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LECTURES ON HYDRODYNAMICAL STABILITY THEORY

by

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Lecture 1

1. Basic Ideas

The first two lectures in this series are intended as an introductory survey for people who have no prior knowledge of the subject.

There is currently much interest in nonlinear or finite-amplitude stability phenomena. Flows are known where all infinitesimal perturbations die out while ones of a certain finite (though perhaps small) amplitude grow. An understanding of this phenomenon is essential for understanding the onset of turbulence and turbulent convection. Probably more powerful mathematical and computational methods than now available will be needed. For an account of some recent work, see Zahn, Toomre, Spiegel, and Gough [1], which describes the result of an impressive amount of numerical work on nonlinear instabilities of plane Poiseuille flow; see also the articles cited therein, also section 2.9 of Monin and Yaglom [2] and the remarks in section 4.6 of Lin [3].

Osborne Reynolds observed nearly a hundred years ago that the equations of motion of a viscous incompressible fluid are invariant under a group of transformations consisting of a rescaling of the quantities that appear in the equations. The formulation of a typical problem contains a characteristic length L (diameter of a pipe, chord of an airfoil, or the like), a characteristic speed V (axial flow velocity averaged over the pipe cross section, speed of the airfoil relative to the ambient air, etc.), and the density ρ and the viscosity coefficient μ of the fluid. The dimensionless combination of these four quantities,

$$R = \frac{LV\rho}{\mu} \tag{1}$$

is called a Reynolds number, after Sommerfeld. L, V, ρ, and μ can be varied by varying the overall size of the arrangement under study without changing shapes, by considering rapid or slow motion, and by considering fluids (air, water, mercury, oil, etc.) with different values of ρ and μ. If the variations are made in such a way as to keep R fixed, then the solution of an initial-value problem, if it is well posed, is unchanged, provided all lengths are scaled like L, all times like L/V, etc. In particular, whether a given laminar flow is stable ought to depend only on the value of R. Many flows are stable for R below a critical value R_{cr} and unstable for R above R_{cr}. However, flows are known that are unstable for all R and others that are stable for all R (with respect to infinitesimal perturbations).

In these lectures, laminar will be taken to mean steady: the velocity vector field is independent of time at each point in space. Hence, the discussion is restricted to problems where the surfaces on which boundary conditions are applied are fixed, in a suitable frame of reference. (However, the physical boundaries may be moving along these surfaces with a constant velocity, as in the case of a sliding plane or a rotating cylinder.)

The idea of stability is to write the velocity and pressure fields as

$$\begin{aligned} &\vec{V}(\vec{x}) + \vec{v}'(\vec{x},t) \\ &P(\vec{x}) + p'(\vec{x},t) \end{aligned} \tag{2}$$

where $\vec{V}$ and P describe the basic flow and $\vec{v}'$, p' an initially very small perturbation. If for all choices of $\vec{v}'$ and p' at $t = 0$, the perturbation remains small for all $t > 0$, the basic flow is stable, otherwise unstable.

Four intuitive ideas, each of which has turned out to be erroneous in certain important cases, have pervaded the subject through much of its history, namely,

(1) It suffices to consider linear stability theory;

(2) It suffices to consider normal modes;

(3) It suffices to consider the simplest normal modes;

(4) A viscous flow is stable if the corresponding inviscid flow is stable;

They will be discussed briefly in order.

If the expressions (2) are substituted into the Navier-Stokes equations, which are nonlinear because of the quadratic advection terms, three types of terms result. The zero order terms containing $\vec{V}$ and P cancel because the basic flow satisfies the equations; the other terms are linear and quadratic in the small quantities $\vec{v}'$ and p'. Dropping the quadratic terms yields the <u>linearized equations</u> for $\vec{v}'$ and p'; they are valid only for "infinitesimal" perturbations, hence do not suffice for investigating finite-amplitude disturbances.

The known functions $\vec{V}(\vec{x})$ and $P(\vec{x})$ appear in the coefficients of the linearized equations, but the variable t does not appear at all. Hence there are <u>normal mode solutions</u>, solutions of the form

$$\begin{aligned} \vec{v}' &= \vec{f}(\vec{x})e^{-i\omega t}, \\ p' &= g(\vec{x})e^{-i\omega t}; \end{aligned} \tag{3}$$

$\vec{f}$ and g satisfy partial differential equations (in $\vec{x} = (x,y,z)$) and boundary conditions; ω appears as an eigenvalue parameter. A normal mode is unstable if $\mathrm{Im}\,\omega > 0$ and stable if $\mathrm{Im}\,\omega \leqslant 0$ (strictly speaking, neutrally stable or metastable if $\mathrm{Im}\,\omega = 0$). It was formerly assumed that the normal modes provide a complete set of functions in which an arbitrary initial disturbance

$\vec{v}'(\vec{x},0)$, $p(\vec{x},0)$ can be expanded, so that stability of all the normal modes would imply stability of the basic flow. Here one must include the continuous spectrum eigenfunctions of certain operators, as in quantum mechanics and Sturm-Liouville theory (these functions are of course not in the Hilbert space of the operator), but in the problems of hydrodynamic stability, the operators are not self-adjoint or even normal, so standard eigenfunction expansion theorems do not apply.

Examples will be mentioned in which the first mode to go unstable is not the simplest one, but one with higher "quantum numbers".

Examples will also be mentioned in which the basic flow is stable in the inviscid limit $\mu = 0$, but not for $\mu > 0$; hence, surprisingly, viscosity can be a cause of instability.

2. Five Classical Problems

Five illustrative classical problems will be discussed briefly. In the Poiseuille flow problems, the boundaries are fixed and the basic flow is maintained by a constant pressure gradient. Two problems of this type are flow in a circular pipe and flow in a channel between two parallel planes. The flow satisfies the Navier-Stokes equations and the no-slip conditions $\vec{V} = \vec{v}' = 0$ on the boundaries. The basic flow is found to have a parabolic velocity profile across the diameter of the pipe and across the channel, respectively. In the Couette flow problems, there is no pressure gradient in the basic flow, and the flow is maintained by relative motion of boundary surfaces. Two problems of this type are flow in a channel between two parallel planes, where one bounding plane is sliding with a uniform velocity relative to the other, and flow between concentric rotating cylinders, whose angular velocities are generally different in magnitude and possibly also in sign. In these problems the no-slip boundary conditions is that $\vec{V}$ agree with the velocity of the sliding surface at the boundary, and $\vec{v}' = 0$ there. The fifth classical problem to be considered here is that of a laminar plane boundary layer; the basic flow is taken to be that given by the Blasius theory, to be described briefly below.

3. Taylor's Investigations of Couette Flow

The first major advance in the subject was made by G.I. Taylor, who reported theoretical and experimental investigations of Couette flow between notating cylinders in 1923. Let r_1 and r_2 be the radii of the surface between which the flow takes place, and let Ω_1 and Ω_2 be their angular velocities about their common axis. The cylinders will be regarded as infinitely long. Let r, ϕ, z be cylindrical coordinates. The basic flow velocity has a ϕ-component only, and that component depends only on r; it can be described by giving the angular velocity $\Omega(r)$, which is easily found from the Navier Stokes equations to have the form

$$\Omega(r) = Ar + \frac{B}{r} \quad (r_1 \leqslant r \leqslant r_2) \tag{4}$$

A and B are determined by the no-slip conditions $\Omega(r_i) = \Omega_i$ $(i = 1, 2)$. The coefficients of the linearized equations for $\vec{v}'$ and p' are here independent not only of t but also of ϕ and z, so that, by a further separation of variables, the normal modes can be taken in the form

$$\begin{aligned} \vec{v}' &= \vec{f}(r)e^{i(kz + n\phi - \omega t)} \\ p' &= g(r)e^{i(kz + n\phi - \omega t)} \end{aligned} \tag{5}$$

where k is an arbitrary real parameter and n is an integer. When the Navier-Stokes equations are written in cylindrical coordinates, and expressions (5) are substituted into them, one finds a 6th order system of ordinary differential equations for g and the components of $\vec{f}$, which, together with suitable boundary conditions at $r = r_1$ and r_2, constitute an eigenvalue problem for ω. The equations are written in full in Monin and Yaglom [2].

Taylor assumed, on the basis of the third intuitive idea mentioned above, that it was only necessary to consider the simplest modes, those with $n = 0$ (As indicated below, that assumption is correct for certain ranges of the parameters that characterize the basic flow, but not for all.) Then the 6th order system is reduced to a 4th order system, which could be handled approximately by the mathematical and computing methods available in Taylor's time. For any value of the parameter k, the eigenvalue problem has a sequence of eigenvalues $\omega_j(k)$, $j = 1, 2, \ldots$. For each j, Im $\omega_j(k)$ is a continuous function of k, which has a single maximum at some critical wave number k. For sufficiently low values of the angular velocities Ω_1 and Ω_2 of the cylinders, Im $\omega_j(k)$ is < 0 for all k, all j. As the rates of rotation of the cylinders increase, the values of the quantities Im $\omega_j(k)$ generally increase until one of the quantities Im $\omega_j(k)$ becomes zero, then positive.

To describe Taylor's results, let X_1 and X_2 be the dimensionless quan-quantities

$$X_1 = \frac{\Omega_1 r_1{}^2 \rho}{\mu}, \qquad X_2 = \frac{\Omega_2 r_2{}^2 \rho}{\mu}, \tag{6}$$

which have the structure of Reynolds numbers, since $\Omega_1 r_1$ and $\Omega_2 r_2$ are velocities. Together with the ratio r_2/r_1 (which is fixed for a given apparatus and had the value 1.136 in Taylor's work), X_1 and X_2 characterize the basic flow. Without loss of generality, it can be assumed that $X_1 \geqslant 0$. According to Taylor's calculations, the basic flow should be stable below the curve in the X_1, X_2 plane shown in Figure 1 and unstable above it, for the case $r_2/r_1 = 1.136$. Similar curves have been obtained for other values of r_2/r_1 by later investigations. Taylor's experiments, made for X_2 in the range -650 to 2200, agreed with his calculations, within experimental error. In this range of X_2, for values of X_1 slightly above the curve, the basic flow is observed to have superposed on it a stable steady flow consisting of vortex rings or rolls in the region between the cylinders, as indicated schematically in Figure 2. This result is a typical bifurcation phenomenon. For X_1 slightly above the curve, there are two states of equilibrium or steady flows that satisfy the Navier-Stokes equations and the boundary conditions: the basic flow and the flow with rolls (the latter is not a single solution, since it can be displaced arbitrarily in the z direction). For such values of X_1 and X_2, the basic flow is an unstable equilibium and the other is a stable one. That does not happen in all flow problems; in many, only chaotic motions are observed when the stability of the basic flow is lost.

Work of Krueger, Gross, and DiPrima, in the early 1960's, using modern computers, showed that if X_2 is taken to have increasingly negative values, the mode with $n = 0$ is no longer the one that first becomes unstable, but rather one with $n = 1$, then $n = 2$, etc., as $-X_2$ increases. In these modes, the rolls are helical, rather than circular.

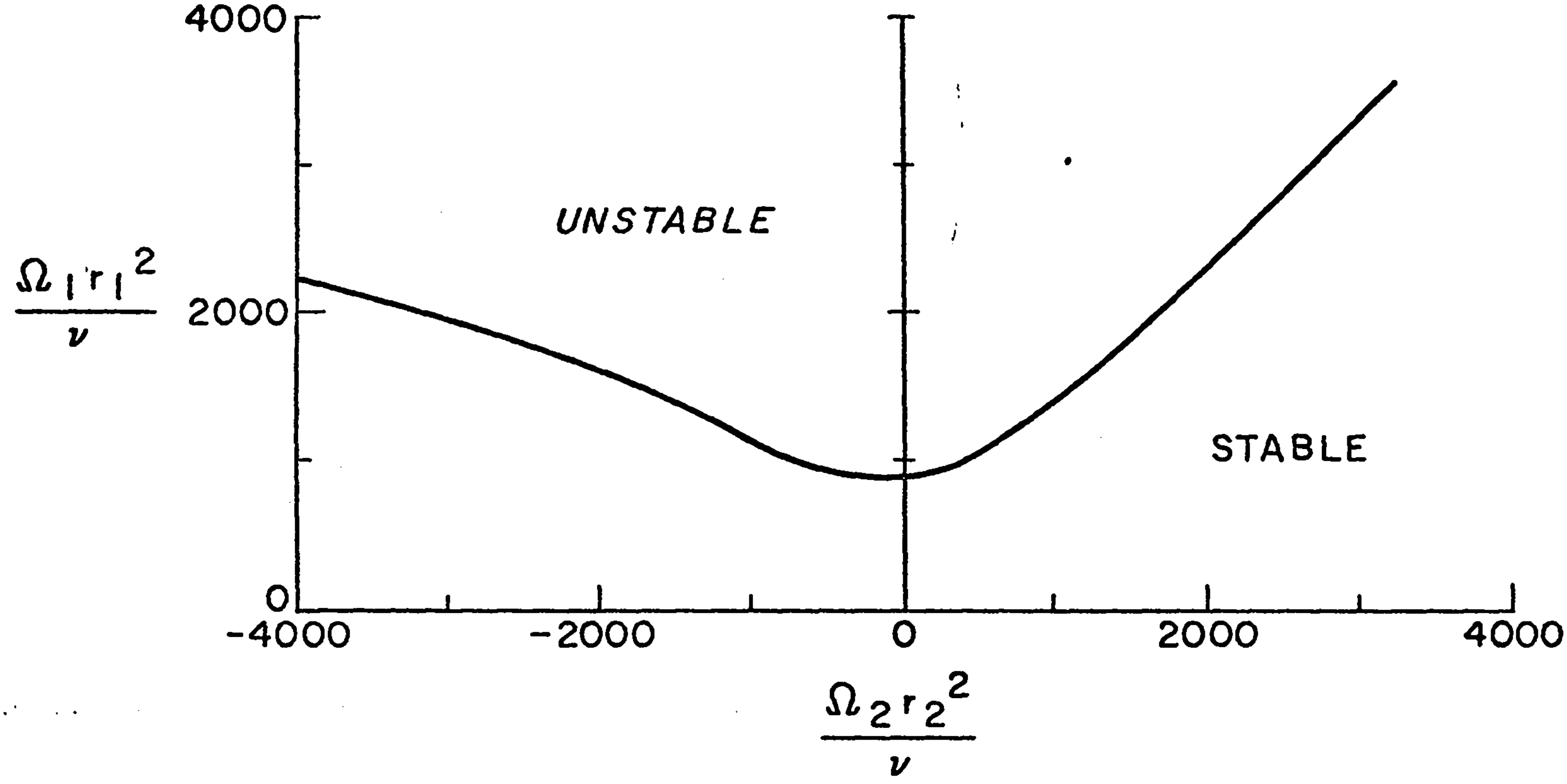


Fig. 1. Stability Diagram of Circular Couette.

