partial\nu_0}+\beta\frac{\partial N_0(\nu_0)}{\partial\nu_0}\right)\nu_0\,d\nu_0 = \gamma^3\frac{\partial\beta}{\partial r}\int(K_0(\nu_0)+\beta N_0(\nu_0))\,d\nu_0 = \gamma^3\frac{\partial\beta}{\partial r}(K_0+\beta N_0)$$

The ν_0-integrated zeroth moment eqn becomes:

$$\frac{\gamma}{c}\left(\frac{\partial J_0}{\partial t}+\beta\frac{\partial H_0}{\partial t}\right)+\gamma\left(\frac{\partial H_0}{\partial r}+\beta\frac{\partial J_0}{\partial r}\right)+\frac{2\gamma}{\gamma}(\beta J_0+H_0)+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}(\beta J_0+H_0)+\gamma^3\frac{\partial\beta}{\partial r}(J_0+\beta H_0)$$
$$+\frac{\gamma}{\gamma}(\beta J_0-\beta K_0)+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}(H_0+\beta K_0)+\gamma^3\frac{\partial\beta}{\partial r}(K_0+\beta N_0)$$
$$=\int(\eta_0(\nu_0)-\chi_0(\nu_0)J_0(\nu_0))\,d\nu_0$$

(95,13)
$$\frac{\gamma}{c}\left(\frac{\partial J_0}{\partial t}+\beta\frac{\partial H_0}{\partial t}\right)+\gamma\left(\frac{\partial H_0}{\partial r}+\beta\frac{\partial J_0}{\partial r}\right)+\frac{\gamma}{r}\left[3\beta J_0+2H_0-\beta K_0\right]+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left[\beta J_0+2H_0+\beta K_0\right]$$
$$+\gamma^3\frac{\partial\beta}{\partial r}\left[J_0+2\beta H_0+K_0\right] = \int_0^\infty\left[\eta_0(\nu_0)-\chi_0(\nu_0)J_0(\nu_0)\right]d\nu_0$$

Now integrate the first moment equation over frequency:

$$\frac{\gamma}{c}\int\left(\frac{\partial H_0(\nu_0)}{\partial t}+\beta\frac{\partial K_0(\nu_0)}{\partial t}\right)d\nu_0+\gamma\int\left(\beta\frac{\partial H_0(\nu_0)}{\partial r}+\frac{\partial K_0(\nu_0)}{\partial r}\right)d\nu_0$$
$$+\frac{\gamma}{\gamma}\int\left[-J_0(\nu_0)+\beta H_0(\nu_0)+3K_0(\nu_0)+\beta N_0(\nu_0)\right]d\nu_0+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\int\left[J_0(\nu_0)+2\beta H_0(\nu_0)-\beta N_0(\nu_0)\right]d\nu_0$$
$$+\gamma^3\frac{\partial\beta}{\partial r}\int\left[3J_0(\nu_0)+2H_0(\nu_0)-N_0(\nu_0)\right]d\nu_0$$
$$-\gamma\frac{\beta}{\gamma}\int\left(\frac{\partial H_0(\nu_0)}{\partial\nu_0}-\frac{\partial N_0(\nu_0)}{\partial\nu_0}\right)\nu_0\,d\nu_0-\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\int\left[\frac{\partial K_0(\nu_0)}{\partial\nu_0}+\beta\frac{\partial N_0(\nu_0)}{\partial\nu_0}\right]\nu_0\,d\nu_0-\gamma^3\frac{\partial\beta}{\partial r}\int\left[\beta\frac{\partial K_0(\nu_0)}{\partial\nu_0}+\frac{\partial N_0(\nu_0)}{\partial\nu_0}\right]\nu$$
$$=-\int\chi_0(\nu_0)H_0(\nu_0)\,d\nu_0$$

The terms in $\partial/\partial\nu_0$ become:

$$-(\gamma\beta/r)\left\{-\int(H_0(\nu_0)-N_0(\nu_0)]\,d\nu_0\right\}=\frac{\gamma}{r}(\beta H_0-\beta N_0)$$
$$-\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left\{-\int[K_0(\nu_0)+\beta N_0(\nu_0)]\,d\nu_0\right\}=\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}(K_0+\beta N_0)$$
$$-\gamma^3\frac{\partial\beta}{\partial r}\left\{-\int[\beta K_0(\nu_0)+N_0(\nu_0)]\,d\nu_0\right\}=\gamma^3\frac{\partial\beta}{\partial r}(\beta K_0+N_0)$$

and the equation becomes

$$\frac{\gamma}{c}\left(\frac{\partial H_0}{\partial t}+\beta\frac{\partial K_0}{\partial t}\right)+\gamma\left(\beta\frac{\partial H_0}{\partial r}+\frac{\partial K_0}{\partial r}\right)+\frac{\gamma}{r}(-J_0+\beta H_0+3K_0+\beta N_0)+\frac{\gamma}{\gamma}(\beta H_0-\beta N_0)$$
$$+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}(J_0+2\beta H_0-\beta N_0)+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}(K_0+\beta N_0)+\gamma^3\frac{\partial\beta}{\partial r}(\beta J_0+2N_0-N_0)+\gamma^3\frac{\partial\beta}{\partial r}(\beta K_0+N_0)$$
$$=-\int\chi_0(\nu_0)H_0(\nu_0)\,d\nu_0$$

$$\frac{\gamma}{c}\left(\frac{\partial H_0}{\partial t}+\beta\frac{\partial K_0}{\partial t}\right)+\gamma\left(\beta\frac{\partial H_0}{\partial r}+\frac{\partial K_0}{\partial r}\right)+\frac{\gamma}{r}\left(-J_0+2\beta H_0+3K_0\right)+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left(J_0+K_0+2\beta H_0\right) \tag{95.14}$$
$$+\gamma^3\frac{\partial\beta}{\partial r}\left(2H_0+\beta J_0+\beta K_0\right) = -\int\chi_0(\nu_0)\,H_0(\nu_0)\,d\nu_0$$

Now check these by using inertial-frame moment equations and transforming $(J, H, K) \to (J_0, H_0, K_0)$ via eqns (91.13)-(91.15)

$$c^{-1}\frac{\partial J}{\partial t}+\frac{\partial H}{\partial r}+\frac{2H}{r} = -\left(\frac{c}{4\pi}\right)G^0$$
$$c^{-1}\frac{\partial H}{\partial t}+\frac{\partial K}{\partial r}+\left(\frac{3K-J}{r}\right) = -\left(\frac{c}{4\pi}\right)G^1$$

Write (91.13)-(91.15) in terms of J, H, K etc using $E=\frac{4\pi}{c}J$, $F=4\pi H$, $P=\frac{4\pi K}{c}$

Then $\frac{4\pi}{c}J = \gamma^2\left[\frac{4\pi}{c}J_0+2\beta\,\frac{4\pi}{c}H_0+\beta^2\frac{4\pi}{c}K\right]$

(91.13′) $\Rightarrow\quad J=\gamma^2\left[J_0+2\beta H_0+\beta^2K_0\right]$ (91.13′)

$$4\pi H=\gamma^2\left[(1+\beta^2)4\pi H_0+v\,\frac{4\pi}{c}J_0+v\,\frac{4\pi}{c}K_0\right]$$

(91.14′) $H=\gamma^2\left[(1+\beta^2)H_0+\beta J_0+\beta K_0\right]$ (91.14′)

$$\frac{4\pi}{c}K=\gamma^2\left(\frac{4\pi}{c}K_0+2\frac{\beta}{c}\,4\pi H_0+\beta^2\frac{4\pi}{c}J_0\right]$$

(91.15′) $K=\gamma^2\left(K_0+2\beta H_0+\beta^2J_0\right)$

Then

$$c^{-1}\frac{\partial}{\partial t}\left\{\gamma^2\left[J_0+2\beta H_0+\beta^2K_0\right]\right\}+\frac{\partial}{\partial r}\left\{\gamma^2\left[(1+\beta^2)H_0+\beta J_0+\beta K_0\right\}\right.$$
$$+\frac{2}{r}\left\{\gamma^2\left[(1+\beta^2)H_0+\beta J_0+\beta K_0\right]\right\} = -\frac{c}{4\pi}G^0$$
$$=c^{-1}\frac{\partial\gamma^2}{\partial t}\left[J_0+2\beta H_0+\beta^2K_0\right]+c^{-1}\gamma^2\left[\frac{\partial J_0}{\partial t}+2\frac{\partial}{\partial t}(\beta H_0)+\frac{\partial}{\partial t}(\beta^2K_0)\right]$$
$$+\frac{\partial\gamma^2}{\partial r}\left[(1+\beta^2)H_0+\beta J_0+\beta K_0\right]+\gamma^2\left\{\frac{\partial}{\partial r}\left[(1+\beta^2)H_0\right]+\frac{\partial}{\partial r}(\beta J_0)+\frac{\partial}{\partial r}(\beta K_0)\right\}$$
$$+\frac{2\gamma^2}{r}\left\{(1+\beta^2)H_0+\beta J_0+\beta K_0\right\} = -\frac{c}{4\pi}G^0$$

now evaluate $\frac{\partial\gamma^2}{\partial t} = 2\gamma\frac{\partial}{\partial t}(1-\beta^2)^{-1/2} = 2\gamma\left(\cancel{-\tfrac{1}{2}}\right)(1-\beta^2)^{-3/2}\left(\cancel{-2}\beta\frac{\partial\beta}{\partial t}\right)$

$$= 2\gamma\cdot\gamma^3\cdot\beta\frac{\partial\beta}{\partial t} = 2\gamma^4\beta\frac{\partial\beta}{\partial t}$$

Similarly $\frac{\partial\gamma^2}{\partial r} = 2\gamma^4\beta\frac{\partial\beta}{\partial r}$

Hence we have:

$$c^{-1}\left(2\gamma^4\beta\frac{\partial\beta}{\partial t}\right)\left(J_0+2\beta H_0+\beta^2K_0\right)+c^{-1}\gamma^2\left(\frac{\partial J_0}{\partial t}+2\beta\frac{\partial H_0}{\partial t}+\beta^2\frac{\partial K_0}{\partial t}\right)+c^{-1}\gamma^2\frac{\partial\beta}{\partial t}\left(2H_0+2\beta K_0\right)$$

$$+2\gamma^4\beta\frac{\partial\beta}{\partial r}\left[(1+\beta^2)H_0+\beta J_0+\beta K_0\right]+\gamma^2\left\{(1+\beta^2)\frac{\partial H_0}{\partial r}+\beta\frac{\partial J_0}{\partial r}+\beta\frac{\partial K_0}{\partial r}\right\}+\gamma^2\frac{\partial\beta}{\partial r}\left[2\beta H_0+J_0+K_0\right]$$

$$+\frac{2\gamma^2}{r}\left\{(1+\beta^2)H_0+\beta J_0+\beta K_0\right\}=-\frac{c}{4\pi}G^0$$

or $$c^{-1}\gamma^2\left(\frac{\partial J_0}{\partial t}+2\beta\frac{\partial H_0}{\partial t}+\beta^2\frac{\partial K_0}{\partial t}\right)+\gamma^2\left\{(1+\beta^2)\frac{\partial H_0}{\partial r}+\beta\frac{\partial J_0}{\partial r}+\beta\frac{\partial K_0}{\partial r}\right\}$$

$$+c^{-1}\gamma^2\frac{\partial\beta}{\partial t}\left\{2\gamma^2\beta\left(J_0+2\beta H_0+\beta^2K_0\right)+\left(2H_0+2\beta K_0\right)\right\}$$

$$+\gamma^2\frac{\partial\beta}{\partial r}\left\{2\gamma^2\beta\left[(1+\beta^2)H_0+\beta J_0+\beta K_0\right]+\left(2\beta H_0+J_0+K_0\right)\right\}$$

$$+\frac{2\gamma^2}{r}\left\{(1+\beta^2)H_0+\beta J_0+\beta K_0\right\}=-\frac{c}{4\pi}G^0=-\frac{c}{4\pi}\gamma\left(G_0^0+\beta G_0^1\right)$$

and $$\frac{1}{4\pi}G_0^0=-c^{-1}\int d\nu_0\left[\eta_0(\nu_0)-\chi_0(\nu_0)J_0(\nu_0)\right]\ldots$$

$$\frac{1}{4\pi}G_0^1=-c^{-1}\int d\nu_0\left[-\chi_0(\nu_0)H_0(\nu_0)\right]$$

hence

$$\frac{c}{4\pi}G_0^0=-\int d\nu_0\left[\eta_0(\nu_0)-\chi_0(\nu_0)J_0(\nu_0)\right]$$

$$\frac{c}{4\pi}G_0^1=+\int\chi_0(\nu_0)H_0(\nu_0)\,d\nu_0$$

$$-\frac{c}{4\pi}G^0=+\gamma\int d\nu_0\left[\eta_0(\nu_0)-\chi_0(\nu_0)J_0(\nu_0)\right]-\gamma\beta\int\chi_0(\nu_0)H_0(\nu_0)\,d\nu_0$$

We now have

$$\frac{\gamma}{c}\left[\left(\frac{\partial J_0}{\partial t}+\beta\frac{\partial H_0}{\partial t}\right)+\beta\left(\frac{\partial H_0}{\partial t}+\beta\frac{\partial K_0}{\partial t}\right)\right]+\gamma\left[\left(\frac{\partial H_0}{\partial r}+\beta\frac{\partial J_0}{\partial r}\right)+\beta\left(\frac{\partial K_0}{\partial r}+\beta\frac{\partial H_0}{\partial r}\right)\right]$$

$$+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left\{2\beta J_0+\overset{2}{\cancel{4}}\beta^2H_0+2\cancel{\beta^3}K_0+2(1-\cancel{\beta^2})H_0+2\beta(1-\cancel{\beta^2})K_0\right\}$$

$$+\gamma^3\frac{\partial\beta}{\partial r}\left\{2\beta(1+\cancel{\beta^2})H_0+\overset{1}{\cancel{2}}\beta^2J_0+\overset{1}{\cancel{2}}\beta^2K_0+2\beta(1-\cancel{\beta^2})H_0+(1-\cancel{\beta^2})J_0+(1-\cancel{\beta^2})K_0\right\}$$

$$+\frac{\gamma}{r}\left\{2H_0+2\beta^2H_0+2\beta J_0+2\beta K_0\right\}=\int d\nu_0\left[\eta_0(\nu_0)-\chi_0(\nu_0)J_0(\nu_0)\right]-\beta\int\chi_0(\nu_0)H_0(\nu_0)\,d\nu_0$$

$$\frac{\gamma}{c}\left[\left(\frac{\partial J_0}{\partial t}+\beta\frac{\partial H_0}{\partial t}\right)+\beta\left(\frac{\partial H_0}{\partial t}+\beta\frac{\partial K_0}{\partial t}\right)\right]+\gamma\left[\left(\frac{\partial H_0}{\partial r}+\beta\frac{\partial J_0}{\partial r}\right)+\beta\left(\frac{\partial K_0}{\partial r}+\beta\frac{\partial H_0}{\partial r}\right]\right.$$

$$+\frac{\gamma}{r}\left[(2H_0+3\beta J_0-\beta K_0)+(-\beta J_0+2\beta^2 H_0+3\beta K_0)\right]$$

$$+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\Big[\underbrace{2\beta J_0+2\beta^2 H_0+2H_0+2\beta K_0}_{(2H_0+\beta J_0+\beta K_0)+\beta(2\beta H_0+J_0+K_0)}\Big]+\gamma^3\frac{\partial\beta}{\partial r}\Big[\underbrace{4\beta H_0+J_0+K_0+\beta^2 J_0+\beta^2 K_0}_{(J_0+K_0+2\beta H_0)+\beta(2H_0+\beta J_0+\beta K_0)}\Big]$$

$$=\frac{\gamma}{c}\left[\left(\frac{\partial J_0}{\partial t}+\beta\frac{\partial H_0}{\partial t}\right)+\beta\left(\frac{\partial H_0}{\partial t}+\beta\frac{\partial K_0}{\partial t}\right)\right]+\gamma\left[\left(\frac{\partial H_0}{\partial r}+\beta\frac{\partial J_0}{\partial r}\right)+\beta\left(\frac{\partial K_0}{\partial r}+\beta\frac{\partial H_0}{\partial r}\right)\right]$$

$$+\frac{\gamma}{r}\left[(2H_0+3\beta J_0-\beta K_0)+\beta(-J_0+K_0+2\beta H_0)\right]+\frac{\gamma^3}{c}\frac{\partial\beta}{\partial t}\left[(2H_0+\beta J_0+\beta K_0)+\beta(J_0+K_0+2\beta H_0)\right]$$

$$+\gamma^3\frac{\partial\beta}{\partial r}\left[(J_0+K_0+2\beta H_0)+\beta(2H_0+\beta J_0+\beta K_0)\right]=\int d\nu_0\left[\eta_0(\nu_0)-\chi_0(\nu_0)J_0(\nu_0)\right]-\beta\int d\nu_0\,\chi_0(\nu_0)H_0($$

In each bracket the 1st parentheses contains the appropriate term from eqn (95.13) and the second parentheses, which is multiplied by β, contains the term from (95.14).