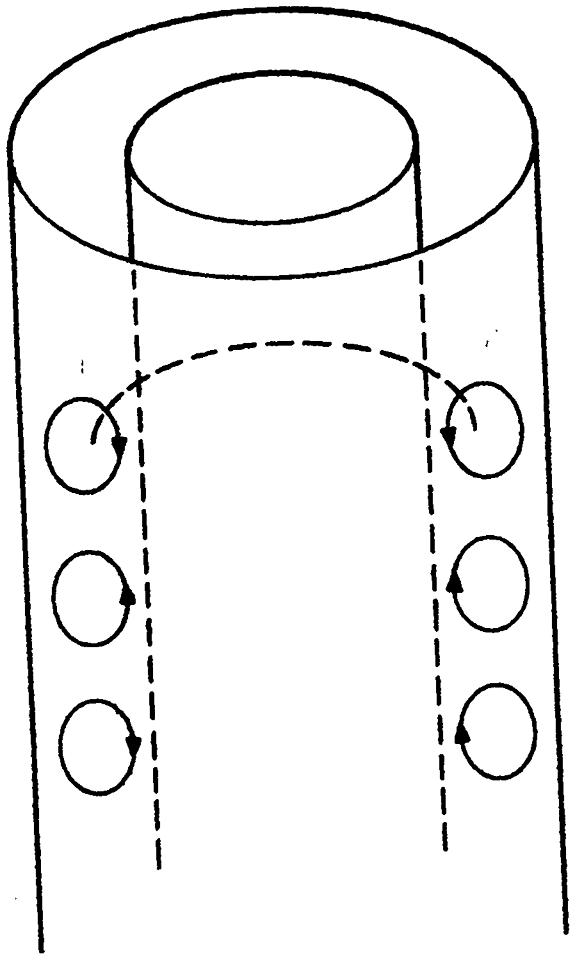


Fig. 2

4. The Plane Laminar Boundary Layer

A brief description of the theory of the laminar boundary layer over a flat plate (Blasius, 1908) is now given, because that is the basic flow for the fifth classical problem, but is less simple and less well known than the other basic flows mentioned above. Suppose that a thin plate occupies the half plane $y = 0$, $x > 0$, and that we seek a steady solution of the Navier-Stokes equations in which the velocity field is asymptotic at large distances to a uniform flow with velocity U in the + x direction, and satisfies the no-slip condition on the plate. (This is a 2-dimensional flow, since the z coordinate does not appear.) It is observed that under usual conditions the flow is almost exactly uniform everywhere except in a thin boundary layer adjacent to the plate. The thickness $\delta = \delta(x)$ of this layer increases slowly with increasing distance x downstream from the edge of the plate, but in the interesting regions, we have $\delta(x) \ll x$, and the boundary layer theory is based on the approximations of regarding δ as a small quantity of the first order. As y increases from zero through the boundary layer for fixed $x > 0$, the x component u of the velocity varies rapidly from 0 to U (which is taken as a quantity of order 1), but it varies slowly with x, and the y component v is small. In fact the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7}$$

shows upon integrating with respect to y from 0 (where $v = 0$) to δ that v is a small quantity of order δ. Underneath each of the other Navier-Stokes equations, we indicate the order of magnitude of each term. (The order of magnitude of ν with respect to δ is not known in advance.) First,

$$\underset{1\cdot 1}{u\frac{\partial u}{\partial x}} + \underset{\delta\cdot\frac{1}{\delta}}{v\frac{\partial u}{\partial y}} = \underset{1\cdot 1}{-\frac{1}{\rho}\frac{\partial p}{\partial x}} + \nu\left(\underset{\nu\cdot 1}{\frac{\partial^2 u}{\partial x^2}} + \underset{\nu\cdot\frac{1}{\delta^2}}{\frac{\partial^2 u}{\partial y^2}}\right) \tag{8}$$

Since the boundary layer is that region of the flow in which the viscous terms are comparable with the other terms, it is seen that ν is of order δ^2, which is simply a backwards way of saying that, other things being equal, the thickness of the layer is proportional to $\sqrt{\nu}$. Then the other Navier-Stokes equation

$$\underset{1\cdot\delta}{u\frac{\partial v}{\partial x}} + \underset{\delta\cdot\frac{\delta}{\delta}}{v\frac{\partial v}{\partial y}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\underset{\delta^2\cdot\delta}{\frac{\partial^2 v}{\partial x^2}} + \underset{\delta^2\cdot\frac{\delta}{\delta^2}}{\frac{\partial^2 v}{\partial y^2}}\right) \tag{9}$$

shows that $\frac{\partial p}{\partial y}$ is of order δ, hence p is constant in the flow to order δ^2. To the lowest order in small quantities, then, the boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{10}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial y^2} \tag{11}$$

It will now be shown that the solution of these equations that satisfies the above-mentioned boundary conditions at infinity and on the plate has a similarity principle. Namely, if

$$u = f(x,y)$$

$$v = g(x,y)$$

is a solution, it is easy to see that for any constant $a > 0$ the functions

$$\tilde{u} = f(ax, \sqrt{a}y)$$
$$\tilde{v} = \sqrt{a}\, g(ax, \sqrt{a}y)$$

also satisfy the same equations and the same boundary conditions. Assuming that the solution is unique (existence and uniqueness of the solutions of various problems of general boundary layer equations have been extensively investigated by the Soviet mathematician Ol'ga Oleinik), we have

$$f(ax, \sqrt{a}y) = f(x,y)\,, \text{ for all } a,x,y. \tag{12}$$

This equation holds in particular for $a = \frac{1}{x}$, hence $f(x,y) = f(1, y/\sqrt{x})$, hence $u = f(x,y)$ depends on x and y only through the combination $\eta = y/\sqrt{x}$. Because of (10), u and v can be expressed through a stream function $\psi = \psi(x,y)$ as

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x}\,,$$

and if $\psi(1,\eta)$ is called $\phi(\eta)$, we have $f(1,\eta) = \phi'(\eta)$. Then, in units such that $\nu = 1$, $U = 1$, it follows from (10) and (11) and the boundary conditions that the universal function $\phi(y)$ is fixed by the differential equation

$$\phi\phi'' + 2\phi''' = 0 \qquad 0 \leqslant \eta < \infty \tag{13}$$

and the boundary conditions

$$\begin{aligned} \phi(0) &= \phi'(0) = 0 \\ \phi'(\infty) &= 1. \end{aligned} \tag{14}$$

The derivative of ϕ gives the velocity profile in the boundary layer according to

$$\frac{u(x,y)}{U} = \phi'\left(\sqrt{\frac{U}{\nu}}\ \frac{y}{\sqrt{x}}\right) \tag{15}$$

and is plotted in Figure 3. As η increases, $\phi' \to 1$ very rapidly. If $\delta(x)$ is defined as the value of y for which $\phi' = .99$ (or in any of various other ways, all of which give the same order of magnitude) it is seen that $\delta(x)$ is proportional to $\sqrt{x}$, which gives a sort of aposteriori justification of the assumption that $\delta(x) << x$ sufficiently far downstream from the leading edge of the plate.

This example contradicts the statement made at the beginng of the lecture that the formulation of a problem always contains a characteristic length L. Since the plate was assumed to be infinitely thin, all linear dimensions here are either 0 or ∞, hence there is no L, hence no Reynolds number. But there is a local Reynolds number, namely

$$\frac{U\delta(x)\rho}{\mu}, \tag{16}$$

which varies from 0 to ∞ as x varies from 0 to ∞. According to our intuitive ideas, then, the boundary layer is expected to be laminar for small x and turbulent for large x. The stability problem of this flow will be discussed in the third lecture in this series, by Steve Orszag.

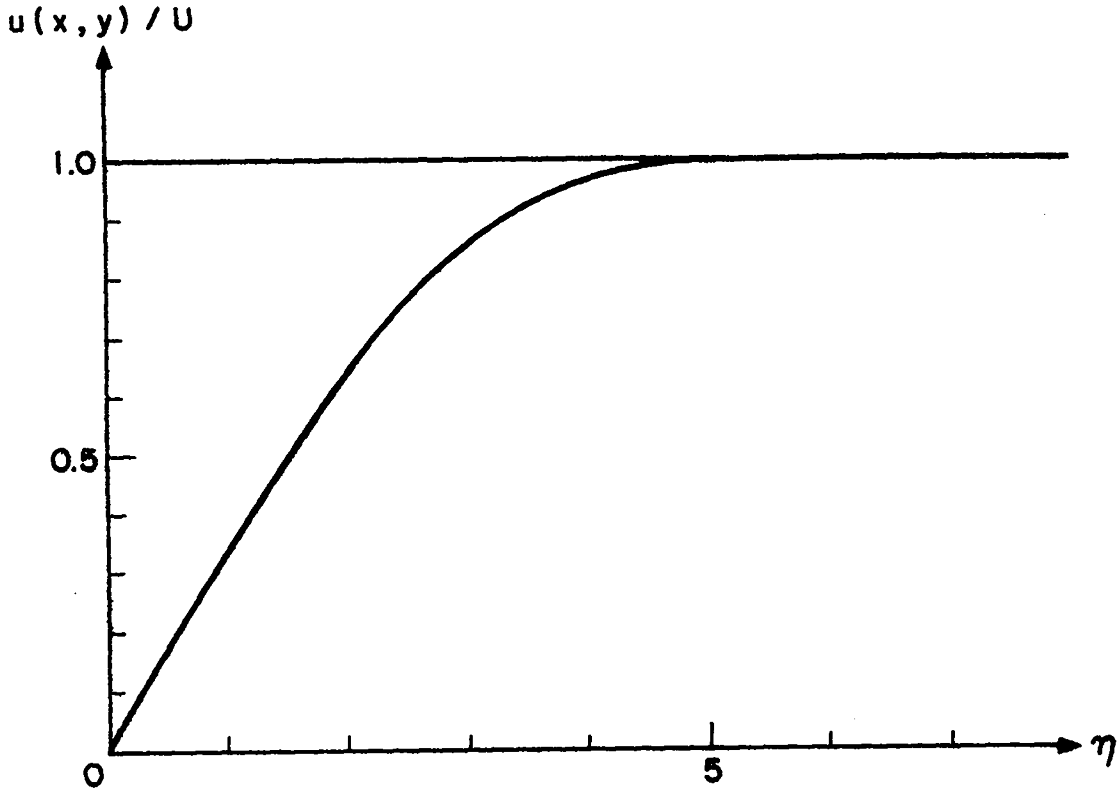


Fig. 3

References

[1] J.-P. Zahn, J. Toomre, E.A. Spiegel, and D.O. Gough, Nonlinear cellular motions in Poiseuille channel flow. J. Fluid Mech. (1974).

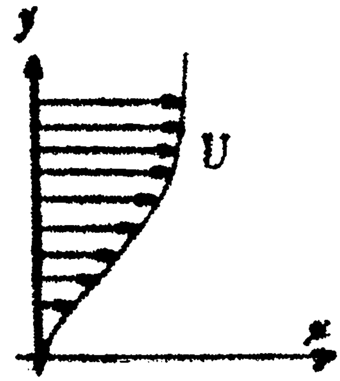
[2] A.S. Monin and A.M. Yaglom, Statistical Fluid Mechanics, M.I.T. Press (1971).

[3] C.C. Lin, The Theory of Hydrodynamic Stability, Cambridge Univ. Press (1955).

Lecture 2

1. <u>The classical linear stability theory of incompressible plane laminar flow.</u>

The classical theory is based on the normal mode concept. Let the x axis be in the direction of the basic flow and the y axis normal to the laminae. Then the basic flow velocity is in the x direction with magnitude $U(y)$. We linearize the Navier-Stokes equations



$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u} \tag{1}$$

by writing $\vec{u} = \vec{U} + \vec{u}'$, $p = P + p'$, finding the linearized equations for $\vec{u}'$ and p', and then dropping the primes. The divergence equation $\nabla\cdot\vec{u}$ holds for $\vec{u}'$ as well as for $\vec{u}$, because $\nabla\cdot\vec{U} = 0$.

If the components of $\vec{u}'$ are called u,v,w, we have

$$\begin{aligned}
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + v\frac{dU}{dy} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w
\end{aligned} \tag{2}$$

Since the coefficients are independent of x,z, and t, we seek normal modes of the form

$$\begin{aligned}
\vec{u} &= \vec{f}(y)e^{i(\alpha_x x + \alpha_z z) + \sigma t} \\
p &= g(y)e^{i(\alpha_x x + \alpha_z z) + \sigma t}
\end{aligned} \tag{3}$$

<u>Squire's theorem</u>: It suffices to consider 2-dimensional disturbances, with $\vec{u}$ and p independent of z, and $\vec{u}$ in the x,z plane. (The interpretation of this statement will be made clearer below).

<u>Proof</u>: We transform the solution (3) by a rotation in the x,z plane: Call

$$\alpha = \sqrt{(\alpha_x^{\ 2} + \alpha_z^{\ 2})} \qquad \cos\Theta = \frac{\alpha_x}{\alpha} \qquad \sin\Theta = \frac{\alpha_z}{\alpha} . \tag{4}$$

Then,

$$\begin{aligned} \hat{x} &= x\cos\Theta + z\sin\Theta & \hat{u} &= u\cos\Theta + w\sin\Theta \\ \hat{y} &= y & \hat{v} &= v \\ z &= -x\sin\Theta + z\cos\Theta, & \hat{w} &= -u\sin\Theta + w\cos\Theta, \end{aligned} \tag{5}$$

$$\begin{aligned} \hat{U} &= U\cos\Theta + 0 \\ \hat{V} &= 0 \\ \hat{W} &= -U\sin\Theta + 0. \end{aligned} \tag{6}$$

After the transformations, the exponent in (3) is simply $i\alpha\hat{x} + \sigma t$. Equations (2) (and also the equation $\nabla\cdot\vec{u} = 0$) are unchanged by the transformation except for the appearance of the circumflex accent and for replacement of the operator $U\frac{\partial}{\partial x}$ by

$$\hat{U}\frac{\partial}{\partial\hat{x}} + \hat{V}\frac{\partial}{\partial\hat{z}} ;$$

however, the solution (3) is independent of $\hat{z}$, so the second term in (7) can be dropped. The normal mode (3) is therefore equivalent to a 2-dimensional normal mode in a basic flow with velocity U(y) reduced by a factor cos Θ. One sometimes sees the argument that this velocity reduction necessarily makes the mode more stable. That argument is invalid because modes are known for certain flows that are stabilized by an increase of velocity. However, if the flow under study has a critcical Reynolds number R_{cr} such that all modes are stable for $R < R_{cr}$, while some mode is unstable in some interval $R_{cr} < R < R_1$, then it clearly suffices to consider 2-dimensional disturbances for finding R_{cr}.

To find the equations for the growth or decay of the 2-dimensional mode, we drop the circumflex accent and ignore the third equation of the set (2), since w does not appear in the first two equations. (The third equation imposes no constraints because p is independent of $\hat{z}$, hence $w \equiv 0$ solves that equation.) In terms of the stream function $\psi = \psi(x,y)$, where $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$, cross-differentiations of the first two equations gives

$$\left(\frac{\partial}{\partial t} + U(y)\frac{\partial}{\partial x}\right)\nabla^2\psi - U''(y)\frac{\partial\psi}{\partial x} = \nu(\nabla^2)^2\psi \tag{8}$$

For the normal mode, $\psi = \phi(y)e^{i\alpha(x-ct)}$, where the quantity σ of (3) has been rewritten as $-i\alpha c$. Substitution into (8) gives the Orr-Sommerfeld equation

$$i\alpha[(U-c)(D^2-\alpha^2) - U'']\phi = \nu(D^2-\alpha^2)^2\phi, \tag{9}$$

where D stands for $\frac{d}{dy}$, and U'' stands for U''(y). (9) is a fourth order linear ordinary differential equation with variable coefficients. For each α, the equation (9), together with boundary conditions, is an eigenvalue problem for c, and the stability condition is that $\mathrm{Im}(c) \leqslant 0$ for all eigenvalues, for all $\alpha \geqslant 0$. To make this a sufficient as well as necessary condition, it would be necessary to establish the completeness of the set of eigenfunctions $\phi_i(y)$ obtained for each α; according to Monin and Yaglom, this difficult question has been investigated by Schensted.

Similar but slightly more complicated equations hold for problems with cylindrical symmetry. There is no analogue of Squire's theorem, however: namely, the modes independent of the angular variable ϕ are not necessarily the most critcial ones for stability.

The classical problems described in the first lecture have been studied by these methods, but only recently with any satisfactory degree of complete completeness.

For Poiseuille flow in a circular pipe, it has been clear from experiments since early in this century that, at least up to quite high Reynolds numbers (now up to 100,000) the flow is stable against infinitesimal disturbances, whereas finite disturbances can be amplified and lead to turbulence at all Reynolds numbers above about 2000. It has very recently been established by theoretical work culminating a half centur of effort that the flow is stable against infinitesimal disturbances for all Reynolds numbers. The most critical mode for stability has n = 1 in the angular factor $e^{in\phi}$, where r, ϕ, z are cylindrical coordinates.

By contrast, Poiseuille flow in the channel between parallel planes has long been known to be unstable for Reynolds numbers above a critical value R_{cr}, whose most accurate value to date, 5772.22, was calculated by Orszag.

The critical mode consists of straight rolls with their axes in the z direction.

Recent work of A. Davey [1], again culminating a half century of effort, has established that Couette's flow in a channel between parallel planes is stable for all Reynolds numbers.

The problem of the stability of boundary layer flow will be discussed in the third lecture, by Steve Orszag, and a few fragmentary results on non-linear stability of the plane Poiseuille flow will be descirbed in the fourth lecture.

2. The inviscid limit; the normal mode approach.

The limiting case obtained by setting $\nu = 0$ in the foregoing equations is of interest for various reasons. Although stability in this limit does not imply stability for $\nu \neq 0$, instability in the inviscid limit implies instability for large enough Reynolds number (small enough ν). Furthermore, the solutions of the inviscid problem appear in the first term of various (asymptotic) expansions in ν or in $1/R$ of the solution of the viscous flow problems.

The stability of inviscid plane laminar flows will be discussed here from two points of view: first the normal mode point of view and then the initial-value-problem point of view. The two are related because the normal modes are associated with the point spectrum of an operator that appears in the second point of view.

Setting $\nu = 0$ in the Orr-Sommerfeld equation (9) gives the Rayleigh equation

$$(U - c)(\phi'' - \alpha^2\phi) - U''\phi = 0. \tag{10}$$

Here, U and ϕ are functions of y. To simplify the discussion, it is supposed that the flow is restricted to a channel $y_1 \leqslant y \leqslant y_2$ between rigid walls. The boundary conditions are

$$\phi(y_1) = \phi(y_2) = 0 \tag{11}$$

For each real α, the equations (10) and (11) constitute an eigenvalue problem for c. Since the time dependence of the mode is given by the factor $\exp(-i\alpha ct)$, the stability condition is that Im(c) be $\leqslant 0$, if we take $\alpha \leqslant 0$, which is possible because only the square of α appears in (10).

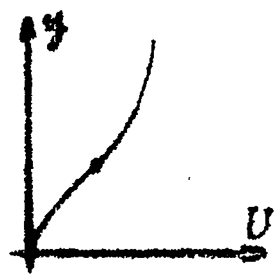
A first necessary condition for the existence of eigenvalues was obtained by Rayleigh, who derived and analyzed equation (10) in 1880. It is customary to write $c = c_r + ic_i$, where $c_r = \mathrm{Re}(c)$ and $c_i = \mathrm{Im}(c)$, and to denote the complex conjugate of ϕ by ϕ^* (in fluid dynamics, the overbar is usually reserved for averages). Equation (10) is divided through by U-c and, $1/(U-c)$ is written as $(U-c)^*/|U-c|^2$. The equation is multiplied by ϕ^* and integrated by parts (the integrated part vanishes because of the boundary conditions (11)). We have

$$-\int_{y_1}^{y_2} (|\phi'|^2 + |\phi|^2)dy - \int_{y_1}^{y_2} \frac{(U-c_r + ic_i)U''|\phi|^2}{(U-c_r)^2 + c_i^2}\, dy = 0. \tag{12}$$

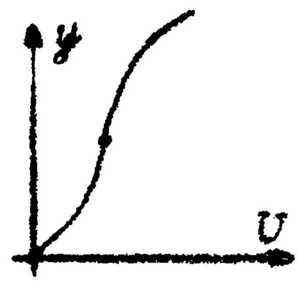
The imaginary part is

$$c_i \int_{y_1}^{y_2} \frac{U''|\phi|^2}{(U-c_r)^2 + c_i^2}\, dy = 0 \tag{13}$$

Hence, either $c_i = 0$, in which case the mode is stable, or U'' must change sign somewhere in the channel, i.e. $|U'|$ must have a maximum or a minimum; $-U'$ is the vorticity of the basic flow. In 1950 Fjørtoft considered the real part of the same equation (12) and showed that $|U'|$ must have a maximum. That is, $U(y)$ must have an inflection point, which must be of the kind shown in (a), not as in (b). This is a necessary but not in general sufficient condition for instability. For further conditions, see the review article by P.G. Drazin and L.N. Howard in Advances in Fluid Mechanics, vol. 9, p. 1 (1966)



(a)



(b)

An important characterzation of the possible eigenvalues c is given by Howard's semicircle theorem, which follows from further manipulations of equation (12): Let U_{min} and U_{max} be the minimum and the maximum of $U(y)$ for $y_1 \leqslant y \leqslant y_2$. Then the possible eigenvalues c lie in the circle in the complex plane whose center is at $\frac{1}{2}(U_{max} + U_{min})$ and whose radius is $\frac{1}{2}(U_{max} - U_{min})$, i.e., whose diameter is the segment $[U_{min}, U_{max}]$ of the real axis. The name of the theorem reflects that we are interested only in the semicircle $Im(c) \geqslant 0$.

Often there are only finitely many eigenvalues c; for plane Couette flow, there are none at all, because (10) reduces to $(U_0y-c)(\phi''-\alpha^2\phi) = 0$, whose only solution satisfying the boundary conditions is $\phi(y) \equiv 0$. Hence, some sort of continuous spectrum must play a role, as has been pointed out by many investigators, starting with Rayleigh in 1894, but has been completely clarified only recently by consideration of the initial-value problem.

The coefficients in Rayleigh's equation (10) are real; hence, if $c = c_r + ic_i$ is an eigenvalue, $c^* = c_r - ic_i$ is also an eigenvalue. Hence, for every damped mode there is also an unstable mode, and conversely. That is no longer true when viscosity is taken into consideration, for the coefficients of the Orr-Sommerfeld equations (9) are not real. There are often damped modes for which no corresponding unstable modes exist. This is particularly important when Rayleigh's equation (10) has no nonreal eigenvalues, so that the inviscid limit gives no clue for the stability at high Reynold's numbers. There are cases in which all normal modes are damped.

The asymptotic theory of the viscous solutions, that is, their behavior as $\nu \to 0$ and their relations to the inviscid solutions, is a quite intricate matter for which the interested person is referred to Lin's book and to the review article by W.H. Reid, The Stability of Parallel Flows, in Basic Developments in Fluid Dynamics, Maurice Holt, editor, Academic Press (1965), page 249 ff.

3. The inviscid limit; the initial-value-problem approach.

We are still concerned with linearized stability theory of incompressible plane laminar flow in the limit $\nu = 0$, but now we set $\psi = \hat{\psi}(y,t)e^{i\alpha x}$ and $\nu = 0$ in (8). The result is

$$(D^2 - \alpha^2)\frac{\partial\hat{\psi}}{\partial t} = i\alpha[U'' - U(D^2 - \alpha^2)]\hat{\psi}. \tag{14}$$

A precise formulation of a linear initial-value problem involves a Banach or Hilbert space $\mathcal{B}$ or $\mathcal{H}$; the instantaneous state of a physical system is represented by a vector $\psi = \psi(t)$ in this space, and the time-dependence of $\mathcal{H}$ is determined by an equation of evolution

$$\frac{d}{dt}\,\psi(t) = A\psi(t), \tag{15}$$

where A is a linear operator. In the present problem, we take the Hilbert space

$$\mathcal{H} + L_2(-\infty < x < \infty,\ y_1 \leqslant y \leqslant y_2), \tag{16}$$

whose elements are quadratically integrable functions $\psi(x,y)$ - stream functions of instantaneous states of flow. The functions $\hat{\psi}$ or $\hat{\psi}(\alpha,y)$ or $\hat{\psi}(\alpha,y,t)$ in (14) is the result of applying the operator of the Fourier transform with respect to x to $\psi(x,y)$ or $\psi(x,y,t)$. That operator is a unitary transformation in $\mathcal{H}$.

To put (14) into the form (15), we apply the operator $(D^2-\alpha^2)^{-1}$ to both sides of (14). $(D^2-\alpha^2)^{-1}$ is simply the integral operator containing the Green's function for the equations $\phi'' - \alpha^2\phi = \chi$ on the interval $[y_1, y_2]$. After a minor rearrangement, the result is

$$-\frac{1}{i\alpha}\frac{\partial\hat{\psi}}{\partial t} = (U + K)\hat{\psi} = A\hat{\psi}, \tag{17}$$

where U is the operator of multiplication of $\hat{\psi}(\alpha,y)$ by the functions $U(y)$, and K is an integral operator

$$(K\hat{\psi})(\alpha,y) = \int_{y_1}^{y_2} G_\alpha(y,\eta)U'(\eta)\hat{\psi}(\alpha,\eta)\,d\eta. \tag{18}$$

To simplify the writing, units and origins are assumed so chosen that

$$[y_1, y_2] = [0, 1]$$

$$[U_{min}, U_{max}] = [0, 1];$$

then the kernel in (18) is

$$G_\alpha(y, \eta) = \begin{Bmatrix} \dfrac{-2 \cosh \alpha \eta}{\sinh \alpha} \sinh\alpha (1-y), & y > \eta \\ \dfrac{2 \sinh \alpha y}{\sinh \alpha} \cosh \alpha (1-\eta), & y < \eta \end{Bmatrix} \tag{19}$$

The operator U is bounded and self-adjoint, while K is compact but not self-adjoint. The properties of the operator A = U + K and the solution of the initial-value problem of (15) are discussed in Rosencrans and Sattinger (see Monin and Yaglom, as usual, for all references not given here). The credit for initiating this approach is due to Case and to Dikii, who independently solved the inital-value problem by a Laplace transformation with respect to t.

The solution of the initial-value problem and the conclusion to be drawn from it will now be very briefly sketched. The solution is of course simply

$$\psi(t) = e^{-i\alpha tA} \psi(0), \tag{20}$$

where, since A is a bounded operator, the exponential is given for all t by the power-series expansion. For our purpose, a more useful representation of the exponential is given by the Cauchy integral

$$\psi(t) = - \frac{1}{2\pi i} \oint_C e^{-i\alpha t\lambda}(A-\lambda)^{-1}\psi(0)d\lambda, \tag{21}$$

where C is a contour in the λ plane that encircles the spectrum of A (which is a bounded set, because A is a bounded operator).

Rosencrans and Sattinger proved that the point spectrum of A consists of the eigenvalues discussed above; they lie in the circle of Howard's semi-circle theorem; the continuous spectrum consists of the interval $[U_{min}, U_{max}]$ on the real axis, and the rest of the λ plane is resolvent set.

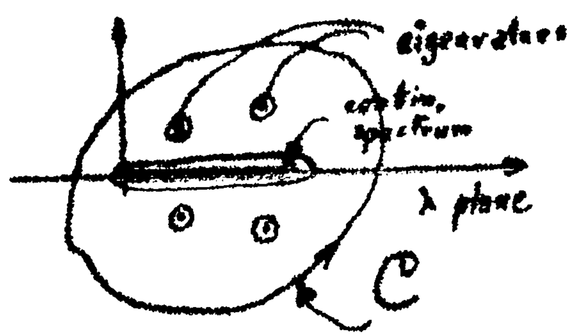
To study the growth of $\psi(t)$, starting from (21), one first represents $(A-\lambda)^{-1}$ as another integral operator. That can be done, because if $(A-\lambda)^{-1}\psi = \chi$, for given ψ, then χ is the solution of the equation

$$(A-\lambda)\chi = \psi. \tag{22}$$

If to both sides of this equation the operator $D^2-\alpha^2$ is applied, then it is seen, from the equation (15) and (17) that define A, that χ satisfies the differential equation

$$\chi'' - \alpha^2\chi - \frac{U''}{U-\lambda}\chi = \frac{\psi''-\alpha^2\psi}{U-\lambda} \tag{23}$$

If the Green's functions for this problem with $\chi = 0$ at $y = y_1$ and y_2, is constructed in the usual way from two solutions of the corresponding homogeneous equation (which is just Rayleigh's equation), one satisfying each boundary condition, then $(A-\lambda)^{-1}$ is obtained as an integral operator. That operator is substituted into (21) and the contour is contracted, as indicated in the accompanying sketch, to small contours encircling the individual eigenvalues plus a contour encircling the continuous spectrum.



For simplicity I make the assumption, which is valid under reasonable assumptions about the function U(y), that there are only finitely many eigenvalues, none real. It can then be shown that each eigenvalue contributes to $\psi(t)$ a normal mode, as described earlier, while the continuous spectrum gives a contribution which decays as $O(1/t)$ with increasing time.

For a considerable class of problems, the procedure just outlined completely justifies the normal mode approach.

Case has applied his Laplace transform method to problems with $\nu \neq 0$ and thus considerably extended the justification of the normal mode approach.

Lecture 3

1. Introduction

The theory of linear hydrodynamic stability raises several important questions that require consideration of nonlinear effects. Among these are:

1. What is the time evolution of a linearly unstable mode? The linear theory predicts exponential growth in time which implies that the approximation of linearization must eventually be violated.
2. Do steady finite-amplitude solutions of the equations of motion exist?
3. What are the stability properties of the linear theory modes? In other words, are the infinitesimal perturbations determined by linear theory themselves stable to perturbations?
4. Do there exist "subcritical" instabilities for Reynolds numbers $R < R_{cr}$, where R_{cr} is the critical Reynolds number determined by linear theory? By their definition, such subcritical instabilities must be of finite amplitude.
5. Do "explosive" instabilities exist which bear little or no resemblance to linear theory modes?
6. By what processes is it possible for a fluid to undergo "transition" from a laminar flow state to turbulence? In particular, what is the relevance of the linear and finite-amplitude modes to these transition processes?

In this lecture, we attempt to survey the known types of nonlinear stability behavior for fluids.

2. Survey of Results

The five basic flow problems surveyed in Lectures 1 and 2 have distinctive nonlinear stability characteristics. Here we summarize these facts:

a. Circular Couette flow. This flow results when two infinite concentric cylinders rotate about their common axis with different angular velocities. If the inner cylinder has radius R_1 and angular velocity Ω_1, while the outer one has radius R_2 and angular velocity Ω_2, the basic (laminar) flow state has azimuthal velocity

$$v(r) = \frac{\Omega_1 - \Omega_2}{R_2{}^2 - R_1{}^2} \frac{R_1{}^2 R_2{}^2}{r} + \frac{\Omega_2 R_2{}^2 - \Omega_1 R_1{}^2}{R_2{}^2 - R_1{}^2} r$$

where r is the radial distance from the axis, and zero radial and axial velocities. It was shown in the previous lectures that this Couette flow is unstable for certain values of Ω_1, Ω_2 for fixed R_1. R_2, and ν, the kinematic viscosity, as first found by Taylor (1923). In the particular limiting case, $d = R_2 - R_1 \ll R_1$, $\Omega_2 = 0$ (fixed outer cylinder), the flow is unstable to infinitesimal and symmetric disturbances for

$$T = \Omega_1{}^2 R_1 d^3/\nu^2 > 1708. = T_{cr}; \tag{2}$$

T is called the Taylor number.