Similarly for (95.16)

$$c^{-1}\partial H/\partial t+\partial K/\partial r+(3K-J)/r=-(c/4\pi)G^1$$

becomes

$$c^{-1}\frac{\partial}{\partial t}\left\{\gamma^2\left[(1+\beta^2)H_0+\beta J_0+\beta K_0\right]\right\}+\frac{\partial}{\partial r}\left\{\gamma^2\left[K_0+2\beta H_0+\beta^2 J_0\right]\right\}$$

$$+\frac{1}{r}\left\{3\gamma^2(K_0+2\beta H_0+\beta^2 J_0)-\gamma^2(J_0+2\beta H_0+\beta^2 K_0)\right\}=-\frac{c}{4\pi}\gamma\left(G_0^1+\beta G_0^0\right)$$

$\Rightarrow$

$$\gamma c^{-1}\left\{\frac{\partial H_0}{\partial t}+\beta^2\frac{\partial H_0}{\partial t}+\beta\frac{\partial J_0}{\partial t}+\beta\frac{\partial K_0}{\partial t}\right\}+\gamma c^{-1}\frac{\partial\beta}{\partial t}\left\{2\gamma^2\beta\left[H_0+\beta^2 H_0+\beta J_0+\beta K_0\right]+2\beta H_0+J_0+K_0\right\}$$

$$+\gamma\left\{\frac{\partial K_0}{\partial r}+2\beta\frac{\partial H_0}{\partial r}+\beta^2\frac{\partial J_0}{\partial r}\right\}+\gamma\frac{\partial\beta}{\partial r}\left\{2\gamma^2\cancel{\beta}(K_0+2\beta H_0+\beta^2 J_0)+2H_0+2\beta J_0\right\}$$

$$+\frac{\gamma}{r}\left\{(3-\beta^2)K_0+4\beta H_0+(3\beta^2-1)J_0\right\}=-\int\chi_0 H_0\,d\nu_0+\beta\int(\eta_0-\chi_0 J_0)\,d\nu_0$$

now $2\gamma^2\beta\,(H_0+\beta^2 H_0+\beta J_0+\beta K_0)+2\beta H_0+J_0+K_0=\gamma^2\{2\beta H_0+2\beta^3 H_0+2\beta^2 J_0+2\beta^2 K_0+2(1-\beta^2)\beta H_0+(1-\beta^2)(J_0+K$

$$=\gamma^2\{2\beta H_0+\cancel{2\beta^3 H_0}+2\beta^2 J_0+2\beta^2 K_0+2\beta H_0-\cancel{2\beta^3 H_0}+J_0+K_0-\beta^2 J_0-\beta^2 K_0\}=\gamma^2\{J_0+K_0+4\beta H_0+\beta^2 J_0+\beta^2 K_0\}$$

and $2\gamma^2\beta(K_0+2\beta H_0+\beta^2 J_0)+2H_0+2\beta J_0=\gamma^2\{2\beta K_0+4\beta^2H_0+2\beta^3J_0+(1-\beta^2)(2H_0+2\beta J_0)\}$

$$=\gamma^2\{2\beta K_0+4\beta^2H_0+2\cancel{\beta^3J_0}+2H_0+2\beta J_0-2\beta^2H_0-2\cancel{\beta^3J_0}\}=\gamma^2\{2H_0+2\beta J_0+2\beta K_0+2\beta^2H_0\}$$

Finally, we can also write these results:

$$\gamma^2(J_0+K_0+4\beta H_0+\beta^2J_0+\beta^2K_0)=\gamma^2\{(J_0+K_0+2\beta H_0)+\beta(2H_0+\beta J_0+\beta K_0)\}$$

and $\gamma^2(2H_0+2\beta J_0+2\beta K_0+2\beta^2H_0)=\gamma^2\{(2H_0+\beta J_0+\beta K_0)+\beta(J_0+K_0+2\beta H_0)\}$

We also find

$$\frac{\gamma}{r}\{(3-\beta^2)K_0+4\beta H_0+(3\beta^2-1)J_0\}=\frac{\gamma}{r}\{3K_0-J_0+4\beta H_0-\beta^2K_0+3\beta^2J_0\}$$

$$=\frac{\gamma}{r}\{(3K_0-J_0+2\beta H_0)+\beta(2H_0+3\beta J_0-\beta K_0)\}$$

Hence the entire equation becomes:

$$c^{-1}\left\{\left(\frac{\partial H_0}{\partial t}+\beta\frac{\partial K_0}{\partial t}\right)+\beta\left(\frac{\partial J_0}{\partial t}+\beta\frac{\partial H_0}{\partial t}\right)\right\}+c^{-1}\gamma^3\frac{\partial\beta}{\partial t}\{(J_0+K_0+2\beta H_0)+\beta(2H_0+\beta J_0+\beta K_0)\}$$

$$+\left\{\left(\frac{\partial K_0}{\partial r}+\beta\frac{\partial H_0}{\partial r}\right)+\beta\left(\frac{\partial H_0}{\partial r}+\beta\frac{\partial J_0}{\partial r}\right)\right\}+\gamma^3\frac{\partial\beta}{\partial r}\{(2H_0+\beta J_0+\beta K_0)+\beta(J_0+K_0+2\beta H_0)\}$$

$$+\frac{\gamma}{r}\{(3K_0-J_0+2\beta H_0)+\beta(2H_0+3\beta J_0-\beta K_0)\}=-\int\chi_0H_0\,d\nu_0+\int(\eta_0-\chi_0J_0)\,d\nu_0$$

Which is exactly (95.14) + β times (95.13).

Equation (95.9) to 1st order in (v/c) becomes:

$$\frac{1}{c}\,\partial I_0/\partial t+(\mu_0+\beta)\,\partial I_0/\partial r+(1-\mu_0^2)\{r^{-1}(1+\beta\mu_0)-c^{-1}\partial\beta/\partial t-(\mu_0\,\partial\beta/\partial r)\}\,\partial I_0/\partial\mu_0$$

$$-\{r^{-1}(1-\mu_0^2)\beta+c^{-1}\mu_0\,\partial\beta/\partial t+\mu_0^2\,\partial\beta/\partial r\}\,\nu_0\,\partial I_0/\partial\nu_0+3\{r^{-1}(1-\mu_0^2)\beta+\mu_0c^{-1}\,\partial\beta/\partial t+\mu_0^2\,\partial\beta/\partial r\}I_0$$

$$=\eta_0-\chi_0I_0$$

or

$$c^{-1}\{\partial I_0/\partial t+v\,\partial I_0/\partial r\}+\mu_0\,\partial I_0/\partial r+(1-\mu_0^2)\{1/r+(\mu_0/c)[v/r-\partial v/\partial r]-a/c^2\}\,\partial I_0/\partial\mu_0$$

$$-c^{-1}\{(1-\mu_0^2)v/r+\mu_0^2\,\partial v/\partial r+\mu_0a/c\}\,\nu_0\,\partial I_0/\partial\nu_0+\frac{3}{c}\{(1-\mu_0^2)v/r+\mu_0^2\,\partial v/\partial r+\mu_0a/c\}I_0$$

$$=\eta_0-\chi_0I_0$$

$$\Rightarrow\frac{1}{c}\frac{DI_0}{Dt}+\mu_0\frac{\partial I_0}{\partial r}+(1-\mu_0^2)\left\{\frac{1}{r}+\frac{\mu_0}{c}\left(\frac{v}{r}-\frac{\partial v}{\partial r}\right)-\frac{a}{c^2}\right\}\frac{\partial I_0}{\partial\mu_0}-\frac{1}{c}\left\{(1-\mu_0^2)\frac{v}{r}+\mu_0^2\frac{\partial v}{\partial r}+\frac{\mu_0a}{c}\right\}\nu_0\frac{\partial I_0}{\partial\nu_0}$$

(95.17) $+\frac{3}{c}\left\{(1-\mu_0^2)\frac{v}{r}+\mu_0^2\frac{\partial v}{\partial r}+\frac{\mu_0a}{c}\right\}I_0=\eta_0-\chi_0I_0$

which is (95.17) written slightly differently than in the text.

Equation (95,10) to 1st order in v/c is

$$c^{-1}\left\{\frac{\partial J_0}{\partial t}+\cancel{\frac{v}{c}\frac{\partial H_0}{\partial t}}\right\}+\left\{\frac{\partial H_0}{\partial r}+\frac{v}{c}\frac{\partial J_0}{\partial r}\right\}-\nu_0\left\{\frac{v}{rc}\left[\frac{\partial J_0}{\partial \nu_0}-\frac{\partial K_0}{\partial \nu_0}\right]+\frac{a}{c^2}\left[\frac{\partial H_0}{\partial \nu_0}\right)+\frac{1}{c}\frac{\partial v}{\partial r}\frac{\partial K_0}{\partial \nu_0}\right\}$$

$$+\left\{\frac{2}{r}\left[H_0+\frac{v}{c}J_0\right]+\frac{a}{c^2}H_0+\frac{1}{c}\frac{\partial v}{\partial r}J_0\right\}=\eta_0-\chi_0 J_0$$

or

$$\frac{1}{c}\frac{DJ_0}{Dt}+\frac{\partial H_0}{\partial r}-\frac{\nu_0}{c}\left\{\frac{v}{r}\frac{\partial J_0}{\partial \nu_0}-\left(\frac{v}{r}-\frac{\partial v}{\partial r}\right)\frac{\partial K_0}{\partial \nu_0}+\frac{a}{c}\frac{\partial H_0}{\partial \nu_0}\right\}+\left\{\left(\frac{2v}{r}+\frac{\partial v}{\partial r}\right)\frac{J_0}{c}+\left(\frac{2}{r}+\frac{a}{c^2}\right)H_0\right\}$$

$$=\frac{1}{c}\frac{\partial J_0}{\partial t}+\frac{\partial H_0}{\partial r}+\frac{2H_0}{r}+\left(\frac{2v}{r}+\frac{\partial v}{\partial r}\right)\frac{J_0}{c}\,{}_{\wedge}\,v-\frac{\nu_0}{c}\left\{\frac{v}{r}\frac{\partial J_0}{\partial \nu_0}-\left(\frac{v}{r}-\frac{\partial v}{\partial r}\right)\frac{\partial K_0}{\partial \nu_0}+\frac{a}{c}\frac{\partial H_0}{\partial \nu_0}\right\}=\eta_0-\chi_0 J_0$$

$$+\frac{a}{c^2}H_0$$

$$(95,18)\qquad =\frac{1}{c}\frac{\partial J_0}{\partial t}+\frac{\partial H_0}{\partial r}+\frac{2H_0}{r}+\left(\frac{2v}{r}+\frac{\partial v}{\partial r}\right)\frac{J_0}{c}+\frac{a}{c^2}H_0-\frac{\nu_0}{c}\left\{\frac{v}{r}\frac{\partial J_0}{\partial \nu_0}-\left(\frac{v}{r}-\frac{\partial v}{\partial r}\right)\frac{\partial K_0}{\partial \nu_0}+\frac{a}{c}\frac{\partial H_0}{\partial \nu_0}\right\}=\eta_0-\chi_0 J_0$$

And (95,12) becomes

$$\frac{1}{c}\frac{\partial H_0}{\partial t}+\frac{\partial K_0}{\partial r}+\frac{v}{c}\frac{\partial H_0}{\partial r}-\frac{\nu_0}{c}\left\{\frac{v}{r}\left(\frac{\partial H_0}{\partial \nu_0}-\frac{\partial N_0}{\partial \nu_0}\right)+\frac{a}{c}\frac{\partial K_0}{\partial \nu_0}+\frac{\partial v}{\partial r}\frac{\partial N_0}{\partial \nu_0}\right\}+\left\{\frac{1}{r}(3K_0-J_0)+\frac{v}{cr}(H_0+N_0)\right\}$$

$$+\frac{a}{c^2}\left[J_0\right]+\frac{1}{c}\frac{\partial v}{\partial r}\left(2H_0-N_0\right)=-\chi_0 H_0$$

$$\frac{1}{c}\left(\frac{\partial H_0}{\partial t}+v\frac{\partial H_0}{\partial r}\right)+\frac{\partial K_0}{\partial r}+\frac{1}{r}(3K_0-J_0)+\frac{1}{c}\left(\frac{v}{r}+2\frac{\partial v}{\partial r}\right)H_0+\frac{1}{c}\left(\frac{v}{r}-\frac{\partial v}{\partial r}\right)N_0+\frac{a}{c^2}J_0$$

$$-\frac{\nu_0}{c}\left\{\frac{v}{r}\frac{\partial H_0}{\partial \nu_0}-\left(\frac{v}{r}-\frac{\partial v}{\partial r}\right)\frac{\partial N_0}{\partial \nu_0}+\frac{a}{c}\frac{\partial K_0}{\partial \nu_0}\right\}=-\chi_0 H_0$$

$$(95,19)\qquad \frac{1}{c}\frac{DH_0}{Dt}+\frac{\partial K_0}{\partial r}+\frac{1}{r}(3K_0-J_0)+\frac{1}{c}\left(\frac{v}{r}+2\frac{\partial v}{\partial r}\right)H_0+\frac{1}{c}\left(\frac{v}{r}-\frac{\partial v}{\partial r}\right)N_0+\frac{a}{c^2}J_0$$

$$-\frac{\nu_0}{c}\left\{\frac{v}{r}\frac{\partial H_0}{\partial \nu_0}-\left(\frac{v}{r}-\frac{\partial v}{\partial r}\right)\frac{\partial N_0}{\partial \nu_0}+\frac{a}{c}\frac{\partial K_0}{\partial \nu_0}\right\}=-\chi_0 H_0$$