Finite amplitude perturbation theory establishes the existence of steady finite amplitude solutions for $T > T_{cr}$ which reduce to the Taylor vortices (linear modes) as $T \downarrow T_{cr}$. These finite amplitude solutions are stable to axisymmetric disturbances for all $T > T_{cr}$ (assuming the validity of second order perturbation theory). In fact, Davey (1962) considered the evolution of various multimode axisymmetric flows and found that they all relaxed to single mode solutions as $t \to \infty$.

On the other hand, for $T \gtrsim (1.08)T_{cr}$ the axisymmetric Taylor vortices are unstable to nonaxisymmetric disturbances. It is inferred that these nonaxisymmetric modifications of the Taylor cortices are stable for a range of T exceeding $(1.08)T_{cr}$.

Coles (1965) performed a careful series of laboratory experiments on cicular Couette flow. He found a series of about 25 discrete flow transitions with increasing Taylor number, as well as a variety of interesting hysteresis effects. For example, in one experiment, Coles found states (k,m) with increasing T, where k is the axial wavenumber and m is the azimuthal wavenumber: (28,0) [axisymmetric Taylor vortices].

In circular Couette flow, discrete transitions to turbulence are found as the Reynolds number increases if the flow state lies within the linearly unstable region of the Ω_1, Ω_2 plane [see Fig. 1]. On the other hand, if the Reynolds number increases within the region of the Ω_1, Ω_2 plane that is stable to infinitesimal disturbances (to the right of the line $X_1=X_2$ in the figure), then an explosive transition to turbulence is observed.

The problem of circular Couette flow is closely analogous to that of Bénard convection, wherein fluid is confined between two heated flat plates. If the bottom plate is hotter than the top then free convection in the region between the plates can result. The analogy between the Bénard problem in which stratification dominates and the Couette which is governed by rotation is general (Veronis 1970). The appropriate non-dimensional parameter governing Bénard convection is the Rayleigh number

$$Ra = \frac{g\alpha\Delta T}{\nu\kappa} d^3,$$

where g is the acceleration due to gravity, α is the thermal expansion coefficient, ν is the kinematic viscosity, κ is the thermal diffusivity, d is the separation between the flat plates, and ΔT is the temperature difference. The critical Rayleigh number in the case of rigid bounding plates is

$$Ra_{cr} = 1708,$$

as for Couette flow.

There is one interesting finite amplitude aspect of the Bénard problem that is not analogous to the circular Couette problem. The linear (infinitesimal) stability theory implies that for $R > R_{cr}$, there exists a range of "horizontal" wavenumbers α corresponding to unstable modes. The horizontal wavenumber of the linear mode u is α if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\alpha^2 u; \tag{3}$$

all horizontal planforms satisfying (3) have equivalent linear stability characteristics. However, not all such horizontal planforms are equally stable to disturbances. It was found by Schlüter, Lortz, and Busse (1965) that the only stable planforms for $(Ra-Ra_{cr})/Ra_{cr} << 1$ are two-dimensional "rolls", i.e. $u \propto \cos(\vec{\alpha}\cdot\vec{x}+\Theta)$ for some horizontal vector $\vec{\alpha}$ and phase shift Θ, and that rolls are stable only within a restricted wavenumber range. Busse (1967) showed the stability of rolls for a more extensive range of Ra, but under the assumption of infinite Prandtl number, i.e., $\nu/\kappa = \infty$.

On the other hand, it is found experimentally that for Ra just above Ra_{cr} that rolls are unstable and hexagon shaped horizontal plan forms are stable. This effect was explained by Palm (1960) using finite-amplitude perturbation theory with temperature dependent transport coefficients (ν,κ). If $d\nu/dT \neq 0$, it was found that there exist a second critical Rayleigh number Ra', such that for $Ra_{cr} < Ra < Ra'$ hexagons are stable, whereas for $Ra > Ra'$, rolls are stable. In addition, there can be hysteresis effects, in the sense that the value of Ra' depends on whether stability of hexagons is studied for Ra increasing or decreasing.

b. Plane Couette flow. This flow has been shown to be linearly stable at all Reynolds numbers in a quasi-rigorous way by Davey (1973). The nature of finite amplitude states was investigated by Watson (1960). However, the nature of the transition process in plane Couette flow remains largely unexplored, due to the difficulty of setting up suitable experiments. E. Mollo-Christensen attempted an experiment using moving belts to simulate the flow, but end effects seemed to give large disturbances to the flow.

c. Plane Poiseuille flow. As discussed in the previous lectures, this flow is linearly unstable for Reynolds numbers $R > R_{cr} \simeq 5772.22$. The finite amplitude features of plane Poiseuille flow are discussed in the next lecture on the basis of the work of Zahn et al. (1974). It seems that steady finite amplitude solutions are possible. However, such steady finite amplitude states have not been observed experimentally to date.

d. Pipe Poiseuille flow. This flow appears stable to all infinitesimal disturbances (Salwen and Grosch, 1972, Metcalfe and Orszag, 1974). The linear problem here is complicated by the fact that there is no analog to

Squire's theorem, so that both axisymmetric and nonaxisymmetric distrubances must be considered. In addition, the equations for nonaxisymmetric disturbances cannot be conveniently reduced to a single high-order differential equation (like the Orr-Sommerfeld equation), so that very large matrix problems must be solved for the eigenvalue. It is found that the least stable mode is usually the nonaxisymmetric mode with azimuthal wave number $n = 1$, i.e. that mode proportional to $e^{\pm i\phi}$.

Champagne and Wygnanski (1973) have done a careful series of experiments to study transition in pipe flows. They find "explosive" transitions to turbulence, with the transition flow having the form of "puffs" or "slugs" of turbulence depending on the level of free stream turbulence. Numerical simulation of this phenomenon seems viable, but this has not yet been attempted. Presumably, an upstream perturbation consisting of an axisymmetric disturbance plus a nonaxisymmetric disturbance to the basic Poiseuille flow is sufficient to drive the explosive transition to turbulence (cf. the discussion of boundary layer transition in sec. 5 below).

e. Flat plate boundary layers. The stability of the flat plate boundary layer is discussed in more detail in sec. 5. The boundary layer is linearly unstable, the linearly growing modes are experimentally observed, but the flow exhibits an explosive transition to turbulence.

In a sense, the flat plate boundary layer flow is intermediate to the cases of circular Couette flow and pipe Poiseuille flow. In Couette flow, linear instability is observed and transition is through a sequence of orderly states. In boundary layers, linear instability is observed but transition is catastrophic. Finally, in pipe flow, linear instability is not observed and transition is catastrophic.

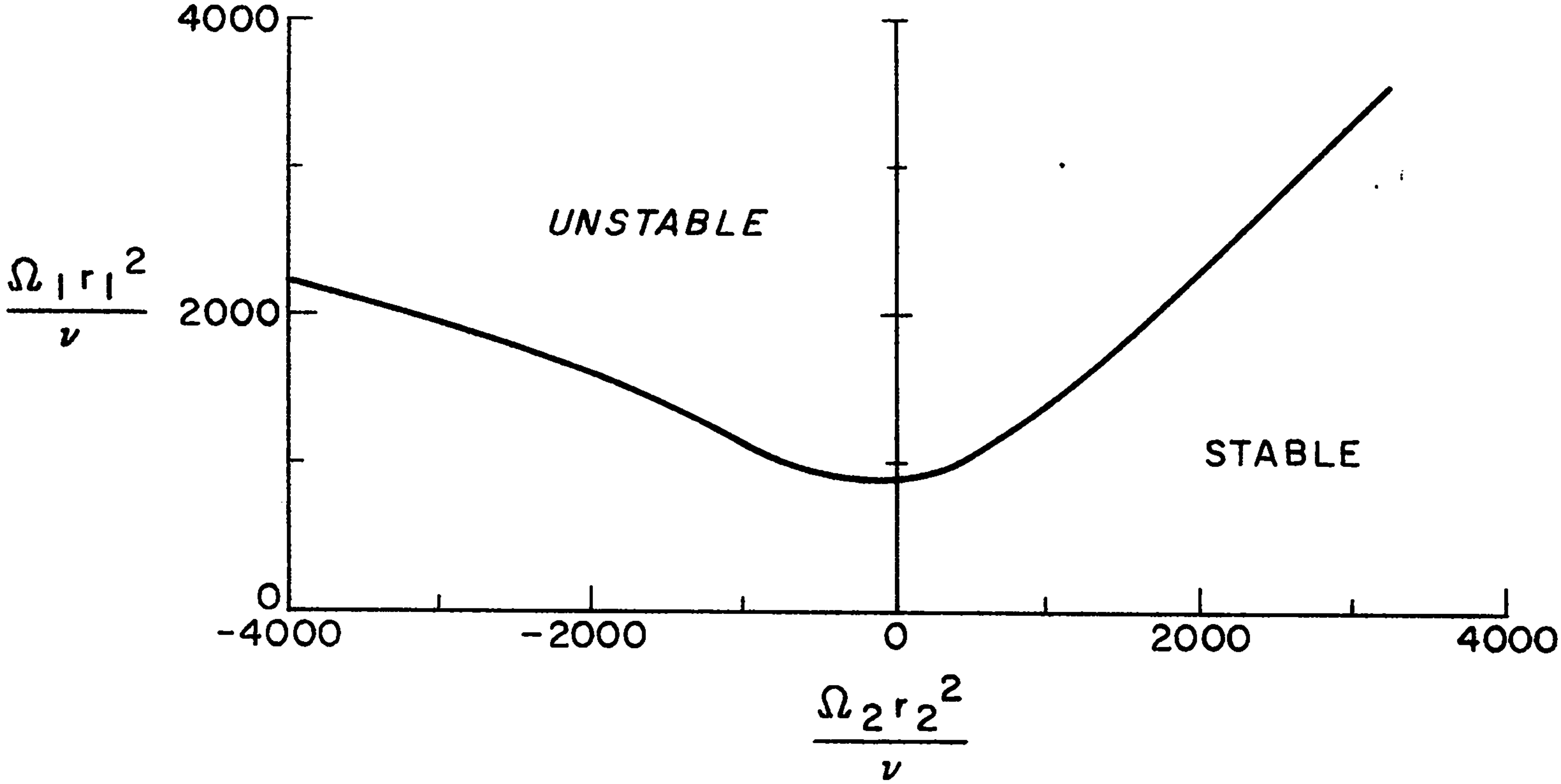


Fig. 1. Stability Diagram of Circular Couette.

3. Landau's Theory

Landau (1944, also Landau and Lifschitz 1959, §27) proposed a theory of finite amplitude stability valid for Reynolds number close to R_{cr}. Landau asserted that linear theory implies modes of the form

$$\vec{u}(\vec{x},t) = A(t)\vec{u}(\vec{x})$$

where $A(t) = \exp(-ict)$ with $c = c_r + ic_i$. If $R > R_{cr}$, there exist growing modes with $c_i > 0$, while if $R < R_{cr}$ all modes decay, i.e. $c_i < 0$. It follows that the amplitude in linear theory grows like

$$\frac{d|A|^2}{dt} = 2c_i|A|^2 \tag{4}$$

where $|A|^2 = AA^*$, and that $c_i \approx c_i'(R-R_{cr}) << c_r$ for the unstable modes with R, close to R_{cr}.

Landau further asserted that nonlinear effects would be such as to modify (4) into an equation of the form

$$\frac{d|A|^2}{dt} = \sum_{n,m=1}^{\infty} a_{nm}A^n A^{*m},$$

where A(t) is now interpreted as the amplitude of that contribution to the finite amplitude mode $\vec{u}(\vec{x},t)$ from the linear mode $\vec{u}(\vec{x})$ in an eigenfunction expansion of $\vec{u}(\vec{x},t)$. The lowest-order correction to (4) must be of the form

$$\frac{d|A|^2}{dt} = 2c_i|A|^2 - k|A|^4 \tag{5}$$

No cubic term can enter (5) because, with the assumption $(R-R_{cr})/R \ll 1$, it follows that $c_r \gg c_i$ so that cubic terms, necessarily of the form A^2A^* or $A(A^*)^2$ oscillate rapidly in time (with period $2\pi/c_r$). These rapid oscillations necessarily average out of the secular behavior of $|A|^2$, giving an equation of the form (5). Corrections to the coefficient k are of higher order than the retained terms because $k \ll c_r$ when $R \approx R_{cr}$. It is important to note that while $c_i = 0$ at $R = R_{cr}$, it is generically true that $k \neq 0$ when $R = R_{cr}$, since there is no a priori reason why $k = 0$.

The character of solutions to (5) is best brought out in the phase plane with axes $|A|^2$ and $d|A|^2/dt$. If $k > 0$ and $R > R_{cr}$, then the phase plane plot shown in Fig. 2(a) implies that the equilibrium point, $d|A|^2/dt = 0$, at

$$|A|^2 = \frac{2c_i}{k} \approx \frac{2c_i'(R-R_{cr})}{k}$$

is stable and that this equilibrium point is approached as $t \to \infty$. In this "supercritical equilibrium" state, $|A| \propto (R-R_{cr})^{1/2}$. If $k > 0$ and $R < R_{cr}$, the nonlinear mode is damped as well as the linear mode.

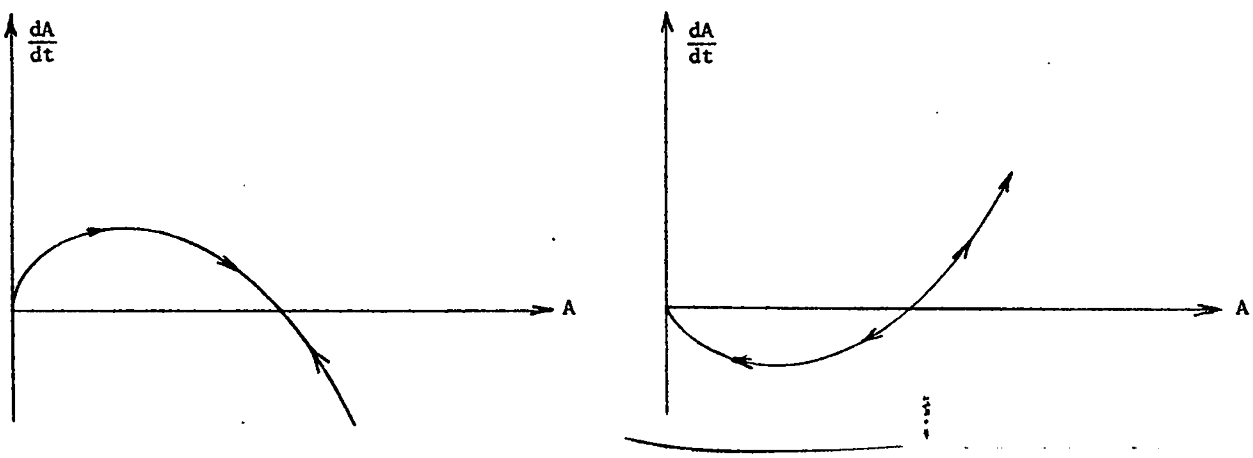


Fig. 2. Phase plane behavior of finite amplitude disturbances
(a) $k > 0$ $R > R_{cr}$.
(b) $k < 0$ $R < R_{cr}$.

If $k < 0$ and $R < R_{cr}$, then the phase plane plot shown in Fig.2(b) implies that the equilibrium point (finite-amplitude steady mode) is unstable. The time evolution predicted by (5) is such that if $|A(0)|^2 > 2c_i/k$, then $|A(t)|^2 \to \infty$, while if $|A(0)|^2 < 2c_i/k$, then $|A(t)|^2 \to 0$. This flow exhibits a "subcritical" finite-amplitude instability. On the other hand, if $k < 0$ and $R > R_{cr}$, then $|A(t)|^2 \to \infty$ for all nonzero initial amplitudes $|A(0)|^2$.

In summary, the sign of the first Landau constant k determines whether there exist finite supercritical equilibria or subcritical finite amplitude instabilities. In the remainder of this lecture, we discuss several special cases of the general theory outlined above.

4. Finite-amplitude theory of circular Couette flow.

As discussed in Lecture 1, the laminar Couette flow between coaxial cylinders of radii $R_1 < R_2$ rotating with angular velocities Ω_1, Ω_2, respectively is

$$U(r) = \frac{\Omega_2 R_2^{\,2} - \Omega_1 R_1^{\,2}}{R_2^{\,2} - R_1^{\,2}}\, r + \frac{(\Omega_1 - \Omega_2) R_1^{\,2} R_2^{\,2}}{R_2^{\,2} - R_1^{\,2}}\, \frac{1}{r} \tag{6}$$

where r is the radial distance from the common axis and U is the aximuthal velocity about this axis.

In the special case where the outer cylinder is fixed ($\Omega_2 = 0$) and the gap $d = R_2 - R_1 \ll R_1$, it may be shown (Chandrasekhar, 1961) that the flow (6) is linearly unstable provided

$$T = \Omega_1^{\,2} R_1 d^3/\nu^2 > 1708 = T_{cr},$$

where the Taylor number T is analogous to the square of the Reynolds number.

When $T > T_{cr}$, instability appears first as so-called Taylor vortices in which the velocity field has the form $\mathrm{Re}(\hat{u}_1(r,t)e^{i\alpha z})$. Taylor vortices are axisymmetric around the z-axis and periodic (with wavenumber α) along the axis. The radial structure of $\hat{u}_1$ is determined by the eigenvalue problem of the linearized stability equation (analogous to the Orr-Sommerfeld equation discussed in Lecture 2).

These linearized Taylor vortex solutions are functions of two parameters α and T. For any value of $T > T_{cr}$, there are values of α for which the amplitude of $\hat{u}_1$ is either growing, neutral (steady), or decaying. It seems clear that those wavenumbers α for which $\hat{u}_1$ is growing most rapidly will eventually dominate a typical solution. However, such growth cannot persist forever because of the bounded energetics of the flow. There is no possibility of stabilizing a growing solution within the linear approximation to the dynamical equations; nonlinear effects must be invoked to stabilize the growth of the Taylor vortices.

Finite-amplitude axisymmetric Taylor vortices are studied by seeking a solution to the Navier-Stokes equations of the form

$$\vec{u}(r,z,t) = \vec{U}(r) + \vec{u}_0(r,t) + \mathrm{Re}(\vec{u}_1(r,t)e^{i\alpha z}) + \mathrm{Re}(\vec{u}_2(r,t)e^{2i\alpha z}) \quad (7)$$

where $\vec{U}(r)$ is the steady Couette flow(6) and it is assumed that the "nonlinear" corrections $\vec{u}_0$, $\vec{u}_2$ are much smaller than the basic Taylor vortex flow $\vec{u}_1$. It is also assumed that

$$\vec{u}_1(r,t) = A(t)\hat{u}_1(r) + \vec{v}_1(r,t) \quad (8)$$

where $\hat{u}_1$ is the linear Taylor mode and $\vec{v}_1$ is a small nonlinear correction.

The Navier-Stokes equations for incompressible flow are most conveniently written

$$\frac{\partial}{\partial t} u_i = P_{ij}(\nabla)[(\vec{u}\cdot\nabla)u_j] + \nu\nabla^2 u_i \tag{9}$$

where $P_{ij}(\nabla)$ is the linear projection operator onto solenoidal vectors. $P_{ij}(\nabla)$ accounts for the effect of pressure; it is defined by the conditions that $v_i = P_{ij}(\nabla)u_j$ satisfies $\nabla\cdot\vec{v} = 0$ for any $\vec{u}$, and $\vec{v} = \vec{u}$ if $\nabla\cdot\vec{u} = 0$. Substituting (8) into (9) and equating coefficients of $e^{in\alpha z}$ gives the equations

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\vec{u}_0 = P(\nabla)[\vec{U}\cdot\nabla\vec{u}_0 + \vec{u}_0\cdot\nabla\vec{U} + |A|^2\hat{u}_1{}^*\cdot\nabla\hat{u}_1] \tag{10}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\vec{u}_1 = P(\nabla)[A\vec{U}\cdot\ \hat{u}_1 &+ A\hat{u}_1\cdot\nabla\vec{U} + \vec{U}\cdot\nabla\vec{v}_1 + \vec{v}_1\cdot\nabla\vec{U} \\ &+ A\hat{u}_1\cdot\nabla\vec{u}_0 + A\vec{u}_0\cdot\nabla\hat{u}_1 + A^*\hat{u}_1{}^*\cdot\nabla\hat{u}_2 + A^*\vec{u}_2\cdot\nabla\hat{u}_1{}^*]\end{aligned} \tag{11}$$

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\vec{u}_2 = P(\nabla)[\vec{U}\cdot\nabla\vec{u}_2 + \vec{u}_2\cdot\nabla\vec{U} + A^2\hat{u}_1\cdot\nabla\hat{u}_1] \tag{12}$$

Terms smaller than those shown explicitly in (10) - (12) have not been written down. For example, in (10), terms like $\vec{U}\cdot\nabla\vec{U}$ disappear because $\vec{U}$ is a steady solution of the Navier-Stokes equations, while terms like $\vec{v}_1\cdot\nabla\vec{v}_1$ and $\vec{u}_2\cdot\nabla\vec{u}_2$ are smaller than retained terms. In fact, in terms of the amplitude A of the Taylor vortex $\hat{u}_1$, the typical magnitude of $\vec{u}_0$ and $\vec{u}_2$ driven by (10) and (12), respectively, is $O(A^2)$, while that of $\vec{v}_1$ is $O(A^3)$ by (11). Despite the fact that $\vec{v}_1$ is higher order than $\vec{u}_0$ and $\vec{u}_2$, it can not be neglected in the determination of the time evolution of A(t).

If (11) is truncated at order A (by eliminating terms involving $\vec{v}_1, \vec{u}_0, \vec{u}_2$) then the linearized stability equation yielding Taylor vortices results. It follows that if the growth rate of the linear theory mode is c_i, retention of only the first two terms on the right-hand side of (11) gives the amplitude equation

$$\frac{d|A|^2}{dt} = 2c_i|A|^2$$

Inclusion of nonlinear effects can be done by solving (10)- (12). However, these equations can be further simplified recalling that $c_i = 0$ at $T = T_{cr}$ so that, in general, $c_i \propto (T-T_{cr})$ is small for $T \sim T_{cr}$. Consequently, it follows from (10) that

$$\frac{\partial\vec{u}_0}{\partial t} = O(|A|^2(T - T_{cr}))$$

and from (12) that

$$\frac{\partial\vec{u}_2}{\partial t} = -2i\alpha c_r\vec{u}_2 + O(|A|^2(T-T_{cr}))$$

where the correction (order) terms are negligible compared to the retained terms in (10), (12). With these approximations, (10) and (12) can be solved for $\vec{u}_0$ and $\vec{u}_2$, respectively, in terms of $\vec{u}_1$. Finally, these solutions are substituted into (11).

In this way, it is easy to see that the terms in (11) involving interaction between $\vec{u}_1$ and $\vec{u}_0$ give rise to a term in the evolution equation of $d|A|^2/dt$ of the form $k_1|A|^4$ as in (5). The Landau constant k_1 due to the interaction of $\vec{u}_1$ with $\vec{u}_0$ is

$$k_1 = -2\,\mathrm{Re}\int \hat{u}_1{}^* \cdot P(\nabla)(\hat{u}_1 \cdot \nabla\vec{u}_0 + \vec{u}_0 \cdot \nabla\hat{u}_1)/|A|^2\, d\vec{r} \Big/ \int \hat{u}_1{}^* \cdot \hat{u}_1 d\vec{r} \tag{13}$$

In the same way, it follows that the terms in (11) involving interaction of $\vec{u}_1$ and $\vec{u}_2$ give rise to a contribution to the first Landau constant of the form

$$k_2 = -2\mathrm{Re}\int \hat{u}_1{}^* \cdot P(\nabla)[\hat{u}_1{}^* \cdot \nabla\vec{u}_2 + \vec{u}_2 \cdot \nabla\hat{u}_1{}^*]/A^2 d\vec{r} \Big/ \int \hat{u}_1{}^* \cdot \hat{u}_1\, d\vec{r} \tag{14}$$

Finally, the terms involving $\vec{v}_1$ in (11) give rise to third-order (in A) corrections to the evolution of dA/dt, so that they too contribute to the first Landau coefficient in the amount k_3 of the form

$$k_3 = -2\mathrm{Re}\int \hat{u}_1{}^* \cdot P(\nabla)[\vec{U} \cdot \nabla\vec{v}_1 + \vec{v}_1 \cdot \nabla\vec{U}]/(|A|^2 A) d\vec{r} \Big/ \int \hat{u}_1{}^* \cdot \hat{u}_1 d\vec{r} \tag{15}$$

In conclusion, (5) is established to hold with the Landau constnat

$$k = k_1 + k_2 + k_3 \tag{16}$$

Here k_1 <u>originates from the nonlinear distortion of the mean flow</u>; k_2 <u>originates from the generation of harmonics</u>; and k_3 is due to the <u>nonlinear distortion of the fundamental</u>. This interpretation of the Landau constant k, given originally by J.T. Stuart (1960), is important to the understanding of nonlinear effects on flows.

In the linearly unstable parameter regions of circular Couette flow, k_1, k_2, k_3 are each positive with k_1 the dominant term. Consequently, Couette flow exhibits supercritical equilibrium states. On the other hand, Reynolds and Potter (1967) showed that plane Poiseuille flow is such that $k_1 > 0$,

$k_2 > 0$, $k_3 < 0$ while $k < 0$. In other words, distortion of the fundamental dominates and subcritical instability is to be expected.

It is observed experimentally that as T increases above T_{cr}, finite amplitude Taylor cells become unstable to nonaxisymmetric disturbances. In fact, if $T/T_{cr} \gtrsim 1.08$, the axisymmetric Taylor vortex develops waviness. This effect was studied by Davey (1962) by considering the perturbed flow

$$\begin{aligned}\vec{u} &= \vec{U}(r) + (A + a(t))\hat{u}(r)\cos \alpha z + b(t)\vec{v}(r)\sin\alpha z \\ &\quad + c(t)\vec{w}(r)\cos\alpha z \cos n\phi + d(t)\vec{x}(r)\sin\alpha z \cos n\phi.\end{aligned} \tag{17}$$

Substituting of (17) into the Navier-Stokes equations, rearranging the nonlinear terms, and finally equating coefficients of like functions gives the (Galerkin) equations

$$\begin{aligned}\dot{a} &= -2c_i a(t) \\ \dot{b} &= 0 \\ \dot{c} &= (\gamma + A\delta_1)c \\ \dot{d} &= (\gamma + A\delta_2)d\end{aligned} \tag{18}$$

It follows from (18) that a(t) is always decreasing for $T > T_{cr}$. Also, it turns out that $\gamma + A\delta_1 < 0$ for all T, but that $\gamma + A\delta_2 > 0$ for $T/T_{cr} \gtrsim 1.08$ and $n = 1$. Hence, d(t) can increase with time and the Taylor cells are unstable to non-axisymmetric disturbances when $T > (1.08)T_{cr}$.

Coles (1965) performed a careful series of experiments on Couette flow and found a complicated series of transitions with increasing Taylor number. He observed that, as T increased, Taylor cells with axial wavenumber α and azimuthal wavenumber n appeared in the sequence

α	n	
28	0	(axisymmetric Taylor vortices)
28	4	
24	5	
22	5	
22	6,	

etc. Detailed resolution of this phenomenon awaits clarification.

5. <u>Boundary-layer transition on a flat plate</u>.

Consider a steady viscous incompressible fluid in the parallel flow state $(u,v,w) = (\bar{u}(z),0,0)$. (see Fig. 3.) The flow can be quasi-parallel as in the case of a boundary layer, since as will be explained later the non-parallel effects are relatively minor. If the perturbed flow state velocities are denoted by $(\bar{u}(z) + \varepsilon u,\ \varepsilon v,\ \varepsilon w)$, where ε is an amplitude parameter, the appropriate equations are

$$u_x + v_y + w_z = 0, \tag{19}$$

$$u_t + \bar{u}u_x + \bar{u}_z w + \varepsilon S_1 = -p_x + \frac{1}{R}\Delta u, \tag{20}$$

$$v_t + \bar{u}v_x + \varepsilon S_2 = -p_y + \frac{1}{R}\Delta v, \tag{21}$$

$$w_t + \bar{u}w_x + \varepsilon S_3 = -p_z + \frac{1}{R}\Delta w. \tag{22}$$

In these equations p is the pressure, R the Reynolds number, Δ the three

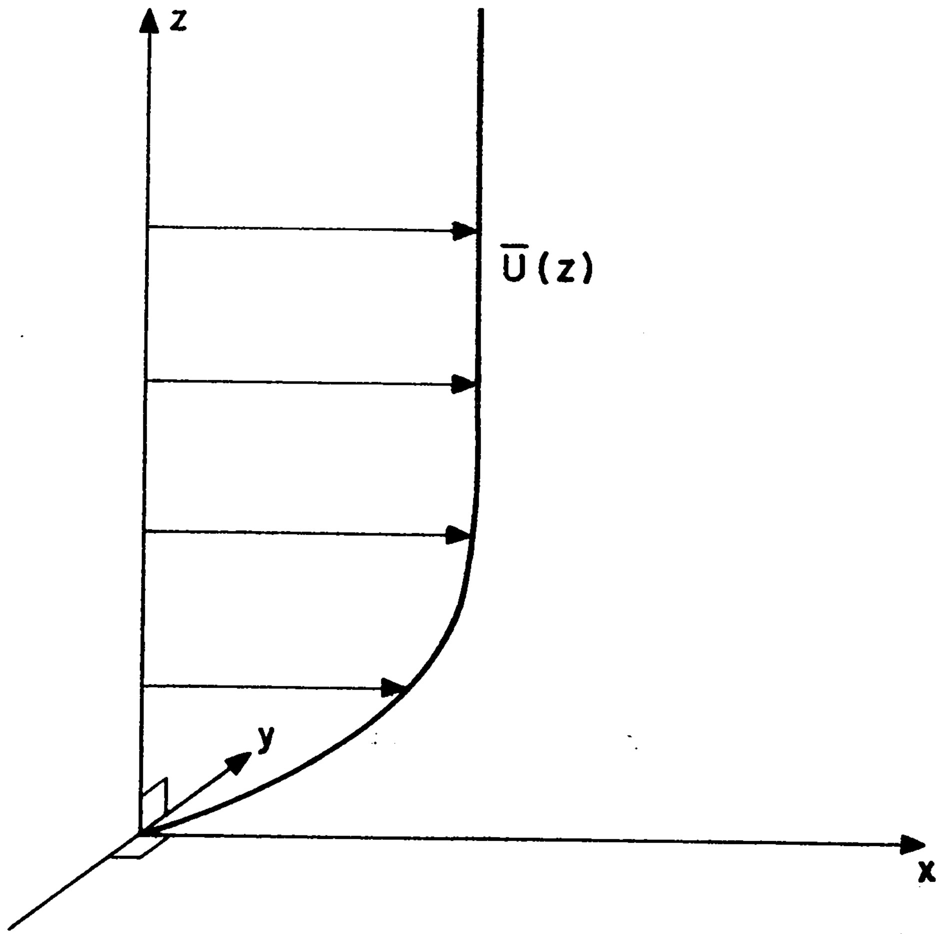


Fig. 3. The Blasius boundary layer profile.