Further, the integrated moment eqns become, (95.13):

$$\frac{1}{c}\left(\frac{\partial J_0}{\partial t}+\cancel{\frac{v}{c}\frac{\partial H_0}{\partial t}}\right)+\frac{\partial H_0}{\partial r}+\frac{v}{c}\frac{\partial J_0}{\partial r}+\left\{\frac{1}{r}\left(2H_0+3\frac{v}{c}J_0-\frac{v}{c}K_0\right)+\frac{a}{c^2}(2H_0)+\frac{1}{c}\frac{\partial v}{\partial r}(J_0+K_0)\right\}$$

$$=\int(\eta_0-\chi_0 J_0)\,d\nu_0$$

or

$$\frac{1}{c}\frac{DJ_0}{Dt}+\frac{\partial H_0}{\partial r}\boxed{\cancel{+\frac{v}{c}\frac{\partial J_0}{\partial r}}}+\frac{2H_0}{r}+\frac{1}{c}\left(\frac{3v}{r}+\frac{\partial v}{\partial r}\right)J_0+\frac{1}{c}\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)K_0+\frac{2a}{c^2}H_0=\int(\eta_0-\chi_0 J_0)\,d\nu$$

$$(95,20)\qquad \frac{1}{c}\frac{DJ_0}{Dt}+\frac{\partial H_0}{\partial r}+\frac{2H_0}{r}+\frac{v}{cr}(3J_0-K_0)+\frac{1}{c}\frac{\partial v}{\partial r}(J_0+K_0)+\frac{2a}{c^2}H_0=\int d\nu_0(\eta_0-\chi_0 J_0)$$

And (95.14) becomes:

$$\frac{1}{c}\left(\frac{\partial H_0}{\partial t}\right)+\left(\frac{\partial K_0}{\partial r}+\frac{v}{c}\frac{\partial H_0}{\partial r}\right)+\frac{1}{r}\left(3K_0-J_0+\frac{2v}{c}H_0\right)+\frac{a}{c^2}(J_0+K_0)+\frac{1}{c}\frac{\partial v}{\partial r}(2H_0)=-\int\chi_0 H_0\, d\nu_0$$

$$\Rightarrow \frac{1}{c}\frac{DH_0}{Dt}+\frac{\partial K_0}{\partial r}+\frac{2v}{cr}H_0+\frac{1}{r}(3K_0-J_0)+\frac{2}{c}\frac{\partial v}{\partial r}H_0+\frac{a}{c^2}(J_0+K_0)=-\int\chi_0 H_0\, d\nu_0$$

or

$$\frac{1}{c}\frac{\partial H_0}{\partial t}+\frac{\partial K_0}{\partial r}+\left(\frac{v}{r}+\frac{\partial v}{\partial r}\right)\frac{2H_0}{c}+\frac{1}{r}(3K_0-J_0)+\frac{a}{c^2}(J_0+K_0)=-\int\chi_0 H_0\, d\nu_0 \tag{95.21}$$

Consistency of inertial frame & co-moving frame eqns to $O(v/c)$ in grey, LTE, nonscattering planar medium; with no acceleration terms:

(95.20) $\Rightarrow$

$$\frac{1}{c}\left(\frac{\partial J_0}{\partial t}+v\frac{\partial J_0}{\partial z}\right)+\frac{\partial H_0}{\partial z}+\frac{1}{c}\frac{\partial v}{\partial z}(J_0+K_0)=\int d\nu_0(\eta_0-\chi_0 J_0)=\chi_0(B_0-J_0)$$

Now take (93.10)

$$\frac{\partial E}{\partial t}+\frac{\partial F}{\partial z}=\chi(4\pi B-cE)+\frac{\chi}{c}vF \Rightarrow \frac{4\pi}{c}\frac{\partial J}{\partial t}+4\pi\frac{\partial H}{\partial z}=\chi_0\left(4\pi B_0-c\frac{4\pi}{c}J\right)+\frac{\chi_0 v}{c}4\pi H$$

or

$$\frac{1}{c}\frac{\partial J}{\partial t}+\frac{\partial H}{\partial z}=\chi_0(B_0-J)+\frac{\chi_0 v}{c}H$$

$$\Rightarrow \frac{1}{c}\frac{\partial}{\partial t}\left[J_0+2\beta H_0\right]+\frac{\partial}{\partial z}\left[H_0+\beta(J_0+K_0)\right]=\chi_0\left[B_0-(J_0+2\beta H_0)\right]+\frac{\chi_0 v}{c}\left[H_0+\beta(J_0+K_0)\right]$$

$$=\frac{1}{c}\frac{\partial J_0}{\partial t}+\cancel{\frac{2v}{c^2}\frac{\partial H_0}{\partial t}}+\frac{2a}{c^2}H_0+\frac{\partial H_0}{\partial z}+\frac{1}{c}\frac{\partial v}{\partial z}(J_0+K_0)+\beta\left(\frac{\partial J_0}{\partial z}+\frac{\partial K_0}{\partial z}\right)=\chi_0(B_0-J_0-2\beta H_0)$$
$$+(\chi_0 v/c)H_0$$

$$\frac{1}{c}\left(\frac{\partial J_0}{\partial t}+v\frac{\partial J_0}{\partial z}\right)+\frac{\partial H_0}{\partial z}+\cancel{\frac{2a}{c^2}}H_0+\frac{v}{c}\left(\frac{\partial J_0}{\partial z}+\frac{\partial K_0}{\partial z}\right)+\frac{1}{c}\frac{\partial v}{\partial z}(J_0+K_0)=\chi_0(B_0-J_0-\beta H_0)$$

$$\Rightarrow \frac{1}{c}\frac{DJ_0}{Dt}+\frac{\partial H_0}{\partial z}+\frac{1}{c}\frac{\partial v}{\partial z}(J_0+K_0)+\frac{v}{c}\left(\cancel{\frac{\partial J_0}{\partial z}}+\frac{\partial K_0}{\partial z}\right)+\chi_0\beta H_0=\chi_0(B_0-J_0)$$

or
$$\frac{1}{c}\frac{DJ_0}{Dt}+\frac{\partial H_0}{\partial z}+\frac{1}{c}\frac{\partial v}{\partial z}(J_0+K_0)=\chi_0(B_0-J_0)-\left[\chi_0\beta H_0+\frac{v}{c}\left(\cancel{\frac{\partial J_0}{\partial z}}+\frac{\partial K_0}{\partial z}\right)\right]$$
$$=\chi_0(B_0-J_0)-\beta\left[\chi_0 H_0+\partial K_0/\partial z\right]$$

But from (95.21)

$$-\chi_0 H_0-\frac{\partial K_0}{\partial z}\cong\frac{1}{c}\frac{\partial H_0}{\partial t}+\frac{v}{c}\left\{\frac{\partial H_0}{\partial t}\right\}+\frac{2}{c}\frac{\partial v}{\partial z}H_0=O(v/c)$$

Hence $\frac{V}{c}\,[\chi_0 H_0 + \partial K_0/\partial z] = \frac{V}{c} \times O(\frac{V}{c}) = O(V^2/c^2)$

and

$$\frac{1}{c}\frac{DJ_0}{Dt} + \frac{\partial H_0}{\partial z} + \frac{1}{c}\frac{\partial V}{\partial z}(J_0+K_0) = \chi_0(B_0-J_0) - O(V^2/c^2)$$

Conversely, starting with (95.21):

$$\frac{1}{c}\frac{\partial H_0}{\partial t} + \frac{\partial K_0}{\partial z} + \frac{V}{c}\frac{\partial H_0}{\partial z} + \frac{2}{c}\frac{\partial V}{\partial z}H_0 = -\chi_0 H_0$$

and (93.11)

$$c^{-2}\,\partial F/\partial t + \partial P/\partial z = -(\chi_0/c)F + (\chi_0 V/c)(4\pi B c) + \chi_0 V P$$

$$\Rightarrow c^{-2}\frac{\partial}{\partial t}(4\pi H) + \frac{\partial}{\partial z}\left(\frac{4\pi K}{c}\right) = -\frac{\chi_0}{c}(4\pi H) + \frac{\chi_0 V}{c}\left(\frac{4\pi B}{c}\right) + \chi_0\frac{V}{c}\left(\frac{4\pi K}{c}\right)$$

$$\Rightarrow \frac{1}{c}\frac{\partial H}{\partial t} + \frac{\partial K}{\partial z} = -\chi_0 H + \frac{\chi_0 V}{c}B + \frac{\chi_0 V K}{c} = -\chi_0 H + (\chi_0 V/c)\,B_0 + (\chi_0 V/c)\,K$$

and thus

$$\frac{1}{c}\frac{\partial}{\partial t}\left[H_0+\beta(\cancel{J_0}+K_0)\right] + \frac{\partial}{\partial z}\left[K_0+2\beta H_0\right] = -\chi_0\left[H_0+\beta(J_0+K_0)\right] + \frac{\chi_0 V}{c}(B_0+K_0+2\beta H_0)$$

$$\left(\frac{1}{c}\frac{\partial H_0}{\partial t} + \frac{V}{c}\frac{\partial H_0}{\partial z}\right) + \frac{\partial K_0}{\partial z} + \frac{V}{c}\frac{\partial H_0}{\partial z} + \frac{2}{c}\frac{\partial V}{\partial z}H_0 = -\chi_0 H_0 - \chi_0\frac{V}{c}(J_0+\cancel{K_0}) + \chi_0\frac{V}{c}(B_0+\cancel{K_0})$$

$$= -\chi_0 H_0 + \chi_0\frac{V}{c}(B_0-J_0)$$

$$\frac{1}{c}\frac{DH_0}{Dt} + \frac{\partial K_0}{\partial z} + \cancel{\frac{V}{c}\frac{\partial H_0}{\partial z}} + \frac{2}{c}\frac{\partial V}{\partial z}H_0 = -\chi_0 H_0 + \chi_0\frac{V}{c}(B_0-J_0) - \frac{V}{c}\frac{\partial H_0}{\partial z}$$

Noninertial Frame Formulation (LH Thomas + J. Castor)

Start with photon Boltzmann eqn (§ 92)

$$m^\alpha \frac{\partial \mathcal{I}}{\partial x^\alpha} + \dot{m}^\alpha \frac{\partial \mathcal{I}}{\partial m^\alpha} = e - a\mathcal{I} = \left.\frac{\delta \mathcal{I}}{\delta \lambda}\right|_{coll} \qquad (95.30)$$

and take

$$\dot{m}^\alpha = \frac{dm^\alpha}{d\lambda} \quad \text{where } \lambda \text{ satisfies } \quad p^\alpha \left(= m^\alpha\right) = \frac{dx^\alpha}{d\lambda}$$

The intrinsic derivative $\frac{\delta m^\alpha}{\delta \lambda} \equiv 0$ along photon trajectory

$$\frac{\delta m^\alpha}{\delta \lambda} = \frac{dm^\alpha}{d\lambda} + \left\{ {\alpha \atop \beta\,\gamma} \right\} m^\beta \frac{dx^\gamma}{d\lambda} = 0$$

Now change indices and write:

$$\frac{\delta m^\beta}{\delta \lambda} = \frac{dm^\beta}{d\lambda} + \left\{ {\beta \atop \alpha\,\gamma} \right\} m^\gamma \frac{dx^\alpha}{d\lambda} = \frac{\partial m^\beta}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} + \left\{ {\beta \atop \alpha\,\gamma} \right\} m^\gamma \frac{dx^\alpha}{d\lambda}$$

now $(dx^\alpha/d\lambda) = m^\alpha$ and $m^\beta = \epsilon^\beta_c m^c \Rightarrow$

$$\frac{\delta m^\beta}{\delta \lambda} = m^\alpha \left[\frac{\partial}{\partial x^\alpha}\left(\epsilon^\beta_c m^c\right) + \left\{ {\beta \atop \alpha\,\gamma} \right\} m^\gamma \right] = m^\alpha \left[\epsilon^\beta_{c,\alpha} m^c + \epsilon^\beta_c \frac{\partial m^c}{\partial x^\alpha} + \left\{ {\beta \atop \alpha\,\gamma} \right\} m^\gamma \right.$$

$$= m^\alpha m^c \left[\epsilon^\beta_{c,\alpha} + \left\{ {\beta \atop \alpha\,\gamma} \right\} \epsilon^\gamma_c \right] + \epsilon^\beta_c m^\alpha \frac{\partial m^c}{\partial x^\alpha}$$

$$= m^a m^c \epsilon^\alpha_a \left[\epsilon^\beta_{c,\alpha} + \left\{ {\beta \atop \alpha\,\gamma} \right\} \epsilon^\gamma_c \right] + \epsilon^\beta_c \frac{dm^c}{d\lambda} = 0$$

now solve for last term:

$$\epsilon^\beta_c \frac{dm^c}{d\lambda} = -\, m^a m^c \epsilon^\alpha_a \left[\epsilon^\beta_{c,\alpha} + \left\{ {\beta \atop \alpha\,\gamma} \right\} \epsilon^\gamma_c \right] \equiv -m^a m^c \epsilon^\alpha_a \epsilon^\beta_{c;\alpha}$$

multiply by ϵ^b_β to get

$$\epsilon^b_\beta \epsilon^\beta_c \frac{dm^c}{d\lambda} = \delta^b_c \frac{dm^c}{d\lambda} = \frac{dm^b}{d\lambda} = -m^a m^c \left(\epsilon^b_\beta \epsilon^\alpha_a \epsilon^\beta_{c;\alpha} \right)$$

Define the <u>Pfaffian</u> <u>derivative</u> $\partial_a \equiv \epsilon^\alpha_a (\partial/\partial x^\alpha)$

and <u>Ricci</u> <u>rotation</u> <u>coefficient</u> $\Gamma^b_{ac} = \epsilon^\alpha_a \epsilon^b_\beta \epsilon^\beta_{c;\alpha}$

The transfer eqn becomes:

~~$m^a \frac{\partial \mathcal{I}}{\partial x^a} = m^a \epsilon^\alpha_a \frac{\partial \mathcal{I}}{\partial x^\alpha} + \frac{dm^b}{d\lambda} \frac{\partial \mathcal{I}}{\partial m^b} = m^a \partial_a \mathcal{I}$~~

$$m^a \frac{D}{Dx^a}\left[\mathcal{I}(x^\alpha, m^a)\right] = \left.\frac{\delta \mathcal{I}}{\delta \lambda}\right|_{coll} = \frac{\partial \mathcal{I}}{\partial x^\alpha}\frac{dx^\alpha}{d\lambda} + \frac{\partial \mathcal{I}}{\partial m^b}\frac{dm^b}{d\lambda}$$

$$= m^a \epsilon^\alpha_a \frac{\partial \mathcal{I}}{\partial x^\alpha} + \frac{\partial \mathcal{I}}{\partial m^b}\frac{dm^b}{d\lambda} = m^a \partial_a \mathcal{I} + \frac{\partial \mathcal{I}}{\partial m^b}\left(-m^a m^c \Gamma^b_{ac}\right)$$

$$= m^a \left[\partial_a \mathcal{I} - m^c \Gamma^b_{ac} \frac{\partial \mathcal{I}}{\partial m^b} \right]$$

$$= m^a \left[\partial_a - m^c \Gamma^b_{ac} \frac{\partial}{\partial m^b} \right] \mathcal{I} = e - a\mathcal{I} \qquad (95.44)$$

$$ds^2 = e^{-2\Phi}dt^2 + e^{2\Lambda}dr^2 + R^2 d\theta^2 + R^2 \sin^2\theta\, d\phi^2$$

$$g_{00} = -\exp(+2\Phi) \quad g_{11} = \exp(2\Lambda) \quad g_{22} = R^2 \quad g_{33} = R^2\sin^2\theta$$

Non-zero elements

$$\begin{Bmatrix} i \\ i\,i \end{Bmatrix} = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^i} \qquad \begin{Bmatrix} i \\ i\,j \end{Bmatrix} = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x_j} \qquad \begin{Bmatrix} i \\ j\,j \end{Bmatrix} = -\frac{1}{2g_{ii}}\frac{\partial g_{jj}}{\partial x_i}$$

Thus

95.46).

$$\begin{Bmatrix} 0 \\ 0\,0 \end{Bmatrix} = \frac{1}{2g_{00}}\frac{\partial g_{00}}{\partial t} = \frac{1}{2g_{00}}\left(+2\frac{\partial\Phi}{\partial t}\right)g_{00} = +\frac{\partial\Phi}{\partial t}$$

$$\begin{Bmatrix} 0 \\ 0\,1 \end{Bmatrix} = \frac{1}{2g_{00}}\frac{\partial g_{00}}{\partial r} = \frac{1}{2g_{00}}\left(2\frac{\partial\Phi}{\partial r}\right)g_{00} = \frac{\partial\Phi}{\partial r}$$

$$\begin{Bmatrix} 0 \\ 1\,1 \end{Bmatrix} = -\frac{1}{2g_{00}}\frac{\partial g_{11}}{\partial t} = -\frac{1}{2[-\exp(2\Phi)]}\cdot 2\frac{\partial\Lambda}{\partial t}e^{2\Lambda} = \exp[2(\Lambda-\Phi)]\frac{\partial\Lambda}{\partial t}$$

$$\begin{Bmatrix} 1 \\ 0\,0 \end{Bmatrix} = -\frac{1}{2g_{11}}\frac{\partial g_{00}}{\partial r} = -\frac{1}{2\exp[2\Lambda]}\left(+2\frac{\partial\Phi}{\partial r}\right)\left[\exp(+2\Phi)\right] = +\exp[-2(\Lambda+\Phi)]\frac{\partial\Phi}{\partial r}$$

$$\begin{Bmatrix} 1 \\ 1\,0 \end{Bmatrix} = \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial t} = \frac{1}{2g_{11}}\left(2\frac{\partial\Lambda}{\partial t}\right)g_{11} = \frac{\partial\Lambda}{\partial t}$$

$$\begin{Bmatrix} 1 \\ 1\,1 \end{Bmatrix} = \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial r} = \frac{1}{2g_{11}}\left(2\frac{\partial\Lambda}{\partial r}\right)g_{11} = \frac{\partial\Lambda}{\partial r}$$

$$\begin{Bmatrix} 0 \\ 2\,2 \end{Bmatrix} = -\frac{1}{2g_{00}}\frac{\partial g_{22}}{\partial t} = +\frac{\exp(+2\Phi)}{2}2R\frac{\partial R}{\partial t} = +\exp(-2\Phi)R\frac{\partial R}{\partial t} = +\exp(-2\Phi)R\frac{\partial R}{\partial t}$$

$$\begin{Bmatrix} 2 \\ 0\,2 \end{Bmatrix} = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial t} = \frac{1}{2R^2}2R\frac{\partial R}{\partial t} = \frac{1}{R}\frac{\partial R}{\partial t}$$

$$\begin{Bmatrix} 1 \\ 2\,2 \end{Bmatrix} = -\frac{1}{2g_{11}}\frac{\partial g_{22}}{\partial r} = -\frac{1}{2\exp(2\Lambda)}\cdot 2R\frac{\partial R}{\partial r} = -\exp(-2\Lambda)R\frac{\partial R}{\partial r}$$

$$\begin{Bmatrix} 2 \\ 2\,1 \end{Bmatrix} = (2g_{22})^{-1}\partial g_{22}/\partial r = (2R^2)^{-1}\partial(R^2)/\partial r = R^{-1}\partial R/\partial r$$

$$\begin{Bmatrix} 0 \\ 3\,3 \end{Bmatrix} = -\frac{1}{2g_{00}}\frac{\partial g_{33}}{\partial t} = -\frac{1}{2g_{00}}\sin^2\theta\cdot 2R\frac{\partial R}{\partial t} = +\exp(-2\Phi)\sin^2\theta\, R\frac{\partial R}{\partial t}$$

$$\begin{Bmatrix} 3 \\ 0\,3 \end{Bmatrix} = \frac{1}{2g_{33}}\frac{\partial g_{33}}{\partial t} = \frac{1}{2R^2\sin^2\theta}\cdot\sin^2\theta\cdot\frac{2R\partial R}{\partial t} = \frac{1}{R}\frac{\partial R}{\partial t}$$

$$\left\{ {1 \atop 3\,3} \right\} = -\frac{1}{2g_{11}} \frac{\partial g_{33}}{\partial r} = \frac{-1}{2\exp(2\Lambda)} \frac{\partial (R^2 \sin^2\theta)}{\partial r} = -\exp(-2\Lambda) \sin^2\theta \, R \frac{\partial R}{\partial r}$$

$$\left\{ {3 \atop 1\,3} \right\} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial r} = \frac{1}{2R^2\sin^2\theta} \frac{\partial}{\partial r}(R^2 \sin^2\theta) = \frac{1}{R}\frac{\partial R}{\partial r}$$

(95.46)

↑

$$\left\{ {2 \atop 3\,3} \right\} = -\frac{1}{2g_{22}} \frac{\partial g_{33}}{\partial \theta} = -\frac{1}{2R^2} \frac{\partial}{\partial \theta}(R^2 \sin^2\theta) = -\sin\theta\cos\theta$$

$$\left\{ {3 \atop 2\,3} \right\} = \frac{1}{2g_{33}} \frac{\partial g_{33}}{\partial \theta} = \frac{1}{2R^2\sin^2\theta} \cdot R^2 \frac{\partial}{\partial \theta}(\sin^2\theta) = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

Orthonormal basis!

$$\underset{\sim}{\epsilon}_0 = e^{-\Phi} \underset{\sim}{\epsilon}_t \qquad \underset{\sim}{\epsilon}_1 = e^{-\Lambda} \underset{\sim}{\epsilon}_r \qquad \underset{\sim}{\epsilon}_2 = R^{-1} \underset{\sim}{\epsilon}_\theta \qquad \underset{\sim}{\epsilon}_3 = (R\sin\theta)^{-1} \underset{\sim}{\epsilon}_\phi$$

or write

$$\underset{\sim}{\epsilon}_a = \epsilon^{\alpha}_{a} \underset{\sim}{\epsilon}_\alpha \qquad \text{where } \underset{\sim}{\epsilon}_a = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$$

$$\underset{\sim}{\epsilon}_\alpha = (\epsilon_t, \epsilon_r, \epsilon_\theta, \epsilon_\phi)$$

Then

(95.48)
(95.49)

$$\epsilon^{\alpha}_{a} = \begin{pmatrix} e^{-\Phi} & & & 0 \\ & e^{-\Lambda} & & \\ & & R^{-1} & \\ 0 & & & (R\sin\theta)^{+} \end{pmatrix} \text{ and } \epsilon^{a}_{\alpha} = \begin{pmatrix} e^{\Phi} & & & 0 \\ & e^{\Lambda} & & \\ & & R & \\ 0 & & & R\sin\theta \end{pmatrix}$$

Photon 4-momentum

M^0

Ricci rotation coefficients

For diagonal transformation matrix $\epsilon^{\alpha}{}_{a}$, we find:

$$\Gamma^{b}_{ac} \equiv \epsilon^{\alpha}{}_{a}\,\epsilon^{b}{}_{\beta}\,\epsilon^{\beta}{}_{c;\alpha} \equiv \epsilon^{\alpha}{}_{a}\,\epsilon^{b}{}_{\beta}\left[\epsilon^{\beta}{}_{c,\alpha} + \left\{{\beta \atop \alpha\gamma}\right\}\epsilon^{\gamma}{}_{c}\right]$$

becomes

$$\Gamma^{b}_{ac} = \epsilon^{a}{}_{a}\,(\epsilon^{-1})^{b}{}_{b}\left[\epsilon^{b}{}_{c,a}\,\delta^{b}_{c} + \left\{{b \atop a c}\right\}\epsilon^{c}{}_{c}\right]$$

Now use (95.48) (95.49) and (95, 46) to evaluate the Γ^{b}_{ac}'s

$$\Gamma^{1}_{00} = \epsilon^{0}{}_{0}(\epsilon^{-1})^{1}{}_{1}\left[\epsilon^{1}{}_{0,0}\,\delta^{1}_{0} + \left\{{1 \atop 0\,0}\right\}\epsilon^{0}{}_{0}\right] = e^{-\Phi}e^{\Lambda}\left[e^{2(\Phi-\Lambda)}\frac{\partial\Phi}{\partial R}e^{-\Phi}\right] = e^{-\Lambda}\frac{\partial\Phi}{\partial R} \equiv D_r\Phi$$

$$\Gamma^{1}_{22} = \epsilon^{2}{}_{2}(\epsilon^{-1})^{1}{}_{1}\left[\epsilon^{1}{}_{2,2}\,\delta^{1}_{2} + \left\{{1 \atop 2\,2}\right\}\epsilon^{2}{}_{2}\right] = R^{-1}e^{\Lambda}\left[-e^{-2\Lambda}R\frac{\partial R}{\partial r}R^{-1}\right] = -R^{-1}e^{-\Lambda}\frac{\partial R}{\partial r} = -R^{-1}\Gamma$$

$$\Gamma^{1}_{33} = \epsilon^{3}{}_{3}(\epsilon^{-1})^{1}{}_{1}\left[\epsilon^{1}{}_{3,3}\,\delta^{1}_{3} + \left\{{1 \atop 3\,3}\right\}\epsilon^{3}{}_{3}\right] = (R\sin\theta)^{-1}e^{\Lambda}\left[-e^{-2\Lambda}R\frac{\partial R}{\partial r}\sin^2\theta\,(R\sin\theta)^{-1}\right] = -R^{-1}e^{-\Lambda}\frac{\partial R}{\partial r} = -R^{-1}D_rR$$
$$= -R^{-1}\Gamma$$

$$\Gamma^{1}_{10} = \epsilon^{1}{}_{1}(\epsilon^{-1})^{1}{}_{1}\left[0 + \left\{{1 \atop 1\,0}\right\}\epsilon^{0}{}_{0}\right] = 1\left[\frac{\partial\Lambda}{\partial t}e^{-\Phi}\right] = D_t\Lambda$$

$$\Gamma^{2}_{33} = \epsilon^{3}{}_{3}(\epsilon^{-1})^{2}{}_{2}\left[0 + \left\{{2 \atop 3\,3}\right\}\epsilon^{3}{}_{3}\right] = (R\sin\theta)^{-2}R(-\sin\theta\cos\theta) = -R^{-1}\cot\theta$$

$$\Gamma^{2}_{20} = \epsilon^{2}{}_{2}(\epsilon^{-1})^{2}{}_{2}\left[0 + \left\{{2 \atop 2\,0}\right\}\epsilon^{0}{}_{0}\right] = 1\left[R^{-1}\frac{\partial R}{\partial t}e^{-\Phi}\right] = R^{-1}e^{-\Phi}\frac{\partial R}{\partial t} = R^{-1}D_tR = R^{-1}U$$

$$\Gamma^{3}_{30} = \epsilon^{3}{}_{3}(\epsilon^{-1})^{3}{}_{3}\left[0 + \left\{{3 \atop 3\,0}\right\}\epsilon^{0}{}_{0}\right] = 1\left[R^{-1}\frac{\partial R}{\partial t}e^{-\Phi}\right] = R^{-1}U$$

$$\Gamma^{2}_{21} = \epsilon^{2}{}_{2}(\epsilon^{-1})^{2}{}_{2}\left[0 + \left\{{2 \atop 2\,1}\right\}\epsilon^{1}{}_{1}\right] = 1\left[R^{-1}\frac{\partial R}{\partial r}e^{-\Lambda}\right] = R^{-1}e^{-\Lambda}\frac{\partial R}{\partial r} = R^{-1}D_rR = R^{-1}\Gamma$$

$$\Gamma^{3}_{31} = \epsilon^{3}{}_{3}(\epsilon^{-1})^{3}{}_{3}\left[0 + \left\{{3 \atop 3\,1}\right\}\epsilon^{1}{}_{1}\right] = 1\left[R^{-1}\frac{\partial R}{\partial r}e^{-\Lambda}\right] = R^{-1}\Gamma$$

$$\Gamma^{3}_{32} = \epsilon^{3}{}_{3}(\epsilon^{-1})^{3}{}_{3}\left[0 + \left\{{3 \atop 3\,2}\right\}\epsilon^{2}{}_{2}\right] = 1\left[\cot\theta\,(R\sin^{-1}\right] = R^{-1}\cot\theta$$

$$\Gamma^{1}_{11} = \epsilon^{1}{}_{1}(\epsilon^{-1})^{1}{}_{1}\left[\epsilon^{1}{}_{1,1} + \left\{{1 \atop 1\,1}\right\}\epsilon^{1}{}_{1}\right] = 1\left[\frac{\partial}{\partial r}(e^{-\Lambda}) + \frac{\partial\Lambda}{\partial r}e^{-\Lambda}\right] = -\frac{\partial\Lambda}{\partial r}e^{-\Lambda} + \frac{\partial\Lambda}{\partial r}e^{-\Lambda} = 0$$

$$\frac{\partial}{\partial m^1} \overset{?}{=} \cos\theta \frac{\partial}{\partial \nu} - \frac{\sin\theta}{\nu} \frac{\partial}{\partial\theta} = \cos\theta \frac{\partial}{\partial \nu} - \frac{\sin\theta}{\nu} \frac{(-\sin\theta)\,\partial}{(-\sin\theta)\,\partial\theta} = \cos\theta \frac{\partial}{\partial \nu} + \frac{\sin^2\theta}{\nu} \frac{\partial}{\partial\mu}$$