dimensional Laplacian and S_i, $i = 1, 2, 3$ are the familiar nonlinear terms:

$$S_1 = uu_x + vu_y + wu_z, \tag{23}$$

$$S_2 = uv_x + vv_y + wv_z, \tag{24}$$

$$S_3 = uw_x + vw_y + ww_z. \tag{25}$$

The elimination of the pressure from (19 - 22) leads to the two basic partial differential equations:

$$(\bar{u}\frac{\partial}{\partial x} + \frac{\partial}{\partial t})w - \bar{u}_{zz}\frac{\partial w}{\partial x} + \varepsilon[\Delta_1 S_3 - \frac{\partial^2 S_1}{\partial x \partial z} - \frac{\partial^2 S_2}{\partial y \partial z}] = \frac{1}{R}\Delta\Delta w. \tag{26}$$

$$(\bar{u}\frac{\partial}{\partial x} + \frac{\partial}{\partial t})\zeta - \bar{u}_z\frac{\partial w}{\partial y} + \varepsilon(\frac{\partial S_2}{\partial x} - \frac{\partial S_1}{\partial y}) = \frac{1}{R}\Delta\zeta. \tag{27}$$

Where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the z component of the perturbation vorticity. For the linear theory, $\varepsilon = 0$, and it is possible to use a Fourier decomposition in x and y so that any amplitude function is written in the form

$$f(x,y,z,t) = \hat{f}(z)\, e^{i\alpha x + i\beta y - i\alpha c t}, \tag{28}$$

and the basic equations become

$$(\bar{u}-c)\left(\frac{d^2}{dz^2} - \alpha^2 - \beta^2\right)\hat{w} - \bar{u}_{zz}\hat{w} = \frac{1}{i\alpha R}\left(\frac{d^2}{dz^2} - \alpha^2 - \beta^2\right)^2 \hat{w}, \tag{29}$$

$$i\alpha(\bar{u}-c)\hat{\zeta} - i\beta\bar{u}_z\,\hat{w} = \frac{1}{R}\left(\frac{d^2}{dz^2} - \alpha^2 - \beta^2\right)\hat{\zeta}. \tag{30}$$

In terms of $\hat{w}$ and $\hat{\zeta}$ the remaining two velocity components u and v are found to be

$$\hat{u} = \frac{i}{\alpha^2+\beta^2} (\beta\hat{\zeta} - \alpha\hat{w}_z), \tag{31}$$

$$\hat{v} = - \frac{i}{\alpha^2+\beta^2} (\alpha\hat{\zeta} - \beta\hat{w}_z) \tag{32}$$

In lecture 2, we briefly reviewed the linear theory of hydrodynamic stability as it pertains to boundary layers. Of course this is merely the first step towards an understanding of the transition process, but it is an important step. The linear theory does give an adequate representation of the motion when the disturbances are small, and under controlled conditions it is capable of predicting the onset of instability and the most unstable mode at any location. However, as the amplitude increases the process is no longer linear and both the mean flow modifications as well as higher harmonics begin to appear. To develop a theory which follows the detailed amplications of the waves is a complicated matter. Fortunately much of the motivation for nonlinear theories is provided by some careful measurements. Schubauer [1957] found that the two dimensional Tollmien-Schlichting waves inevitably became three dimensional before leading to breakdown and the formation of the turbulent spots studied earlier by Emmons [1951]. The importance of three dimensionality had been reported by many experimenters using water and dye methods: Hama, Long and Hegarty [1957], Fales [1957], Weske [1957] and by Meyer and Kline [1961]. Indeed the simple fact that turbulence is essentially three dimensional shows that any two dimensional theoretical approach is doomed to failure. Using hot wire techniques detailed and definitive information on the nonlinear processes up to and including breakdown was obtained by Klebanoff, Tidstrom and Sargent [1962] and by Kovasznay, Komada and Vasudeva [1962]. It was found that events take place in the following sequence:

1. Two dimensional linear waves.
2. Waves become strongly three dimensional.
3. Longitudinal vortices redistribute momentum in the boundary layer.
4. Breakdown in the form of an initial burst of high frequency appears as a secondary instability at predictable positions and times corresponding to where and when the modified profile is locally inflectional.
5. Development and cross contamination of turbulent spots to form a fully developed turbulent boundary layer.

The length and existence of some of these stages depends on whether the initiation is natural or controlled and on the tunnel turbulence level. The mechanisms are most easily identified in the controlled experiments where an excitation of a definite frequency and spanwise length is introduced by means of a vibrating ribbon with periodic spacers.

Various theoretical models have been proposed to explain this nonlinear regime. Among these was the Görtler Witting model [1957] in which the formation of the observed longitudinal vortices is based on an instability of the curved flow due to the two dimensional waves. However, experiments showed that the location of the most intense vortex structure was exactly out of phase with the theoretical prediction.

The Landau interactions of nonlinear stability outlined in sections 3 and 4, while certainly present in boundary layer transition, are not dominant and do not lead to the explosive small scale instabilities characteristic of boundary layer transition. The Landau concepts are very relevant and important in the case of slow (as opposed to fast) transition processes e.g. thermal convection and rotating cylinders where the large scale structure is persistent. Along similar lines, the possibility of wave resonances being responsible for transition has been suggested by Raetz [1959].

The most plausible theory compatible with the National Bureau of Standards experiments, is the one proposed by Lin and Benney [1961, 1962, 1964]. In this approach a two and three dimensional wave are allowed to interact and the second order effects are calculated. Typical mean cross flow patterns are shown in Fig. 4 in which the sequence is ordered as the ratio of three dimensional amplitude ε_3 to two dimensional amplitude ε_2 is increased. The existence and movement of these longitudinal vortices and other general predictions are in remarkably good agreement with the Klebanoff experiments. Associated with the secondary mean flow the local boundary profile undergoes spanwise modifications and tends to develop a point of inflection at about $z = .6\delta$. Once each cycle this tendency is accentuated by the oscillatory vortex structure of the three dimensional wave. The most intense vortex occurs when the streamline is convex, contrary to the Görtler Witting theory. It is at these well-defined positions and times that the first burst of high frequency originates as a new instability near inflectional points in the evolved mean velocity profile. A linear stability analysis of this type of shear layer was made by Greenspan and Benney [1963]. The fact that the energy associated with the new instability can increase by a factor of 100 over one cycle of the primary wave gives convincing evidence that it is the mechanism responsible for breakdown.

The simple picture of processes 4 and 5 is the continuous creation of local instabilities (leading to the birth of turbulent spots) at favorable positions and times corresponding to the most intense shear layer. At a

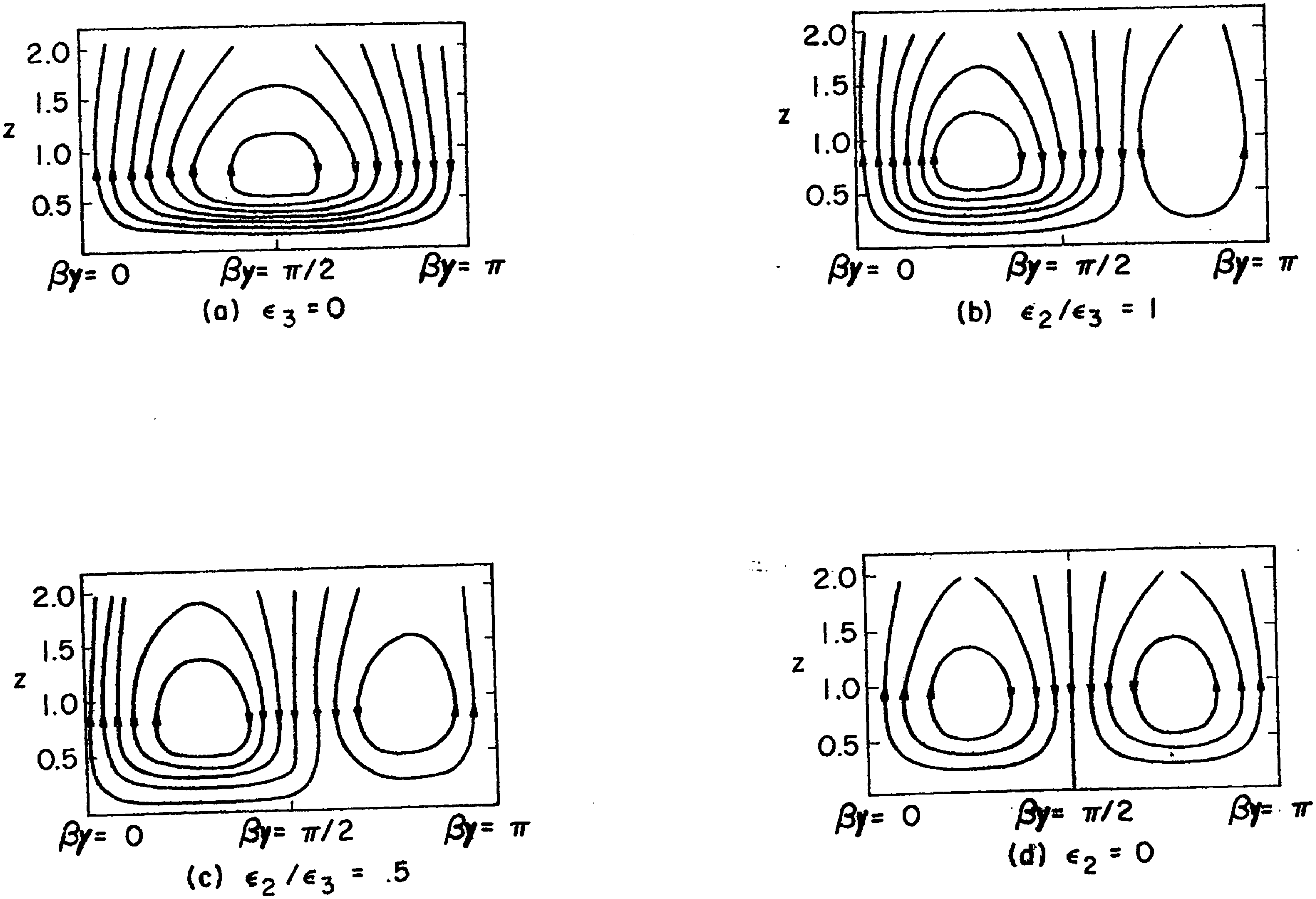


Fig. 4. Streamlines of velocity.

fixed position optimal conditions will occur once during each cycle of the primary wave. This new instability wave is convected downstream with a speed c_2 (corresponding to the local inflectional speed; c_2 being approximately twice the speed c_1 of the Tollimein-Schlichting wave. A convenient representation of this idealization is given in the x-t plane as indicated in Fig. 5. Here $x = 0$ represents the positions of the onset of the secondary instability. Measurements made just beyond $x = 0$ show bursts of the new instability which have a scale consistent with experiments. Two primary wave lengths beyond the initial breakdown, the flow is essentially a fully developed turbulent boundary layer.

While it may well be that a purely two dimensional wave can produce transition if its amplitude is large enough the inevitable three dimensionality certainly accelerates the chain of events. In natural transition, where the three dimensionality is not as organized as in the controlled case it is expected (and found) that the primary instability will be longer in extent and the breakdown will occur at less regular spanwise positions. On the other hand when the initial disturbance is of relatively large amplitude, breakdown is attained rapidly. A precise theoretical explanation of the vortex spacing remains elusive, but experimentally it is found that there is definitely a preferred spacing. Several qualitative arguments have been advanced to explain this fact. One such argument notes that any linear three dimensional fluctuation component has the form $\beta e^{(\alpha_0 - k^2\beta^2)t}$ where β is the spanwise wave number. This quantity has a maximum when $k\beta\sqrt{2t}=1$, and for interaction times of two or three periods, this value of β has the right order of magnitude. Another proposal is to use the amplitude equations and invoke nonlinearity to select β. This calculation is involved and has not yet been made. In any case the

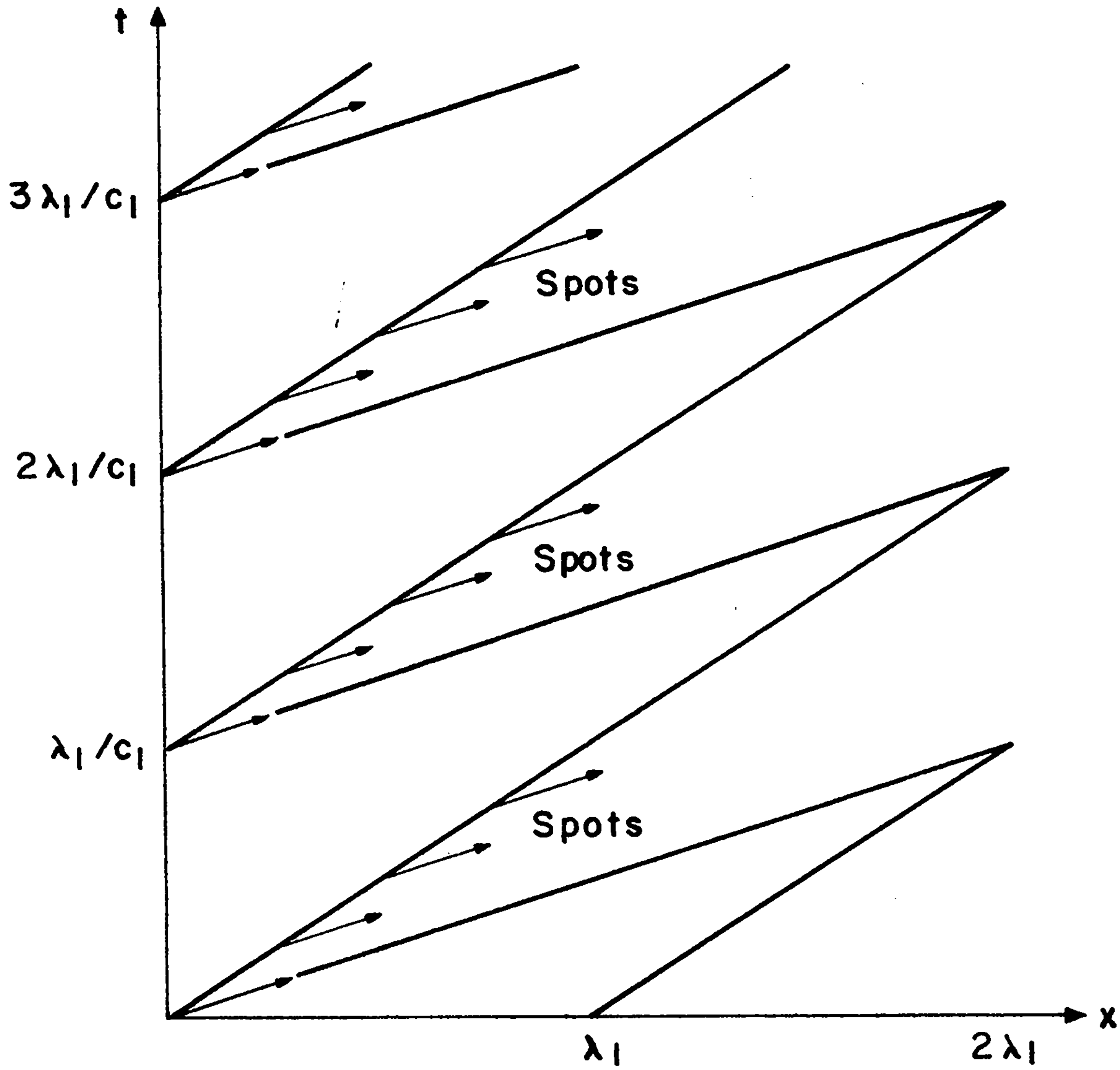


Fig. 5. x-t representation showing favorable locations for onset of secondary instability.

cross flow velocities are larger than would be expected since a factor of $(\alpha R)^{1/3}$ or $\varepsilon^{-1/2}$ multiplies these amplitudes depending on whether the theoretical development is viscous or nonlinear. In this sense the presence of three dimensional wave motions is not unexpected.

Other studies along these lines have been performed by Benney and Bergeron [1969] and by Landahl [1972] and while the results are of considerable theoretical interest they provide no contradictions to the basic ideas already outlined.

The ability to give a precise and deterministic prediction for the transition point is the fundamental goal of this investigation. No ad hoc assumptions are included so that our approach is a direct numerical attack, guided by the known theoretical and experimental evidence. For this reason it is to be distinguished from the other "reliable rule of thumb" type methods (see Smith [1957] and Kaplan [1974]. All the evidence suggests that the preceding theoretical picture is the correct one. Recent numerical calucations have been performed (Orszag, 1974) with the goal of obtaining a good simulation of the known experimental and theoretical facts in the nonlinear wave regime up to and including breakdown. Some preliminary results from this study are shown in Figs. 6 - 8. These results were obtained by direct numerical solution of the Navier-Stokes equations. Details of these calculations are presented elsewhere (Orszag, 1974).

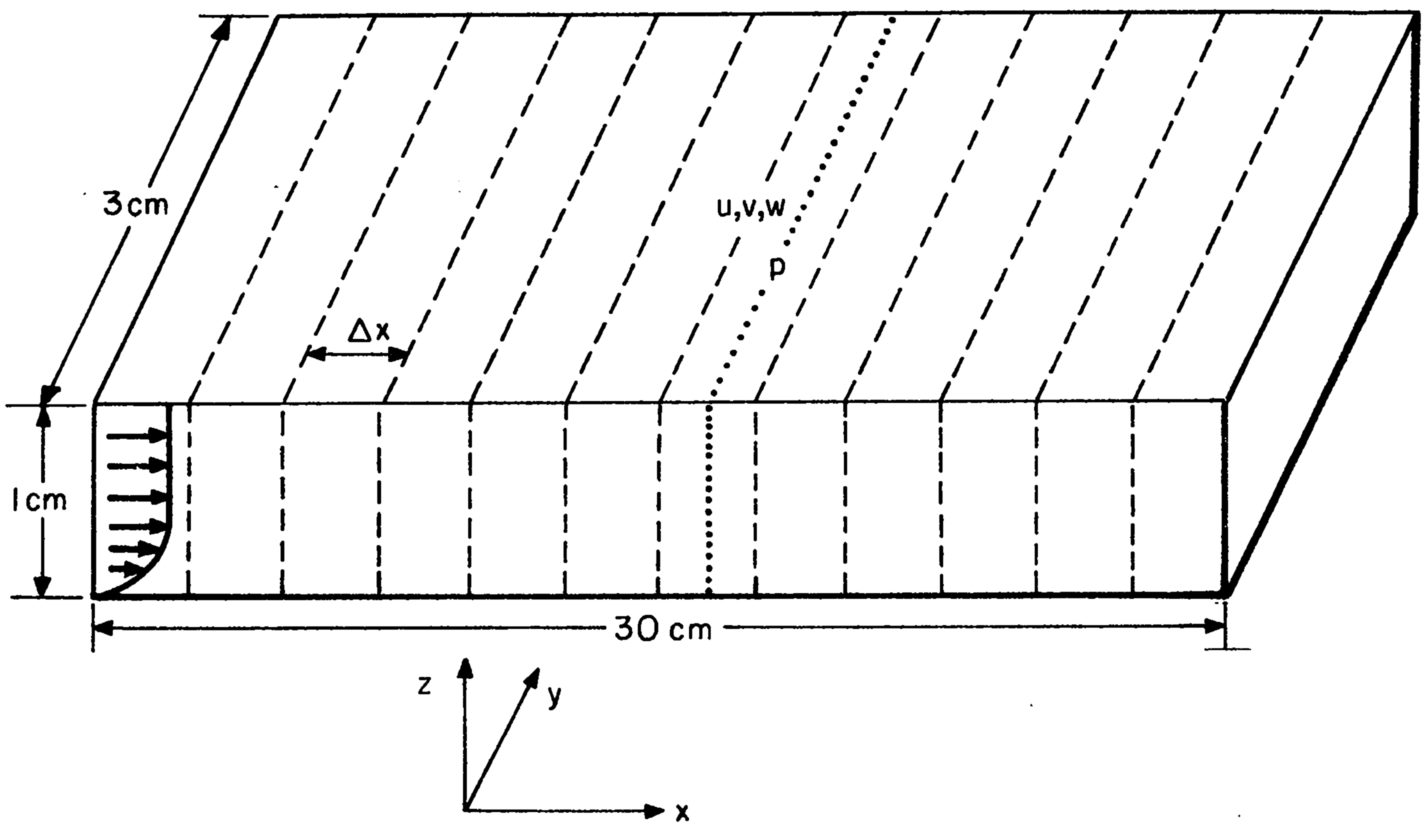


Fig. 6. Computational domain for transition experiments: Δx=30 cm/128, 8 Fourier modes in y, 65 Chebyshev polynomials in z. Inflow conditions: $\delta = \sqrt{\nu X/U} \simeq .1$ cm, $U \simeq 1500$ cm/s, $R \simeq 10^3$, $X \simeq 100$ cm.

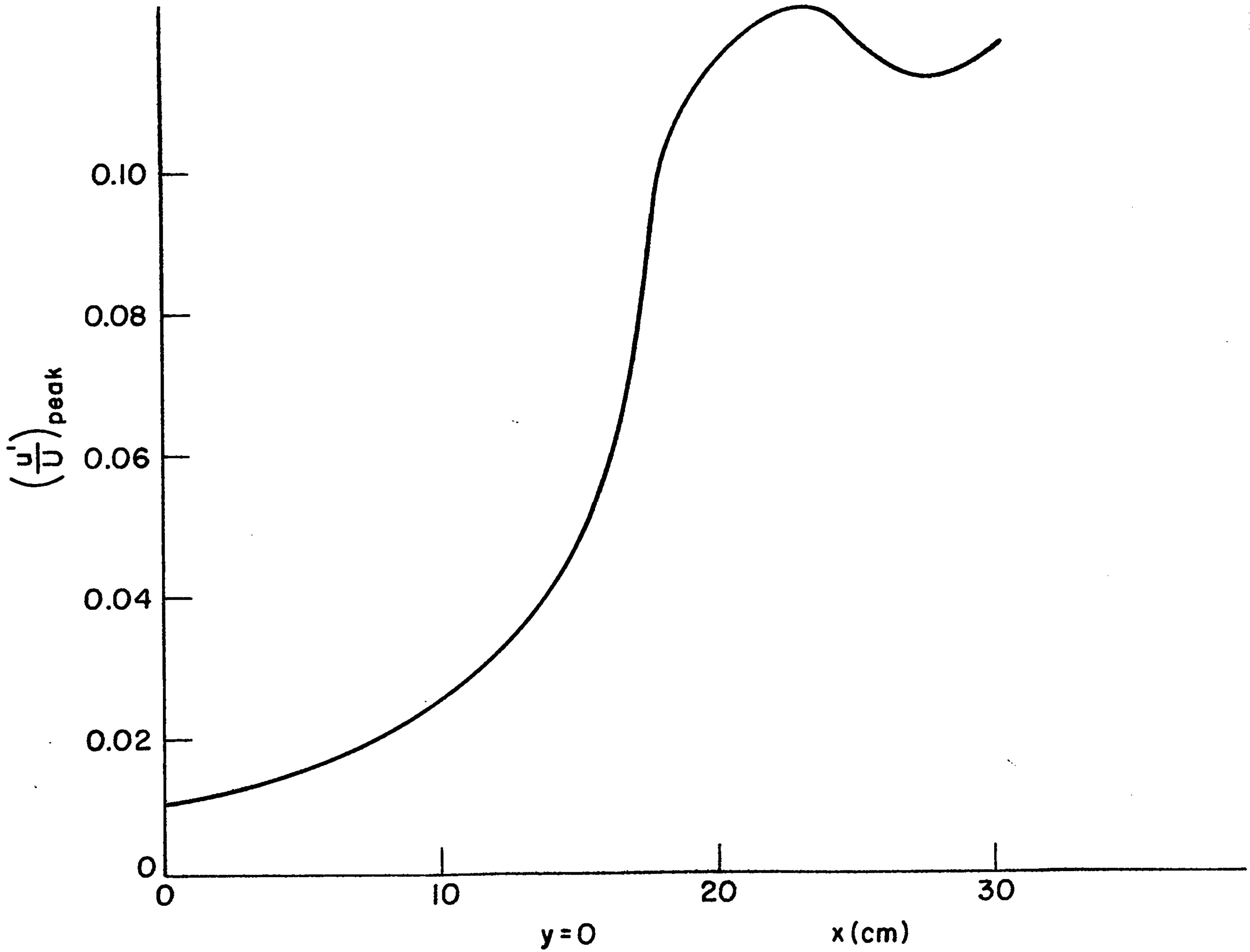


Fig. 7. Downstream variation of peak disturbance amplitude. At x=0 (upstream boundary), disturbance is 1% two-dimensional Tollmien-Schlicting wave with .15% three-dimensional perturbation.

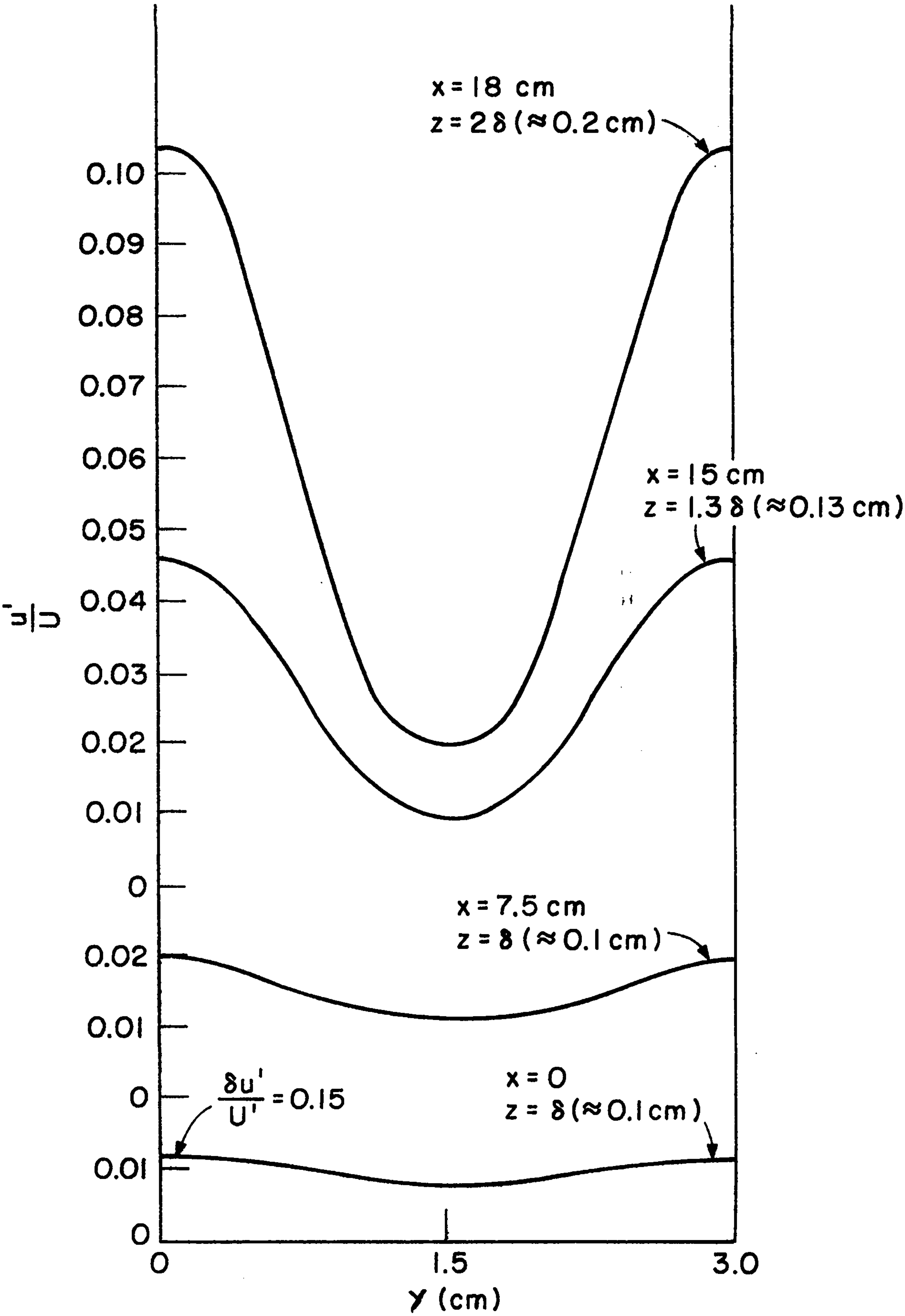


Fig. 8. Cross-stream (y) variation of disturbance amplitude for several downstream (x) locations.

REFERENCES

Benney, D.J. (1961), J. Fluid Mech. 10, 209.

Benney, D.J. (1964), Phys. Fluids 7, 319.

Benney, D.J. and Bergeron, R.F. (1969), Stud. in Appl. Math. 48, 181.

Busse, F.H. (1967) J. Math. and Phys. 46, 140.

Champagne, F.H. and Wygnanski, I. (1973), J. Fluid Mech.

Chandrasekhar, S. (1961), Hydrodynamic and Hydromagnetic Stability, Oxford University Press.

Coles, D. (1965), J. Fluid Mech. 21, 385.

Davey, A. (1962), J. Fluid Mech. 14, 336.

Davey, A. (1973), J. Fluid Mech. 57, 369.

Emmons, H.W. (1951), J. Aero. Sci. 18, 490.

Fales, E.N. (1957), J. Franklin Inst. 259, 491.

Görtler, H. and Witting, H. (1957), Bound Layer Res. Symp., Freiburg, p. 110.

Hama, F.R., Long, J.D. and Hegarty, J.C. (1957), J. Appl. Phys. 28, 388.

Kaplan, R.E. (1974), to be published.

Klebanoff, P.S., Tidstrom, K.D. and Sargent, L.M. (1962), J. Fluid Mech. 12, 1.

Kovasznay, L.S.G., Komada, H., and Vasudeva, B.R. (1962), Proc. 1962 Heat Transfer and Fluid Mech. Inst., p. 1.

Landahl, M.T. (1972), J. FLuid Mech. 56, 775.

Landau, L. (1944), DAN SSSR 44, 339

Landau, L. and Lifschitz, E.M. (1959), Fluid Mechanics, Addison-Wesley.

Lin, C.C. and Benney, D.J. (1962), Proc. Symp. Appl. Math. 13, 1.

Metcalfe, R.W. and Orszag, S.A. (1974), to be published.

Meyer, K.A. and Kline, S.J. (1961), Thermoscience Div., Stanford Univ., Report MD-7.

Orszag, S.A. (1974), to be published.

Palm, E. (1960), J. Fluid Mech. 8, 183.

Raetz, G.S. (1959), Northrop Report NOR-59-383.

Reynolds, W.C. and Potter, M.C. (1967), J. Fluid Mech. 27, 465.