$$= \mu \frac{\partial}{\partial \nu} + \frac{(1-\mu^2)}{\nu} \frac{\partial}{\partial\mu}$$

$$\frac{\partial}{\partial m^2} = \sin\theta \cos\Phi \frac{\partial}{\partial \nu} + \nu^{-1} \cos\theta \cos\Phi \frac{\partial}{\partial\theta} = \sin\theta \cos\Phi \frac{\partial}{\partial \nu} + \frac{\cos\theta \cos\Phi}{\nu} \frac{(-\sin\theta)\,\partial}{(-\sin\theta)\,\partial\theta}$$

$$= \cos\Phi \left[\sin\theta \frac{\partial}{\partial \nu} - \frac{\cos\theta}{\nu} \sin\theta \frac{\partial}{\partial\mu} \right] = (1-\mu^2)^{1/2} \cos\Phi \left(\frac{\partial}{\partial \nu} - \frac{\mu}{\nu} \frac{\partial}{\partial\mu} \right)$$

$$\frac{\partial}{\partial m^3} = \sin\theta \sin\Phi \frac{\partial}{\partial \nu} + \frac{\cos\theta \sin\Phi}{\nu} \frac{\partial}{\partial\theta} = \sin\Phi \left[\sin\theta \frac{\partial}{\partial \nu} + \frac{\cos\theta}{\nu} (-\sin\theta) \frac{\partial}{(-\sin\theta)\,\partial\theta} \right]$$

$$= (1-\mu^2)^{1/2} \sin\Phi \left[\frac{\partial}{\partial \nu} - \frac{\mu}{\nu} \frac{\partial}{\partial\mu} \right]$$

$$m^a\partial_a \equiv m^a \epsilon^\alpha_a \frac{\partial}{\partial x^\alpha} = m^0 \epsilon^0_0 \frac{\partial}{\partial t} + m^1 \epsilon^1_1 \frac{\partial}{\partial r}$$
$$= \nu e^{-\Phi}\frac{\partial}{\partial t} + \nu\cos\Theta\, e^{-\Lambda}\frac{\partial}{\partial r} = \nu e^{-\Phi}\frac{\partial}{\partial t} + \nu\mu e^{-\Lambda}\frac{\partial}{\partial r} \equiv \nu D_t + \mu\nu D_r$$

$$m^a m^c \Gamma^b_{ac}\frac{\partial}{\partial m^b} = m^a m^c \Gamma^1_{ac}\,\partial/\partial m^1 + m^a m^c \Gamma^2_{ac}\,\partial/\partial m^2 + m^a m^c \Gamma^3_{ac}\,\partial/\partial m^3$$
$$= (m^0 m^0 \Gamma^1_{00} + m^2 m^2 \Gamma^1_{22} + m^3 m^3 \Gamma^1_{33} + m^1 m^0 \Gamma^1_{10})\,\partial/\partial m^1$$
$$+ (m^3 m^3 \Gamma^2_{33} + m^2 m^0 \Gamma^2_{20} + m^2 m^1 \Gamma^2_{21})\,\partial/\partial m^2 + (m^3 m^0 \Gamma^3_{30} + m^3 m^1 \Gamma^3_{31} + m^3 m^2 \Gamma^3_{32})\,\partial/\partial m^3$$

These coefficients become:

$$m^0 m^0 \Gamma^1_{00} + m^2 m^2 \Gamma^1_{22} + m^3 m^3 \Gamma^1_{33} + m^1 m^0 \Gamma^1_{10} = \nu^2\left[D_r\Phi + \mu D_t\Lambda - (1-\mu^2)\Gamma/R\right]$$
$$m^3 m^3 \Gamma^2_{33} + m^2 m^1 \Gamma^2_{21} + m^2 m^0 \Gamma^2_{20} = (1-\mu^2)^{1/2}\frac{\nu^2}{R}\left[-\mu\sin^2\Phi + (U+\mu\Gamma)\cos\Phi\right]$$
$$m^3 m^0 \Gamma^3_{30} + m^3 m^1 \Gamma^3_{31} + m^3 m^2 \Gamma^3_{32} = (1-\mu^2)^{1/2}\frac{\nu^2}{R}\sin\Phi\left[U+\mu\Gamma+\mu\cos\Phi\right]$$

Then using

$$\frac{\partial}{\partial m^1} = \mu\frac{\partial}{\partial\nu} + \frac{(1-\mu^2)}{\nu}\frac{\partial}{\partial\mu}$$
$$\frac{\partial}{\partial m^2} = (1-\mu^2)^{1/2}\cos\Phi\left(\frac{\partial}{\partial\nu} - \frac{\mu}{\nu}\frac{\partial}{\partial\mu}\right)$$
$$\frac{\partial}{\partial m^3} = (1-\mu^2)^{1/2}\sin\Phi\left(\frac{\partial}{\partial\nu} - \frac{\mu}{\nu}\frac{\partial}{\partial\mu}\right)$$

we have

$$\nu^{-1} m^a m^c \Gamma^b_{ac}\frac{\partial}{\partial m^b} = \nu\left[D_r\Phi + \mu D_t\Lambda - (1-\mu^2)\frac{\Gamma}{R}\right]\left[\mu\frac{\partial}{\partial\nu} + \frac{(1-\mu^2)}{\nu}\frac{\partial}{\partial\mu}\right]$$
$$+ (1-\mu^2)^{1/2}\frac{\nu}{R}\left[-\mu\sin^2\Phi + (U+\mu\Gamma)\cos\Phi\right](1-\mu^2)^{1/2}\cos\Phi\left(\frac{\partial}{\partial\nu} - \frac{\mu}{\nu}\frac{\partial}{\partial\mu}\right)$$
$$+ (1-\mu^2)^{1/2}\frac{\nu}{R}\sin\Phi\left[U+\mu\Gamma+\mu\cos\Phi\right](1-\mu^2)^{1/2}\sin\Phi\left(\frac{\partial}{\partial\nu} - \frac{\mu}{\nu}\frac{\partial}{\partial\mu}\right)$$

$$= \nu\left\{\mu D_r\Phi + \mu^2 D_t\Lambda - \mu(1-\mu^2)\frac{\Gamma}{R} + \frac{(1-\mu^2)}{R}\cos\Phi\left[-\mu\sin^2\Phi + (U+\mu\Gamma)\cos\Phi\right] + \frac{(1-\mu^2)}{R}\sin^2\Phi(U+\mu\Gamma+\mu\cos\Phi)\right\}\frac{\partial}{\partial\nu}$$
$$+ (1-\mu^2)\left\{D_r\Phi + \mu D_t\Lambda - (1-\mu^2)\frac{\Gamma}{R} + \frac{\cos\Phi}{R}\left[-\mu\sin^2\Phi + (U+\mu\Gamma)\cos\Phi\right](-\mu) + \frac{\sin^2\Phi}{R}(U+\mu\Gamma+\mu\cos\Phi)(-\mu)\right\}\frac{\partial}{\partial\mu}$$

$$= \nu\left\{\mu D_r\Phi + \mu^2 D_t\Lambda - \mu(1-\mu^2)\frac{\Gamma}{R} + \frac{(1-\mu^2)}{R}\left[\cos\Phi\left(-\mu\sin^2\phi + (U+\mu\Gamma)\cos\Phi\right) + \sin^2\Phi(U+\mu\Gamma+\mu\cos\Phi)\right]\right\}\frac{\partial}{\partial\nu}$$
$$+ (1-\mu^2)\left\{D_r\Phi + \mu D_t\Lambda - (1-\mu^2)\frac{\Gamma}{R} - \frac{\mu}{R}\left[\cos\Phi\left(-\mu\sin^2\Phi + (U+\mu\Gamma)\cos\Phi\right) + \sin^2\Phi(U+\mu\Gamma+\mu\cos\Phi)\right]\right\}\frac{\partial}{\partial\mu}$$

now $\cos\Phi\,(-\mu\sin^2\phi + U\cos\Phi + \mu\Gamma\cos\Phi) + \sin^2\Phi\,(U+\mu\Gamma+\mu\cos\Phi) = -\cancel{\mu\sin^2\Phi}\cos\Phi + U\cos^2\Phi + \mu\Gamma\cos^2\Phi$
$+\cancel{\mu\sin^2\Phi\cos\Phi} + \sin^2\Phi\, U + \mu\Gamma\sin^2\Phi = (U+\mu\Gamma)$

Hence $\nu^{-1} m^a m^c \Gamma^b_{ac} \frac{\partial}{\partial m^b} = \nu\{\mu D_r \Psi + \mu^2 D_t \Lambda \cancel{-\mu(1-\mu^2)}\frac{\Gamma}{R} + \frac{(1-\mu^2)}{R}(U + \cancel{\mu\Gamma})$

$$+ (1-\mu^2)\{D_r \Psi + \mu D_t \Lambda - (1-\mu^2)\frac{\Gamma}{R} - \frac{\mu}{R}(U+\mu\Gamma)\}$$

$$= \nu\{\mu D_r \Psi + \mu^2 D_t \Lambda + (1-\mu^2)\frac{U}{R}\}\frac{\partial}{\partial \nu} + (1-\mu^2)\{D_r \Psi + \mu D_t \Lambda - \frac{\Gamma}{R} - \mu \frac{U}{R}\}\frac{\partial}{\partial \mu}$$

and we then have

$$m^a \partial_a - m^a m^c \Gamma^b_{ac} \frac{\partial}{\partial m^b} = \nu D_t + \mu\nu D_r - \nu^2\{\mu D_r \Psi + \mu^2 D_t \Lambda + (1-\mu^2)\frac{U}{R}\}\frac{\partial}{\partial \nu}$$
$$+ \nu(1-\mu^2)\{\frac{\Gamma}{R} - D_r \Psi + \mu(\frac{U}{R} - D_t \Lambda)\}\frac{\partial}{\partial \mu}$$

S(61) $\Rightarrow$

$$\frac{1}{\nu}(e - af) = D_t f + \mu D_r f - \nu\{\mu D_r \Psi + \mu^2 D_t \Lambda + (1-\mu^2)\frac{U}{R}\}\frac{\partial f}{\partial \nu} + (1-\mu^2)\{\frac{\Gamma}{R} - D_r \Psi + \mu(\frac{U}{R} - D_t \Lambda)\}\frac{\partial f}{\partial \mu}$$

Take (t', r, θ, ϕ) = inertial frame coordinates (t, M_r, θ, ϕ) = Lagrangian coordinates

$$M_r(r,t') = \int_0^r 4\pi (r')^2 \rho_0(r',t')\, dr'$$

$$t(r,t') = t' - c^2 \int_0^r v(r',t')\, dr'$$

and $v \equiv \frac{\partial r}{\partial t'} = -(4\pi r^2\rho_0)^{-1} \frac{\partial M_r}{\partial t'}\Big|_r \Rightarrow \frac{\partial M_r}{\partial t'} = -(4\pi r^2 \rho_0)\, v$

$$dM_r = \frac{\partial M_r}{\partial r} dr + \frac{\partial M_r}{\partial t'} dt' = 4\pi r^2 \rho_0(r,t)\, ar + (-4\pi r^2 \rho_0 v)\, dt'$$

(95.64) $$\frac{dM_r}{4\pi r^2 \rho_0} = dr - v\,dt' \equiv dx$$

$$dt = \frac{\partial t}{\partial t'} dt' + \frac{\partial t}{\partial r} dr = dt' - \frac{1}{c^2} v(r,t')\, dr - \frac{1}{c^2}\left[\int_0^r \frac{\partial v(r',t')}{\partial t'}\, dr'\right] dt'$$

(95.66) now define $I \equiv \int_0^r (\partial v/\partial t')\, dr'$, then

(95.65) $$dt = dt'(1 - I/c^2) - (v/c^2)\, dr$$

Solve for dr and dt':

$$dr = dx + v\,dt'$$

$$\Rightarrow dt = dt'(1 - I/c^2) - (v/c^2)(dx + v\,dt')$$

$$= dt'(1 - I/c^2) - (v^2/c^2)\,dt' - (v/c^2)\,dx$$

$$dt' = [dt + (v/c^2)dx] / [1 - I/c^2 - v^2/c^2]$$

(95.68) $$\boxed{dt' = D^{-1}[dt + (v/c^2)dx]}$$

$$dr = dx + vD^{-1}[at + (v/c^2)\,dx]$$

$$= dx(1 + D^{-1}v^2/c^2) + D^{-1} v\,dt$$

$$= dx[1 - I/c^2 \cancel{-v^2/c^2} \cancel{+v^2/c^2}]D^{-1} + D^{-1}v\,dt$$

(95.67) $$\boxed{dr = D^{-1}(1 - I/c^2)\,dx + D^{-1} v\,dt}$$

(95.68) $$D = 1 - I/c^2 - v^2/c^2$$

Substitute into the inertial frame metric

$$ds^2 = -c^2(dt')^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= -c^2[dt + v\,dx/c^2]^2/D^2 + [(1 - I/c^2)dx + v\,dt]^2/D^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$ds^2 = -c^2 D^{-2} dt^2 - (v^2/c^2) D^{-2} dx^2 - 2\,v\,dx\,dt\,D^{-2}$$
$$+ v^2 D^{-2} dt^2 + (1 - I/c^2) D^{-2} dx^2 + 2v(1 - I/c^2)\,dx\,dt\,D^{-2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

95.71) lmost
$$ds^2 = (v^2 - c^2) D^{-2} dt^2 + [(1 - I/c^2)^2 - v^2/c^2] D^{-2} dx^2 - (2vI/c^2) D^{-2} dx\,dt + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$
$$= -G\,dt^2 + F\,dx^2 - 2H\,dM_r\,dt + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

now to $O(v/c)$
$$G = (v^2 - c^2) D^{-2} = -c^2(1 - v^2/c^2)/[1 - (I + v^2)/c^2]^2 \cong -c^2/[1 - 2(I + v^2)/c^2 + (I + v^2)^2/c^4]$$

(95.73) (95.75)
$$-G = \approx -c^2/[1 - 2I/c^2] \approx -c^2(1 + 2I/c^2) \approx -c^2 - 2I$$

(95.72)
$$F = [(1 - I/c^2)^2 - v^2/c^2]/[1 - (I + v^2)/c^2]^2 = \frac{[1 - 2I/c^2 - v^2/c^2 + I^2/c^4]}{1 - 2I/c^2 - 2v^2/c^2 + \frac{I^2}{c^4} + \frac{2Iv^2}{c^4} + \frac{v^4}{c^4}}$$

$$F \cong \frac{(1 - 2I/c^2)}{(1 - 2I/c^2)} + O(v^2/c^2) = 1 + O(v^2/c^2)$$

(95.74)
$$2vI/c^2 = 2v \int \frac{\partial v}{\partial t} dr'/c^2 = O(v^2/c^2) = 2H(4\pi r^2 \rho_0)$$

and $dx \equiv dM_r / 4\pi r^2 \rho_0$

Thus we have the $O(v/c)$ metric

(v/c) 95.71)
$$\boxed{ds^2 = (-c^2 - 2I)dt^2 + \frac{dM_r^2}{(4\pi\rho_0 r^2)^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)}$$

Can compare this with Castor's metric eq (18) of Castor 1972.