Salwen, H. and Grosch, C.E. (1972), J. Fluid Mech. 54, 93.

Schubauer, G.B. (1957), Bound. Layer Res. Symp., Freiburg, p. 85.

Smith, A.M.O. (1957), Int. Cong. Appl. Mech. 4, 234.

Stuart, J.T. (1960), J. Fluid Mech. 9, 353.

Taylor, G.I. (1923), Phil. Trans. Roy. Soc. London A223, 289.

Veronis, G. (1970), Ann. Rev. Fluid Mech. 2, 37.

Watson, J. (1960), J. Fluid Mech. 9, 371.

Weske, J.R. (1957), Univ. Maryland Tech. Note BN-91.

Zahn, J.P., Toomre, J., Spiegel, E.A. and Gough, D.O. (1974), J. Fluid Mech. 64, 319.

Lecture 4

Nonlinear Instabilities

In this lecture, a single article [1] on nonlinear instabilities will be described briefly, in the hope of giving a little of the flavor of recent work in that area.

It is recalled from lectures 2 and 3 that plane Poiseuille flow (pressure-driven flow between fixed parallel plane walls) is stable with respect to infinitesimal 2-dimensional disturbances (rolls), in which the perturbing stream function ψ is of the form

$$\psi(x, z, t) = \psi_0(z)\, e^{-i\alpha(x - ct)}$$

provided that the point R,α lies outside the shaded region of the R,α plane shown in Figure 1. Here, R is the Reynolds number based on the half thickness of the channel and on the fluid speed at its center; x is the downstream coordinate, and z the transverse coordinate. According to Squire's Theorem, it suffices to consider 2-dimensional disturbances for determining the onset of instability, which occurs for Reynolds number $R = 5772.22$ and wave number $\alpha = 1.0206$, according to accurate calculations of Orszag [2] using Chebyshev polynomials to solve the Orr-Sommerfeld eigenvalue problem for this case.

The main conclusion of the article under discussion is that, for certain points R,α outside the shaded area, the flow is metastable or unstable with respect to rolls that have the same general form as the infinitesimal ones, but are of finite amplitude.

The authors consider three types of disturbance, called cases I, II, and III; the first two of them are 2-dimensional and will be discussed first. In all three, the disturbance is assumed periodic in x, hence representable by a Fourier series. The Navier-Stokes equations are

$$\begin{aligned}
(\partial_t + u\partial_x + w\partial_z)u &= -\frac{1}{\rho}\partial_x p + \nu\nabla^2 u \quad , \\
(\partial_t + u\partial_x + w\partial_z)w &= -\frac{1}{\rho}\partial_x p + \nu\nabla^2 w \quad , \\
\partial_x u + \partial_z w &= 0 \quad ,
\end{aligned} \tag{1}$$

in the usual notation. They hold for all x and for $-1 \leqslant z \leqslant 1$; the half thickness of the channel has been taken as the unit of length. The boundary condition is that u and w are $= 0$ for $z = \pm 1$. The Fourier series are written in the form

$$\begin{aligned}
u &= U(z,t) + \mathrm{Re}\sum_{n=1}^{\cdots} u_n(z,t)\, e^{-i\alpha nx} = U + u' \\
w &= \cancel{W(z,t)} + \mathrm{Re}\sum_{n=1}^{\cdots} w_n(z,t)\, e^{-i\alpha nx} = W + w' \quad .
\end{aligned} \tag{2}$$

We show now that W(z,t) is zero, as indicated. The third equation of (1) is

$$\partial_x U + \partial_x u' + \partial_z W + \partial_z w' = 0 \quad .$$

The first term in this equation is zero, because U does not depend on x. We average the other terms with respect to x over a period (e.g., for $0 \leqslant x \leqslant 2\pi/\alpha$). The second and fourth terms average to zero, because they are Fourier series with the constant term missing. Hence, the average of $\partial_z W$ is zero, but $\partial_z W$ does not depend on x, so it is zero. Therefore W is a constant, which is $= 0$ by the boundary conditions.

Equations (1) show that $\partial_x p$ is periodic, but not of course p itself. In fact, p is of the form

$$p = - P_0 x + \text{Fourier series} \quad , \tag{3}$$

where P_0 is the pressure gradient that drives the undisturbed flow.

We come now to a question that is somewhat puzzling to nonspecialists reading papers in this field. When one speaks of an infinitesimal disturbance, it is perfectly clear what is understood as the undisturbed or basic flow. Not so for finite disturbances, for one can add a finite velocity field to what is called the disturbance and subtract the same field from what is called the basic flow. In the present problem, one might insist that the basic flow have the same form as for the linearized theory, $U = U_0(1 - z^2)$, but it is no longer clear what value of the Reynolds number to ascribe to the finitely disturbed flow. At least three conventions are possible, as follows: The Reynolds number is $R = LV/(\mu/\rho)$, where in any case L is taken as the half width of the channel. Three choices for the characteristic speed V are:

(experimentalists): V = average x component of the actual velocity at the center of the channel.

(theorists) : V = the value of U_0 for a basic flow $U_0(1 - z^2)$ having the same pressure gradient P_0 as the flow under study.

(engineers) : V = the value of U_0 for a basic flow $U_0(1 - z^2)$ having the same flow rate (gallons per minute) as the flow under study.

The second choice is the one used in the paper. The equation for the laminar flow is, from (1),

$$\frac{P_0}{\rho} + \nu \frac{d^2U}{dz^2} = 0 \quad ,$$

which gives the parabolic profile $U = U_0(1 - \frac{z^2}{L^2})$. In units such that $L = 1$, $U_0 = 1$, we have

$$\frac{P_0}{\rho} = 2\nu \; , \quad R = \frac{1}{\nu} = \frac{\rho}{\mu} \tag{4}$$

The idea of the calculation is to truncate the Fourier series (2), substitute them into the Navier-Stokes equation, then equate to zero the net coefficient of $e^{-i\alpha nx}$, for each n, thus obtaining partial differential equations for the functions $U(z,t)$, $u_n(z,t)$, and $w_n(z,t)$. The truncation is at $n = 1$ and $n = 2$ in cases I and II, respectively. Such drastic truncation is presumably justifiable if conditions are quite close to the stability limit, for then the magnitude of the $n^{\underline{th}}$ harmonic ought to contain the $n^{\underline{th}}$ power of some small but finite quantity ε, according to (1). Note that truncation at $n = 1$ does not bring us back to the linear theory, because the nonlinear terms give a reaction of the term in $n = 1$ back on the term $U(z,t)$.

To carry out this program, we next equate to zero the net coefficient of the $n = 0$ terms in the first equation of (2). The term $u\partial_x u$ contributes nothing, because $\frac{1}{2}u^2$ is a Fourier series and its derivative has no constant term. In the pressure gradient term, the first equation of (4) is used, and we find

$$\partial_t U - \nu(\partial_z^2 + 2)U + \mathrm{Re} \sum_{n=1}^{\cdots} w_n^* \partial_z u_n = 0 \qquad (5)$$

The last term in this equation gives the reaction of the disturbance on the basic flow $U(z,t)$.

In the remaining equations it is convenient to introduce the stream function ψ, such that

$$u' = \partial_z \psi \ , \quad w' = -\partial_x \psi \quad ,$$

and then to write its Fourier series as

$$\psi = \mathrm{Re} \sum_{n=1}^{\cdots} \phi_n (z,t)\, e^{-i\alpha n x} \quad ; \qquad (6)$$

then

$$u_n = \partial_z \phi_n \qquad w_n = i\alpha n \phi_n \quad ,$$

and (5) is rewritten as

$$\partial_t U - \nu(\partial_z^2 + 2)U + \alpha \operatorname{Im} \sum_{n=1}^{\cdots} n \phi_n^* \partial_z^2 \phi_n \tag{7}$$

For Case I, there is only one term in the summation, above, and it can be rewritten as simply $\alpha \operatorname{Im} \phi^* \partial_z^2 \phi$, by dropping the subscripts. In this case, there is only one other equation, which, after a Galilean transformation $x \to x - ct$, is

$$\partial_t(\partial_z^2 - \alpha^2)\phi + i\alpha\left[\partial_z^2 U - (U - c)(\partial_z^2 - \alpha^2)\right]\phi - \nu\left(\partial_z^2 - \alpha^2\right)^2 \phi = 0 \quad . \tag{8}$$

It is seen that ϕ satisfies a linear equation, but influences U through (7) and the resulting alteration of U reacts back on ϕ through (8), so that the whole system (7) + (8) is nonlinear. The system is of the fifth order, and the corresponding difference equation is chosen to be implicit in time, so that one must solve a large 5×5 block tridiagonal system. The corresponding steady-state equations were also used in the study. If the ∂_t terms are set = 0, (8) is just the Orr-Sommerfeld equation, but of course the coefficients U and $\partial_z^2 U$ are not the same as for the linearized problem. (The linearized equations have a steady-state solution for the critical case if c is chosen as the velocity with which the rolls move downstream.)

The main conclusions for Case I are the following: For fixed values of R, α corresponding to any point outside the shaded area in Figure 1, if the initial disturbance $\phi(z,0)$ is small enough and $U(z,0)$ is close enough to the basic flow

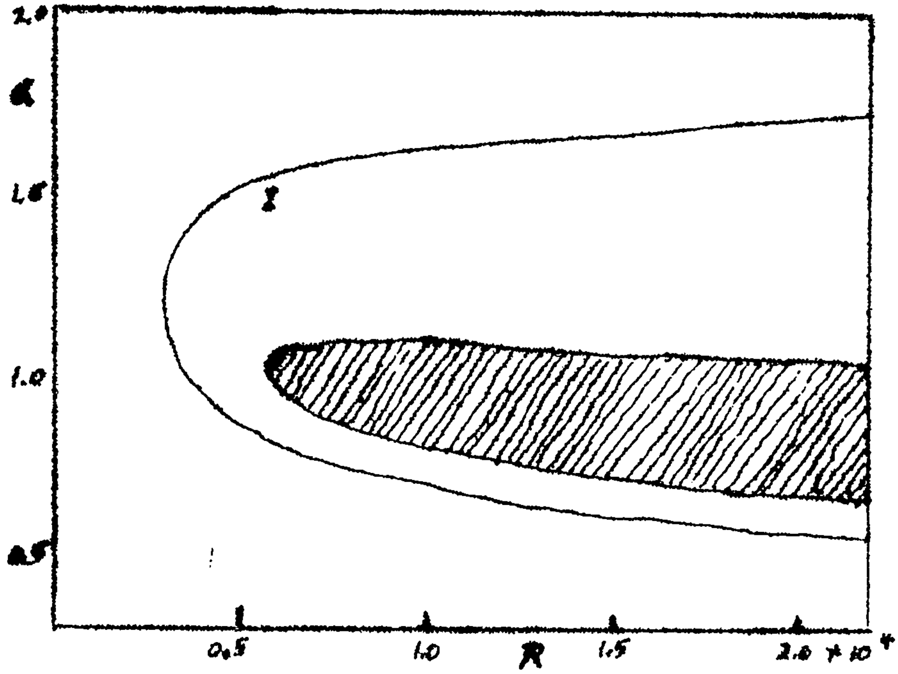


Fig. 1

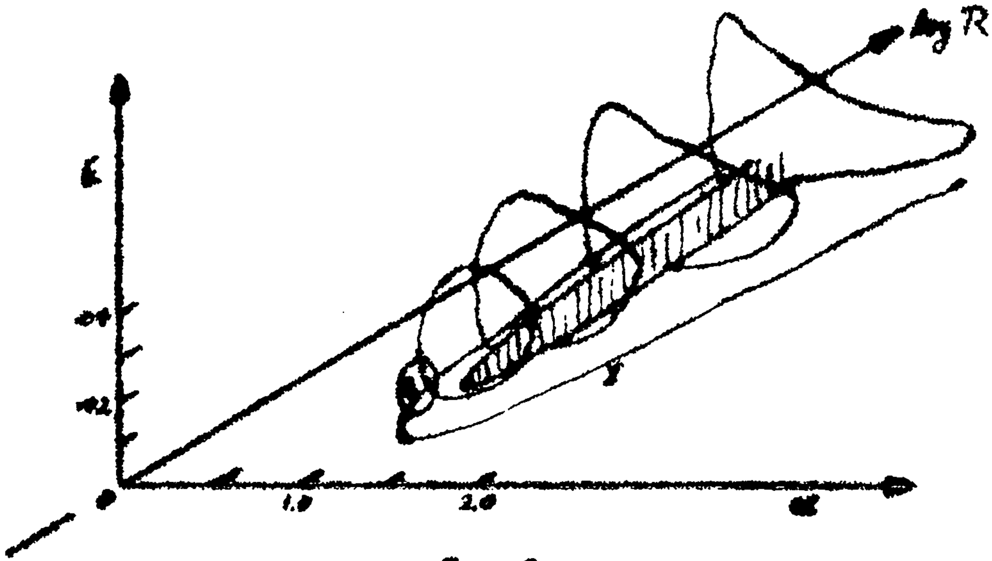


Fig. 2

$1 - z^2$, then the disturbance dies out, as t increases. On the other hand, if the initial disturbance is large enough, and if R, α is any point in the larger area enclosed by the solid curve labeled I, the solution settles down to a set of finite amplitude rolls, as $t \to \infty$. If the velocity c, which appears in (8) because of the Galilean transformation that was made, is set (it is adjustable, in the authors' code) equal to the speed with which the rolls move downstream, then the solution is asymptotically independent of t. Such stationary solutions were then studied by eliminating the time dependence in the code. The magnitude of the disturbance can be expressed in various ways, for example by giving the energy

$$E = \text{const.} \int_{-1}^{1} \left(u'^2 + w'^2\right) dz$$

of the disturbance in the moving reference frame. When that is done, the various steady solutions that were found lie on a surface in the 3-dimensional space R, α, E indicated in Figure 2. It has roughly the shape of the front of a submarine hull lying over the R, α plane but resting on that plane on the shaded area shown in the figures, where the linearized theory predicts instability. The projection of the surface onto the R, α plane is the area enclosed by the solid curve in Figure 1. Points on the under side of the surface represent unstable equilibria, in the sense that there are corresponding steady solutions but a small departure from such solution grows in time, while points on the upper surface represent

REFERENCES

[1] J.P. Zahn, J. Toomre, E.A. Spiegel, and D.O. Gough, "Non-linear cellular motions in Poiseuille channel flow," J. Fluid Mech. 64, 319 (1974).

[2] S.A. Orszag, "Accurate solution of the Orr-Sommerfeld stability equation," J. Fluid Mech. 50, 689 (1971).

stable equilibria toward which a solution can tend either from above or from below (with respect to the energy E) for given R and α.

Case II has two Fourier terms containing ϕ_1, ϕ_2, in addition to U(z,t). The results are qualitatively similar but somewhat more complicated in certain respects; for a detailed description the reader is referred to the original article.

In Case III, the disturbance is three dimensional. There is a single Fourier term in x, but it depends also on y and has the form

$$\cos \beta y \, e^{-i\alpha x} \quad .$$

If the quantity α of Figures 1 and 2 is reinterpreted as the total horizontal wave number $\sqrt{\alpha^2 + \beta^2}$, then, it is found, the "submarine hull" shrinks down in size, as β is increased, thus suggesting a sort of finite-amplitude version of Squire's theorem: the 2-dimensional disturbances are the most unstable, in a sense.

As pointed out by the authors, the rolls obtained by their calculations are not observed in experiments; instead, there is a more chaotic motion. In order to explain the discrepancy, it may be necessary to consider disturbances of a still more complicated form, or possibly to refine the experiments, for example by reducing edge and end effects, or reducing the ambient level of residual turbulence that enters the channel.