Compare this with:

(95.45)
$$ds^2 = -e^{2\Phi} dt^2 + e^{2\Lambda} dx^2 + R^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$
$dr \rightarrow dM_r$

to get

(95.76)
$$e^{2\Phi} = c^2 + 2I \Rightarrow \Phi = \tfrac{1}{2}\ln(c^2 + 2I)$$
$$e^{2\Lambda} = (4\pi r^2 \rho_0)^{-2} \Rightarrow 2\Lambda = -2\ln(4\pi r^2 \rho_0)$$
$$\Lambda = -\ln(4\pi r^2 \rho_0)$$
$$R = r$$
$$d\pi = dM_r$$

Then calculate derivatives

(95.77) $$D_t \equiv e^{-\Phi}\frac{\partial}{\partial t} = (c^2+2I)^{-1/2}\frac{\partial}{\partial t} = \frac{1}{c(1+2I/c^2)^{1/2}}\frac{\partial}{\partial t}$$
$$= \frac{1}{c}\left(1-\frac{I}{c^2}\right)\frac{\partial}{\partial t} \cong \frac{1}{c}\frac{\partial}{\partial t}$$

(95.77) $$D_r = e^{-\Lambda}\frac{\partial}{\partial r} = (4\pi r^2\rho_0)\frac{\partial}{\partial m_r} = \frac{\partial}{\partial r}$$

Then

$$U = D_t R = \frac{1}{c}\frac{\partial r}{\partial t} = v/c$$
$$\Gamma = D_r R = \frac{\partial}{\partial r}(r) = 1$$

(95.78) $$D_r\Phi = D_r\left[\tfrac{1}{2}\ln(c^2+2I)\right] = \tfrac{1}{2}(c^2+2I)^{-1}\frac{\partial}{\partial r}(2I) = (c^2+2I)^{-1}\frac{\partial}{\partial r}\int_0^r \frac{\partial v}{\partial t'}\,dr'$$
$$= (c^2+2I)^{-1}\frac{\partial v}{\partial t'} = \frac{1}{c^2}\frac{\partial v}{\partial t'} - \frac{1}{c^2}(2I/c^2)\frac{\partial v}{\partial t'} \cong \frac{1}{c^2}\frac{\partial v}{\partial t'} + O(v^2/c^2)$$
$$= a/c^2$$

(95.79) $$D_t\Lambda = \frac{1}{c}\frac{\partial}{\partial t}\left[-\ln(4\pi r^2\rho_0)\right] = -\frac{1}{c}\frac{1}{(4\pi r^2\rho_0)}\frac{\partial}{\partial t}(4\pi r^2\rho_0) = -\frac{1}{cr^2\rho_0}\frac{\partial}{\partial t}(r^2\rho_0)$$

$\frac{\partial}{\partial t} \leftrightarrow \frac{D}{Dt}$??

$$= -\frac{1}{c}\frac{1}{r^2\rho_0}\left(2r\frac{\partial r}{\partial t}\rho_0 + r^2\frac{\partial\rho_0}{\partial t}\right) = -\frac{1}{c}\left[\frac{2v}{r} + \frac{\partial(\ln\rho_0)}{\partial t}\right]$$

Now return to (95.61):

$$\nu^{-1}(e-a\mathcal{I}) = D_t\mathcal{I} + \mu D_r\mathcal{I} - \nu\left[\mu D_r\Phi + \mu^2 D_t\Lambda + (1-\mu^2)\frac{U}{R}\right]\frac{\partial\mathcal{I}}{\partial\nu} + (1-\mu^2)\left\{\frac{\Gamma}{R} - D_r\Phi + \mu\left(\frac{U}{R} - D_t\Lambda\right)\right\}\frac{\partial\mathcal{I}}{\partial\mu}$$

and first substitute in (95.76)-(95.79)

$$\nu^{-1}(e-a\mathcal{I}) = c^{-1}\frac{\partial\mathcal{I}}{\partial t}\left(\frac{D\mathcal{I}}{Dt}?\right) + \mu\frac{\partial\mathcal{I}}{\partial r} - \nu\left[\mu\frac{a}{c^2} - \frac{\mu^2}{c}\left(\frac{2v}{r} + \frac{\partial}{\partial t}(\ln\rho_0)\right) + (1-\mu^2)\frac{v}{rc}\right]\frac{\partial\mathcal{I}}{\partial\nu}$$
$$+ (1-\mu^2)\left\{\frac{1}{r} - \frac{a}{c^2} + \mu\left(\frac{v}{cr} + \frac{2v}{rc} + \frac{1}{c}\frac{\partial\ln\rho_0}{\partial t}\right)\right\}\frac{\partial\mathcal{I}}{\partial\mu}$$

mpare w/
astor (1972) eqn (20)

$$\nu^{-1}(e-a\mathcal{I}) = \frac{1}{c}\frac{\partial\mathcal{I}}{\partial t} + \mu\frac{\partial\mathcal{I}}{\partial r} - \nu\left[\mu\frac{a}{c^2} - \frac{\mu^2}{c}\frac{\partial\ln\rho_0}{\partial t} + (1-3\mu^2)\frac{v}{rc}\right]\frac{\partial\mathcal{I}}{\partial\nu}$$
$$+ (1-\mu^2)\left\{\frac{\mu}{c}\left(\frac{3v}{r} + \frac{\partial\ln\rho_0}{\partial t}\right) + \frac{1}{r} - \frac{a}{c^2}\right\}\frac{\partial\mathcal{I}}{\partial\mu}$$

Now replace: $\mathcal{I} = I_0/\nu_0^3 \quad e = \eta_0/\nu_0^2 \quad a = \nu_0\chi_0$ and explicitly note that μ is μ_0, ν is ν_0

$$\nu_0^{-1}(\eta_0/\nu_0^2 - \nu_0\chi_0 I_0/\nu_0^3) = \nu_0^{-3}(\eta_0 - \chi_0 I_0) = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{I_0}{\nu_0^3}\right) + \mu_0\frac{\partial}{\partial r}\left(\frac{I_0}{\nu_0^3}\right) - \nu_0\left[\mu_0\frac{a}{c^2} - \frac{\mu_0^2}{c}\frac{\partial\ln\rho_0}{\partial t} + \frac{(1-3\mu_0^2)v}{rc}\right]\frac{\partial}{\partial\nu_0}\left(\frac{I_0}{\nu_0^3}\right)$$
$$+ (1-\mu_0^2)\left\{\frac{\mu_0}{c}\left(\frac{3v}{r} + \frac{\partial\ln\rho_0}{\partial t}\right) + \frac{1}{r} - \frac{a}{c^2}\right\}\frac{\partial}{\partial\mu_0}\left(\frac{I_0}{\nu_0^3}\right)$$

$$\nu_0^{-3}(\eta_0 - \chi_0 I_0) = \nu_0^{-3}\left(\frac{1}{c}\frac{\partial I_0}{\partial t} + \mu_0 \frac{\partial I_0}{\partial r}\right) - \nu_0^{-2}\left[\mu_0 \frac{a}{c^2} - \frac{\mu_0^2}{c}\frac{\partial \ln\rho_0}{\partial t} + \frac{(1-3\mu_0^2)v}{rc}\right]\frac{\partial I_0}{\partial \nu_0} + \frac{3\nu_0}{\nu_0^4}\left[\frac{\mu_0 a}{c^2} - \frac{\mu_0^2}{c}\frac{\partial \ln\rho_0}{\partial t} + \frac{(1-3\mu_0^2)v}{rc}\right] I_0$$

$$+ \nu_0^{-3}(1-\mu_0^2)\left\{\frac{\mu_0}{c}\left(\frac{3v}{r} + \frac{\partial \ln\rho_0}{\partial t}\right) + \frac{1}{r} - \frac{a}{c^2}\right\}\frac{\partial I_0}{\partial \mu_0}$$

and thus

$$\eta_0 - \chi_0 I_0 = \frac{1}{c}\frac{\partial I_0}{\partial t} + \mu_0 \frac{\partial I_0}{\partial r} + (1-\mu_0^2)\left\{\frac{\mu_0}{c}\left(\frac{3v}{r} + \frac{\partial \ln\rho_0}{\partial t}\right) + \frac{1}{r} - \frac{a}{c^2}\right\}\frac{\partial I_0}{\partial \mu_0} - \nu_0\left\{\frac{(1-3\mu_0^2)v}{rc} - \frac{\mu_0^2}{c}\frac{\partial \ln\rho_0}{\partial t} + \mu_0\frac{a}{c^2}\right\}\frac{\partial I_0}{\partial \nu_0}$$

$$+ 3\left\{\frac{(1-3\mu_0^2)v}{rc} - \frac{\mu_0^2}{c}\frac{\partial \ln\rho_0}{\partial t} + \frac{\mu_0 a}{c^2}\right\} I_0$$

note also D_r can be written $4\pi r^2\rho_0 \frac{\partial}{\partial M_r}$, yielding $\mu D_r I = 4\pi r^2 \rho_0 \mu_0 \frac{\partial I}{\partial M_r} = \frac{4\pi r^2 \rho_0 \mu_0}{\nu_0^3}\frac{\partial I_0}{\partial M_r}$

hence

(95.80)

'ompare Castor (21)

$$\eta_0 - \chi_0 I_0 = \frac{1}{c}\frac{\partial I_0}{\partial t} + 4\pi r^2 \rho_0 \mu_0 \frac{\partial I_0}{\partial M_r} + (1-\mu_0^2)\left\{\frac{\mu_0}{c}\left(\frac{3v}{r} + \frac{\partial \ln\rho_0}{\partial t}\right) + \frac{1}{r} - \frac{a}{c^2}\right\}\frac{\partial I_0}{\partial \mu_0} - \nu_0\left\{\frac{(1-3\mu_0^2)v}{cr} - \frac{\mu_0^2}{c}\frac{\partial \ln\rho_0}{\partial t} + \mu_0\frac{a}{c^2}\right\}\frac{\partial I_0}{\partial \nu_0}$$

$$+3\left\{\frac{(1-3\mu_0^2)v}{rc} - \frac{\mu_0^2}{c}\frac{\partial \ln\rho_0}{\partial t} + \frac{\mu_0 a}{c^2}\right\} I_0$$

Where the derivatives $\frac{\partial}{\partial t}$ are Lagrangean derivatives, hence can be written $\frac{D}{Dt}$ as in DMM eqn (95.80)

The continuity eqn gives $\frac{D\ln\rho_0}{Dt} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v) = -\frac{2v}{r} - \frac{\partial v}{\partial r}$ and thus

$$-\frac{\mu_0^2}{c}\frac{D\ln\rho_0}{Dt} = +2\frac{\mu_0^2}{c}\frac{v}{r} + \frac{\mu_0^2}{c}\frac{\partial v}{\partial r}$$

e retrieve

95.17)

$$\eta_0 - \chi_0 I_0 = \frac{1}{c}\frac{DI_0}{Dt} + \mu_0\frac{\partial I_0}{\partial r} + (1-\mu_0^2)\left\{\frac{\mu_0}{c}\left(\frac{v}{r} - \frac{\partial v}{\partial r}\right) + \frac{1}{r} - \frac{a}{c^2}\right\}\frac{\partial I_0}{\partial \mu_0} - \nu_0\left\{\frac{(1-\mu_0^2)v}{rc} + \frac{\mu_0^2}{c}\frac{\partial v}{\partial r} + \mu_0\frac{a}{c^2}\right\}\frac{\partial I_0}{\partial \nu_0}$$

$$+ 3\left\{\frac{(1-\mu_0^2)v}{rc} + \frac{\mu_0^2}{c}\frac{\partial v}{\partial r} + \frac{\mu_0 a}{c^2}\right\} I_0$$

Integrate this equation over $d\mu_0$ and over $\mu_0 d\mu_0$ and write $(J_0, H_0, K_0) \to (E_0, F_0, P_0)$
$N_0 \to Q_0$

$$\int(\eta_0 - \chi_0 I_0)\,d\mu_0 = \frac{1}{c}\int\frac{DI_0}{Dt}\,d\mu_0 + 4\pi r^2\rho_0\int \mu_0\frac{\partial I_0}{\partial M_r}\,d\mu_0 + \int(1-\mu_0^2)\left\{\frac{\mu_0}{c}\left(\frac{3v}{r} + \frac{D\ln\rho_0}{Dt}\right) + \frac{1}{r} - \frac{a}{c^2}\right\}\frac{\partial I_0}{\partial \mu_0}\,d\mu_0$$

$$-\nu_0\int\left\{\frac{v}{cr}(1-3\mu_0^2) - \frac{\mu_0^2}{c}\frac{D\ln\rho_0}{Dt} + \mu_0\frac{a}{c^2}\right\}\frac{\partial I_0}{\partial \nu_0}\,d\mu_0 + 3\int\left\{\frac{(1-3\mu_0^2)v}{cr} - \frac{\mu_0^2}{c}\frac{D\ln\rho_0}{Dt} + \frac{\mu_0 a}{c^2}\right\} I_0\,d\mu_0$$

$$\eta_0 - \chi_0 J_0(\nu_0) = \frac{1}{c}\frac{DJ_0(\nu_0)}{Dt} + 4\pi r^2\rho_0\frac{\partial H_0}{\partial M_r} - \int\left\{\frac{\partial}{\partial \mu_0}\left\{(1-\mu_0^2)\left[\frac{\mu_0}{c}\left(\frac{3v}{r} + \frac{D\ln\rho_0}{Dt}\right) + \frac{1}{r} - \frac{a}{c^2}\right]\right\}\right\} I_0\,d\mu_0$$

$$-\nu_0\int\left\{\frac{v}{cr}\frac{\partial I_0}{\partial \nu_0} - \frac{3v}{cr}\mu_0^2\frac{\partial I_0}{\partial \nu_0} - \frac{D\ln\rho_0}{Dt}\frac{\mu_0^2}{c}\frac{\partial I_0}{\partial \nu_0} + \frac{a}{c^2}\mu_0\frac{\partial I_0}{\partial \nu_0}\right\}d\mu_0 + \int\left\{\frac{3v}{cr}I_0 - 9\frac{\mu_0^2 v}{cr}I_0 - 3\frac{\mu_0^2}{c}\frac{D\ln\rho_0}{Dt}I_0 + 3\frac{\mu_0 a}{c^2}I_0\right\}d\mu_0$$

$$\eta_0 - \chi_0 J_0(\nu_0) = c^{-1}\frac{DJ_0(\nu_0)}{Dt} + 4\pi\rho_0\frac{\partial}{\partial r}(r^2H_0) - 4\pi\rho_0 H_0\cdot 2r - \int\left\{(1-\mu_0^2)\frac{1}{c}\left(\frac{3v}{r}+\frac{D\ln\rho_0}{Dt}\right)I_0 + (-2\mu_0)\left[\frac{\mu_0}{c}\left(\frac{3v}{r}+\frac{D\ln\rho_0}{Dt}\right)+\frac{1}{r}-\frac{a}{c^2}\right]I_0\right\}d\mu_0$$

$$-\nu_0\left\{\frac{v}{cr}\frac{\partial J_0(\nu_0)}{\partial\nu_0} - \frac{3v}{cr}\frac{\partial K_0}{\partial\nu_0} - \frac{1}{c}\frac{D\ln\rho_0}{Dt}\frac{\partial K_0}{\partial\nu_0} + \frac{a}{c^2}\frac{\partial H_0}{\partial\nu_0}\right\} + \left\{\frac{3v}{cr}J_0 - 9\frac{v}{cr}K_0 - \frac{3}{c}\frac{D\ln\rho_0}{Dt}K_0 + 3\frac{a}{c^2}H_0\right\}$$

$$= c^{-1}\frac{DJ_0}{Dt} + 4\pi\rho_0\frac{\partial}{\partial r}(r^2H_0) - 4\pi\cdot 2r\rho_0H_0 - \left\{\frac{1}{c}\left(\frac{3v}{r}+\frac{D\ln\rho_0}{Dt}\right)J_0 - \frac{1}{c}\left(\frac{3v}{r}+\frac{D\ln\rho_0}{Dt}\right)K_0 - \frac{2}{c}\left(\frac{3v}{r}+\frac{D\ln\rho_0}{Dt}\right)K_0 - \frac{2}{r}H_0 + \frac{2a}{c^2}H_0\right\}$$

$$-\nu_0\left\{\frac{v}{cr}\frac{\partial J_0}{\partial\nu_0} - \frac{3v}{cr}\frac{\partial K_0}{\partial\nu_0} - \frac{1}{c}\frac{D\ln\rho_0}{Dt}\frac{\partial K_0}{\partial\nu_0} + \frac{a}{c^2}\frac{\partial H_0}{\partial\nu_0}\right\} + \left\{\frac{3v}{cr}J_0 - \frac{9v}{cr}K_0 - \frac{3}{c}\frac{D\ln\rho_0}{Dt}K_0 + \frac{3a}{c^2}H_0\right\}$$

Non derivative terms are:

$$-4\pi(2r\rho_0H_0) - \frac{1}{c}\left(\frac{3\cancel{v}}{\cancel{r}}+\frac{D\ln\rho_0}{Dt}\right)J_0 + \frac{1}{c}\left(\frac{3v}{r}+\frac{D\ln\rho_0}{Dt}\right)K_0 + \frac{6v}{cr}K_0 + \frac{2}{c}\frac{D\ln\rho_0}{Dt}K_0 + \frac{2H_0}{r} - \frac{2a}{c^2}H_0$$
$$+\frac{3\cancel{v}}{\cancel{cr}}J_0 \qquad\qquad -\frac{9v}{cr}K_0 - \frac{3}{c}\frac{D\ln\rho_0}{Dt}K_0 + \frac{3a}{c^2}H_0$$

$$= -4\pi(2r\rho_0H_0) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}J_0 + \cancel{\frac{3v}{cr}K_0} + \cancel{\frac{6v}{cr}}K_0 - \cancel{\frac{9v}{cr}}K_0 + \frac{1}{c}\cancel{\frac{D\ln\rho_0}{Dt}}K_0 + \frac{2}{c}\cancel{\frac{\partial\ln\rho_0}{Dt}}K_0 - \frac{3}{c}\cancel{\frac{D\ln\rho_0}{Dt}}K_0$$
$$+\frac{2H_0}{r} - \frac{2a}{c^2}H_0 + \frac{3a}{c^2}H_0$$

$$= -\frac{1}{c}\frac{D\ln\rho_0}{Dt}J_0 - 4\pi(2r\rho_0H_0) + \frac{2H_0}{r} + \frac{a}{c^2}H_0$$

$$\eta_0 - \chi_0 J_0 = \frac{1}{c}\frac{DJ_0}{Dt} + 4\pi\rho_0\frac{\partial}{\partial r}(r^2H_0) - \nu_0\left\{\frac{v}{cr}\frac{\partial J_0}{\partial\nu_0} - \left(\frac{3v}{cr}+\frac{1}{c}\frac{D\ln\rho_0}{Dt}\right)\frac{\partial K_0}{\partial\nu_0} + \frac{a}{c^2}\frac{\partial H_0}{\partial\nu_0}\right\} - \frac{1}{c}\frac{D\ln\rho_0}{Dt}J_0 - 4\pi(2r\rho_0H_0)$$
$$+\frac{2H_0}{r} + \frac{a}{c^2}H_0$$

$$= \frac{1}{c}\frac{\partial J_0}{Dt} + 4\pi\rho_0\frac{\partial}{\partial r}(r^2H_0) - \frac{\partial}{\partial\nu_0}\left\{\nu_0\left(\frac{v}{cr}\right)(J_0-3K_0) - \frac{\nu_0}{c}\frac{D\ln\rho_0}{Dt}K_0 + \frac{a\nu_0}{c^2}H_0\right\}$$
$$-\frac{1}{c}\frac{D\ln\rho_0}{Dt}J_0 - 4\pi\frac{(2r\rho_0H_0)}{4\pi r^2\rho_0} + \frac{2H_0}{r} + \frac{a}{c^2}H_0$$
$$+\left\{\frac{v}{cr}\left(\cancel{\frac{\partial J_0}{\partial\nu_0}} - 3K_0\right) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}K_0 + \frac{a}{c^2}H_0\right\}$$

$$= \frac{1}{c}\frac{DJ_0}{Dt} + 4\pi\rho_0\frac{\partial}{\partial m_r}(r^2H_0) - \frac{\partial}{\partial\nu_0}\left\{\nu_0\left[\frac{v}{cr}(J_0-3K_0) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}K_0 + \frac{a}{c^2}H_0\right]\right\}$$
$$+\frac{v}{cr}(J_0-3K_0) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}(J_0+K_0) - \cancel{4\pi\frac{(2r\rho_0H_0)}{4\pi r^2\rho_0}} + \cancel{\frac{2H_0}{r}} + \frac{2a}{c^2}H_0$$

Replace $E_0 = \frac{4\pi}{c}J_0$ $F_0 = 4\pi H_0$ and $P_0 = \frac{4\pi}{c}K_0$ $\Rightarrow$ $J_0 = \frac{c}{4\pi}E_0$ $H_0 = \frac{1}{4\pi}F_0$ $K_0 = \frac{c}{4\pi}P_0$

$$\eta_0 - \chi_0 J_0 = \frac{1}{4\pi}\frac{DE_0}{Dt} + \rho_0\frac{\partial}{\partial m_r}(r^2F_0) - \frac{\partial}{\partial\nu_0}\left\{\frac{c\nu_0}{4\pi}\left[\frac{v}{cr}(E_0-3P_0) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}P_0\right] + \frac{a}{c^2}\frac{\nu_0}{4\pi}F_0\right\}$$
$$+\frac{v}{cr}\left(\frac{c}{4\pi}\right)(E_0-3P_0) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}\left(\frac{c}{4\pi}\right)(E_0+P_0) + \frac{2a}{c^2}\frac{F_0}{4\pi}$$

$$4\pi\eta_0 - \chi_0E_0c = \frac{DE_0}{Dt} + 4\pi\rho_0\frac{\partial}{\partial m_r}(r^2F_0) - \frac{\partial}{\partial\nu_0}\left\{\nu_0\left[\frac{v}{r}(E_0-3P_0) - \frac{D\ln\rho_0}{Dt}P_0 + \frac{a}{c^2}F_0\right]\right\}$$
$$+\frac{v}{r}(E_0-3P_0) - \frac{D\ln\rho_0}{Dt}(E_0+P_0) + \frac{2a}{c^2}F_0$$

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So final equation is:

$$4\pi\eta_0 - c\chi_0 E_0 = \frac{DE_0}{Dt} + 4\pi\rho_0\frac{\partial}{\partial M_r}(r^2F_0) - \frac{\partial}{\partial\nu_0}\left\{\nu_0\left[\frac{v}{r}(E_0-3P_0) - \frac{D\ln\rho_0}{Dt}P_0 + \frac{a}{c^2}F_0\right]\right\}$$

(95.82)

$$+ \frac{v}{r}(E_0-3P_0) - \frac{D\ln\rho_0}{Dt}(E_0+P_0) + \frac{2a}{c^2}F_0$$

Now integrate (95.80) over $\mu_0 d\mu_0$

$$\int(\eta_0-\chi_0I_0)\mu_0d\mu_0 = \frac{1}{c}\int\mu_0d\mu_0\frac{DI_0}{Dt} + 4\pi r^2\rho_0\int\mu_0^2d\mu_0\frac{\partial I_0}{\partial M_r} + \int\mu_0d\mu_0(1-\mu_0^2)\left\{\frac{\mu_0}{c}\left(\frac{3v}{r}+\frac{D\ln\rho_0}{Dt}\right)+\frac{1}{r}-\frac{a}{c^2}\right\}\frac{\partial I_0}{\partial\mu_0}$$

$$-\nu_0\int\left\{\frac{(1-3\mu_0^2)v}{cr} - \frac{\mu_0^2}{c}\frac{D\ln\rho_0}{Dt} + \mu_0\frac{a}{c^2}\right\}\frac{\partial I_0}{\partial\nu_0}\mu_0d\mu_0 + 3\int\mu_0d\mu_0\left\{\frac{(1-3\mu_0^2)v}{cr} - \frac{\mu_0^2}{c}\frac{D\ln\rho_0}{Dt} + \frac{\mu_0a}{c^2}\right\}I_0$$

$$\Rightarrow\ -\chi_0H_0 = \frac{1}{c}\frac{DH_0}{Dt} + 4\pi r^2\rho_0\frac{\partial K_0}{\partial M_r} + \frac{1}{c}\left(\frac{3v}{r}+\frac{D\ln\rho_0}{Dt}\right)\left[\int(\mu_0^2-\mu_0^4)d\mu_0\frac{\partial I_0}{\partial\mu_0}\right] + \left(\frac{1}{r}-\frac{a}{c^2}\right)\int(\mu_0-\mu_0^3)\frac{\partial I_0}{\partial\mu_0}d\mu_0$$

$$-\nu_0\left\{\frac{v}{cr}\left(\frac{\partial H_0}{\partial\nu_0} - 3\frac{\partial N_0}{\partial\nu_0}\right) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}\frac{\partial N_0}{\partial\nu_0} + \frac{a}{c^2}\frac{\partial K_0}{\partial\nu_0}\right\} + \frac{3v}{cr}(H_0-3N_0) - \frac{3}{c}\frac{D\ln\rho_0}{Dt}N_0 + \frac{3a}{c^2}K_0$$

$$-\chi_0H_0 = \frac{1}{c}\frac{DH_0}{Dt} + 4\pi r^2\rho_0\frac{\partial K_0}{\partial M_r} + \frac{1}{c}\left(\frac{3v}{r}+\frac{D\ln\rho_0}{Dt}\right)(-)\int I_0(2\mu_0-4\mu_0^3)d\mu_0 + \left(\frac{1}{r}-\frac{a}{c^2}\right)(-)\int I_0(1-3\mu_0^2)d\mu_0$$

$$-\nu_0\frac{\partial}{\partial\nu_0}\left\{\frac{v}{cr}(H_0-3N_0) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}N_0 + \frac{a}{c^2}K_0\right\} + \frac{3v}{cr}(H_0-3N_0) - \frac{3}{c}\frac{D\ln\rho_0}{Dt}N_0 + \frac{3a}{c^2}K_0$$

and thus $-\chi_0H_0 = \frac{1}{c}\frac{DH_0}{Dt} + 4\pi\rho_0r^2\frac{\partial K_0}{\partial M_r} + \frac{1}{c}\left(\frac{3v}{r}+\frac{D\ln\rho_0}{Dt}\right)(4N_0-2H_0) + \left(\frac{1}{r}-\frac{a}{c^2}\right)(3K_0-J_0)$

$$-\frac{\partial}{\partial\nu_0}\left\{\nu_0\left[\frac{v}{cr}(H_0-3N_0) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}N_0 + \frac{a}{c^2}K_0\right]\right\} + \frac{v}{cr}(H_0-3N_0) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}N_0 + \frac{aK_0}{c^2} + \frac{3v}{cr}(H_0-3N_0) - \frac{3}{c}\frac{D\ln\rho_0}{Dt}N_0 + \frac{3aK_0}{c^2}$$

$$= \frac{1}{c}\frac{DH_0}{Dt} + 4\pi\rho_0\frac{\partial(K_0r^2)}{\partial M_r} - 4\pi\rho_0\frac{K_0\cdot 2r}{4\pi\rho_0r^2} + \left(\frac{a}{c^2}-\frac{1}{r}\right)J_0 + \left(\frac{-1}{c}\right)\left(\frac{3v}{r}+\frac{D\ln\rho_0}{Dt}\right)(2H_0) + \left(\frac{1}{r}-\frac{a}{c^2}\right)3K_0 + \frac{1}{c}\left(\frac{3v}{r}+\frac{D\ln\rho_0}{Dt}\right)\cdot 4N_0$$

$$-\frac{\partial}{\partial\nu_0}\left\{\nu_0\left[\frac{v}{cr}(H_0-3N_0) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}N_0 + \frac{a}{c^2}K_0\right]\right\} \qquad + \frac{4v}{cr}H_0 \quad + \cancel{\frac{3a}{c^2}}K_0 + \left(\cancel{-\frac{3v}{cr}N_0}\right) - \cancel{\frac{9v}{cr}N_0} - \frac{4}{c}\frac{D\ln\rho_0}{Dt}N_0 + \frac{\text{[illegible]}}{c^2}$$

$$-\chi_0H_0 = \frac{1}{c}\frac{DH_0}{Dt} + 4\pi\rho_0\frac{\partial(r^2K_0)}{\partial M_r} + \left(\frac{a}{c^2}-\frac{1}{r}\right)J_0 - \left(\frac{2v}{cr}+\frac{2}{c}\frac{D\ln\rho_0}{Dt}\right)H_0 + \frac{K_0}{r}\overset{\cancel{+\frac{4a}{c^2}K_0}}{\wedge} + \frac{aK_0}{c^2} - \frac{\partial}{\partial\nu_0}\left\{\nu_0\left[\frac{v}{cr}(H_0-3N_0) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}N_0 + \frac{aK_0}{c^2}\right]\right\}$$

Substitute $E_0 = \frac{4\pi}{c}J_0$ $\quad F_0 = 4\pi H_0$ $\quad P_0 = \frac{4\pi}{c}K_0$ $\quad Q_0 = \frac{4\pi}{c}N_0$

$$-\chi_0\frac{F_0}{4\pi} = \frac{1}{4\pi c}\frac{DF_0}{Dt} + \frac{4\pi\rho_0c}{4\pi}\frac{\partial}{\partial M_r}(r^2P_0) - \left(\frac{2v}{cr}+\frac{2}{c}\frac{D\ln\rho_0}{Dt}\right)\frac{F_0}{4\pi} + \frac{c}{4\pi}P_0 - \frac{a}{c^2}\frac{c}{4\pi}P_0 + \frac{c}{4\pi}\left(\frac{a}{c^2}-\frac{1}{r}\right)E_0$$

$$-\frac{\partial}{\partial\nu_0}\left\{\nu_0\left[\frac{v}{cr}(F_0-3Q_0)\frac{1}{4\pi} - \frac{1}{c}\frac{D\ln\rho_0}{Dt}\frac{N_0}{4\pi} + \frac{1}{4\pi}\frac{aP_0}{c^2}\right]\right\}$$

or

$$-\chi_0F_0 = \frac{1}{c}\frac{DF_0}{Dt} + 4\pi\rho_0c\frac{\partial}{\partial M_r}(r^2P_0) - \left(\frac{2v}{cr}+\frac{2}{c}\frac{D\ln\rho_0}{Dt}\right)F_0 + \frac{cP_0}{r} + \frac{acP_0}{c^2} - \frac{\partial}{\partial\nu_0}\left\{\nu_0\left[\frac{v}{cr}(F_0-3Q_0) - \frac{1}{c}\frac{D\ln\rho_0}{Dt}Q_0 + \frac{a}{c^2}P_0\right]\right\}$$

$$+ c\left(\frac{a}{c^2}-\frac{1}{r}\right)E_0$$

$$-\chi_0 F_0 = \frac{1}{c}\frac{DF_0}{Dt} + 4\pi\rho_0 c\frac{\partial}{\partial M_r}(r^2 P_0) - \frac{2}{c}\left(\frac{v}{r} + \frac{D\ln\rho_0}{Dt}\right)F_0 + \frac{cP_0}{r} + \frac{a}{c^2}(P_0 + E_0) - \frac{cE_0}{r} - \frac{\partial}{\partial\nu_0}\left\{\nu_0\left[\frac{v}{cr}(F_0 - 3Q_0) - \frac{D\ln\rho_0}{c\,Dt}Q_0 + \frac{a}{c^2}P_0\right]\right\}$$

~~Multiply~~ Divide by c

$$-c\chi_0 F_0 = \frac{1}{c^2}\frac{DF_0}{Dt} + 4\pi\rho_0\frac{\partial}{\partial M_r}(r^2 P_0) + \frac{1}{r}(P_0 - E_0) + \frac{a}{c^2}(P_0 + E_0) - \frac{2}{c^2}\left(\frac{v}{r} + \frac{D\ln\rho_0}{Dt}\right)F_0 - \frac{1}{c^2}\frac{\partial}{\partial\nu_0}\left\{\nu_0\left[\frac{v}{\cancel{c}r}(F_0 - 3Q_0) - \frac{D\ln\rho_0}{Dt}Q_0 + \frac{a}{\cancel{c^2}}P_0\right]\right\}$$

$$\text{(95.83)}\qquad -\frac{\chi_0 F_0}{c} = \frac{1}{c^2}\frac{DF_0}{Dt} + 4\pi\rho_0\frac{\partial}{\partial M_r}(r^2 P_0) + \frac{1}{r}(P_0 - E_0) + \frac{a}{c^2}(P_0 + E_0) - \frac{2}{c^2}\left(\frac{v}{r} + \frac{D\ln\rho_0}{Dt}\right)F_0 + \frac{1}{c^2}\frac{\partial}{\partial\nu_0}\left\{\nu_0\left[\frac{v}{r}(3Q_0 - F_0) + \frac{D\ln\rho_0}{Dt}Q_0 - aP_0\right]\right\}$$

Integrate (95.82) over frequency:

$$\int(4\pi\eta_0 - c\chi_0 E_0)\,d\nu_0 = \int d\nu_0\frac{DE_0}{Dt} + 4\pi\rho_0\frac{\partial}{\partial M_r}\left(r^2\int F_0\,d\nu_0\right) - \int\frac{\partial}{\partial\nu_0}\left\{\qquad\right\}d\nu_0 + \frac{v}{r}\int(E_0 - 3P_0)\,d\nu_0$$
$$- \frac{D\ln\rho_0}{Dt}\int(E_0 + P_0)\,d\nu_0 + \frac{2a}{c^2}\int F_0\,d\nu_0$$

$$\text{(95.84)}\qquad \int(4\pi\eta_0 - c\chi_0 E_0)\,d\nu_0 = \frac{DE_0}{Dt} + 4\pi\rho_0\frac{\partial}{\partial M_r}(r^2 F_0) + \frac{v}{r}(E_0 - 3P_0) - \frac{D\ln\rho_0}{Dt}(E_0 + P_0) + \frac{2a}{c^2}F_0$$

Integrate (95.83) over frequency:

$$\text{(95.85)}\qquad -\frac{1}{c}\int\chi_0 F_0\,d\nu_0 = \frac{1}{c^2}\frac{DF_0}{Dt} + 4\pi\rho_0\frac{\partial}{\partial M_r}(r^2 P_0) + \frac{1}{r}(P_0 - E_0) + \frac{a}{c^2}(P_0 + E_0) - \frac{2}{c^2}\left(\frac{v}{r} + \frac{D\ln\rho_0}{Dt}\right)F_0$$

Relativistic momentum eqn in comoving frame

$$\rho_{000}\frac{D\underset{\sim}{v}}{Dt} = \underset{\sim}{f} - \underset{\sim}{\nabla} p + \underset{\sim}{G}_0$$

take nonrelativistic limit in spherically symmetric case:

$$\rho_{000} \to \rho_0$$

$$\rho_0 \frac{D\underset{\sim}{v}}{Dt} = -\frac{GM_r\rho_0}{r^2} - \frac{\partial p}{\partial r} + \frac{1}{c}\int \chi_0(\nu_0)F_0(\nu_0)\,d\nu_0$$

now from (95.85),

$$-\frac{1}{c}\int \chi_0(\nu_0)F_0(\nu_0)\,d\nu_0 = \frac{1}{c^2}\frac{DF_0}{Dt} + 4\pi\rho_0 r^2\frac{\partial P_0}{\partial M_r} + \frac{3P_0-E_0}{r} - \frac{2}{c^2}\left(\frac{v}{r}+\frac{D\ln\rho_0}{Dt}\right)F_0 + \frac{a}{c^2}(E_0+P_0)$$

Drop the $\frac{a}{c^2}$ terms and write $\partial M_r = 4\pi\rho_0 r^2 dr$ to obtain

$$-\frac{1}{c}\int \chi_0(\nu_0)F_0(\nu_0)\,d\nu_0 = \frac{1}{c^2}\frac{DF_0}{Dt} + \frac{\partial P_0}{\partial r} + \frac{3P_0-E_0}{r} - \frac{2}{c^2}\left(\frac{v}{r}+\frac{D\ln\rho_0}{Dt}\right)F_0$$

Then

$$\rho_0\frac{D\underset{\sim}{v}}{Dt} = -\frac{GM_r\rho_0}{r^2} - \frac{\partial p}{\partial r} - \frac{1}{c^2}\frac{DF_0}{Dt} - \frac{\partial P_0}{\partial r} - \frac{3P_0-E_0}{r} + \left(\frac{2v}{c^2r} + \frac{2}{c^2}\overset{1/2}{\frac{D\ln\rho_0}{Dt}}\right)F_0$$

Now $\frac{D\ln\rho_0}{Dt} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2v) = -\frac{2v}{r} - \frac{\partial v}{\partial r}$ $\Rightarrow \frac{1}{c^2}\left(\frac{\cancel{2}v}{r} + \cancel{2}\frac{D\ln\rho_0}{Dt}\right) = \frac{1}{c^2}\left(\frac{2v}{r} - \frac{4v}{r} - \frac{\partial v}{\partial r}\right) = \frac{-1}{c^2}\left(\frac{2v}{r} - \frac{\partial v}{\partial r}\right)$

Thus

$$\rho_0\frac{D\underset{\sim}{v}}{Dt} + \frac{1}{c^2}\frac{DF_0}{Dt} - \frac{1}{c^2}\frac{D\ln\rho_0}{Dt}F_0 = -\frac{GM_r\rho_0}{r^2} - \frac{\partial}{\partial r}(p+P_0) - \frac{(3P_0-E_0)}{r} + \left(\frac{2v}{c^2r} - \frac{2v}{c^2r} - \frac{1}{c^2}\frac{\partial v}{\partial r}\right)F_0$$

(96.4) $$\rho_0\frac{D\underset{\sim}{v}}{Dt} + \frac{\rho_0}{c^2}\left[\frac{D}{Dt}\left(\frac{F_0}{\rho_0}\right)\right] = -\frac{GM_r\rho_0}{r^2} - \frac{\partial}{\partial r}(p+P_0) - \frac{(3P_0-E_0)}{r} - \frac{1}{c^2}\frac{\partial v}{\partial r}F_0$$

On fluid-flow timescale $\frac{\rho_0}{c^2}\frac{D}{Dt}\left(\frac{F_0}{\rho_0}\right)$ and $\frac{1}{c^2}\frac{\partial v}{\partial r}F_0$ are $O\left(\frac{v}{c}\right)$ wrt $\frac{\partial P_0}{\partial r}$ in both streaming limit and diffusion limit. We can then write:

$$\rho_0\frac{D\underset{\sim}{v}}{Dt} = \underset{\sim}{f} - \underset{\sim}{\nabla} p$$

Gas Energy Equation:

$$\rho_0\left[\frac{De}{Dt}+p\frac{D\rho_0^{-1}}{Dt}\right]=c\left(F_0^0+G_0^0\right)$$

cF_0^0 = rate of nonmechanical energy input $=\rho_0\epsilon$

$$cG_0^0=\int_0^\infty\{c\chi_0(\nu_0)E_0(\nu_0)-4\pi\eta_0(\nu_0)\}d\nu_0$$

thus $\rho_0\left[\frac{De}{Dt}+p\frac{D}{Dt}\left(\frac{1}{\rho_0}\right)\right]=\rho_0\epsilon+\int[c\chi_0(\nu_0)E_0(\nu_0)-4\pi\eta_0(\nu_0)]d\nu_0$

but from (95.84)

$$\int[4\pi\eta_0-c\chi_0E_0]d\nu_0=\frac{DE_0}{Dt}-E_0\frac{D\ln\rho_0}{Dt}+4\pi\rho_0\frac{\partial}{\partial m_r}(r^2F_0)-P_0\frac{D\ln\rho_0}{Dt}-\frac{v}{r}(3P_0-E_0)+\frac{2aF_0}{c^2}$$

$$=\rho_0\frac{D}{Dt}\left(\frac{E_0}{\rho_0}\right)+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_0)+P_0\rho_0\frac{D}{Dt}\left(\frac{1}{\rho_0}\right)-\frac{v}{r}(3P_0-E_0)+\frac{2aF_0}{c^2}$$

$$=\rho_0\left\{\frac{D}{Dt}\left(\frac{E}{\rho_0}\right)+P_0\frac{D}{Dt}\left(\frac{1}{\rho_0}\right)-\frac{v}{\rho_0r}(3P_0-E_0)\right\}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_0)+\frac{2aF_0}{c^2}$$

Drop the $\frac{a}{c^2}$ term; and substitute into gas energy eqn:

$$\rho_0\left[\frac{De}{Dt}+p\frac{D}{Dt}\left(\frac{1}{\rho_0}\right)\right]=\rho_0\epsilon-\rho_0\left\{\frac{D}{Dt}\left(\frac{E_0}{\rho_0}\right)+P_0\frac{D}{Dt}\left(\frac{1}{\rho_0}\right)-\frac{v}{\rho_0r}(3P_0-E_0)\right\}-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_0)$$

$$\Rightarrow\rho_0\left[\frac{D}{Dt}(e+E_0/\rho_0)+(p+P_0)\frac{D}{Dt}\left(\frac{1}{\rho_0}\right)-\frac{v}{\rho_0r}(3P_0-E_0)\right]=\rho_0\epsilon-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_0)$$

further

$$\frac{1}{\rho_0r^2}\frac{\partial}{\partial r}(r^2F_0)=\frac{1}{4\pi\rho_0r^2}\frac{\partial}{\partial r}(4\pi r^2F_0)=\frac{\partial}{\partial m_r}(4\pi r^2F_0)$$

hence

96.9) $$\frac{D}{Dt}(e+E_0/\rho_0)+(p+P_0)\frac{D}{Dt}\left(\frac{1}{\rho_0}\right)-\frac{v}{\rho_0r}(3P_0-E_0)=\epsilon-\frac{\partial}{\partial m_r}(4\pi r^2F_0)$$

for isotropic radiation (diffusion), $3P_0-E_0\approx0$, $L_r^0=4\pi r^2F_0$

96.10) then $$\frac{D}{Dt}(e+E_0/\rho_0)+(p+P_0)\frac{D}{Dt}\left(\frac{1}{\rho_0}\right)=\epsilon-\frac{\partial L_r^0}{\partial m_r}$$

Mechanical Energy Equation: take v·(96.2):

$$v\left\{\rho_0\frac{Dv}{Dt}\right\}=v\left\{-\frac{Gm_r\rho_0}{r^2}-\frac{\partial p}{\partial r}+\frac{1}{c}\int\chi_0(\nu_0)F_0(\nu_0)\,d\nu_0\right.$$

$$\Rightarrow\rho_0\frac{D}{Dt}\left(\tfrac{1}{2}v^2\right)=-\frac{Gm_r\rho_0v}{r^2}-v\frac{\partial p}{\partial r}+\frac{v}{c}\int\chi_0F_0\,d\nu_0$$

÷ by ρ_0 : $$\frac{D}{Dt}\left(\frac{v^2}{2}\right)=-\frac{Gm_rv}{r^2}-\frac{v}{\rho_0}\frac{\partial p}{\partial r}+\frac{v}{c\rho_0}\int\chi_0F_0\,d\nu_0$$

now $v=\frac{Dr}{Dt}$ and $\frac{v}{\rho_0}\frac{\partial p}{\partial r}=\frac{1}{\rho_0}\frac{\partial(pv)}{\partial r}-\frac{p}{\rho_0}\frac{\partial v}{\partial r}=\frac{1}{4\pi\rho_0}\frac{\partial}{\partial r}(4\pi pv)-\frac{p}{\rho_0}\frac{\partial v}{\partial r}$

So $$\frac{D}{Dt}\left(\frac{v^2}{2}\right) + \frac{GM_r}{r^2}\frac{Dr}{Dt} + \frac{1}{4\pi\rho_0}\frac{\partial}{\partial r}(4\pi p v) - \frac{P}{\rho_0}\frac{\partial v}{\partial r} = \frac{v}{c\rho_0}\int \chi_0 E_0 \, d\nu_0$$

$$= \frac{D}{Dt}\left(\frac{v^2}{2}\right) - GM_r\frac{D}{Dt}\left(\frac{1}{r}\right) + \frac{r^2\partial}{\partial M_r}(4\pi p v) - \frac{P}{\rho_0}\frac{\partial v}{\partial r} = \frac{v}{c\rho_0}\int \chi_0 E_0 \, d\nu_0$$