Conf-760805--18

LA-UR-76-1518

TITLE: PERTURBATION METHODS APPLIED TO PROBLEMS IN DETONATION PHYSICS

AUTHOR(S): John B. Bdzil

SUBMITTED TO: The Sixth Symposium on Detonation, San Diego, California, August 24-27, 1976.

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87545

MASTER

Form No. 836
St. No. 2629
1/75

UNITED STATES
ENERGY RESEARCH AND
DEVELOPMENT ADMINISTRATION
CONTRACT W-7405-ENG. 36

PERTURBATION METHODS APPLIED TO PROBLEMS IN DETONATION PHYSICS*

J. B. Bdzil
Los Alamos Scientific Laboratory
University of California
Los Alamos, New Mexico 87545

A theoretical study of an explosive which releases a small fraction, δ^2, of its total energy via resolved reactions is presented. Two separate problems are treated. First, a time-dependent one-dimensional unsupported detonation is considered. It is shown that to $O(\delta)$ the detonation is a reactive simple wave. The particle velocity profiles are calculated for a model explosive. Second, the detonation edge effect for a steady-state semi-infinite unconfined detonation is considered. It is shown that the near-field flow is dominated by the Prandtl-Meyer singularity, whereas the far-field flow is controlled by the reactivity and streamline divergence. The shock locus, sonic locus, and limiting characteristic are calculated and the effects of confinement are discussed.

I. INTRODUCTION

Detonation physics is primarily concerned with understanding the rather complex subject of reactive nonlinear hydrodynamics. As a result of this, the number of analytic solutions describing detonation problems is very small. In part, this has led to a heavy reliance on numerical solution methods for these problems. By their very nature, these methods are only marginally suitable for parameter variation studies; thus they often provide little or no guidance as to the nature of the governing physics.

For a small class of problems, analytical solutions can be found using modern singular perturbation theory. The purpose of this paper is to describe the underlying scaling principles of these methods, and to show how they filter the relevant physics from the full governing equations. As examples, we consider two problems that have as a small parameter δ^2, the fraction of the total energy released via resolved reactions.

In section II, we examine a one-dimensional time-dependent detonation. We show that to lowest order in the perturbation (δ), the evolution of the detonation proceeds as if it were a simple wave with independent variables x and δt. A straightforward application of the method of characteristics to the resulting equations allows us to study a wide class of simple wave problems.

In section III, we examine a steady two-dimensional detonation. We show that for an unconfined semi-infinite detonation, the physical space divides itself into two distinct regions. Very near the edge (inner problem) the flow is nearly a free unreactive expansion with $x/\delta^{1/2}$ and $\delta^{1/2}y$ being the independent variables, where y is the distance into the charge from the edge. Away from the edge (outer problem) the reactivity and streamline divergence enter equally with x and δy being the independent variables. We determine the shape of the shock and sonic loci and study the effects of confinement on the detonation.

II. A TIME-DEPENDENT DETONATION

A. Statement of the Problem

Most of the experiments that have been performed on explosives were designed to measure the parameters contained in the Chapman-Jouguet theory. When applied to unsupported detonation, this theory makes the following assumptions: (1.) Initially the right half-space ($x \geq 0$) is occupied by a quiescent fluid at a density ρ_o which is in a state of metastable chemical equilibrium; (2.) At time $t = 0$ a piston, which is originally at $x = 0$, is impulsively brought to a velocity $u^* > 0$ and then withdrawn producing a planar shock wave followed by a rarefaction; (3.) On passing over the initially quiescent fluid the shock initiates an instantaneous chemical reaction, of specific internal energy $q(1-\delta^2)$, which then supports a classical detonation with a pressure P^* and a velocity D^*; (4.) In terms of this model the parameters ρ_o, D^*, and P^* completely characterize the flow. In this section, we will consider the consequences of releasing an additional small amount of energy $q\delta^2$ to the flow on a relatively slow time scale.

We limit our discussion to the following constitutive relations: a polytropic equation of state

$$E = \frac{1}{\gamma-1}\,\frac{P}{\rho} + q\delta^2(1-\lambda) - q, \qquad (2.1)$$

where E is the specific internal energy, P is the pressure, ρ is the density, γ is the adiabatic exponent, and a state-independent square-root rate law ($0 \leq \lambda \leq 1$)

$$r = k(1-\lambda)^{1/2}, \qquad (2.2)$$

where k is a constant rate multiplier. Neglecting all transport processes, the field equations for our time-dependent one-dimensional flow (shock fixed coordinates) are

$$\frac{Do}{Dt'}\ln(P'/\rho'^{\gamma}) = \frac{\gamma+1}{2\gamma}\,\frac{\delta^2 r'}{c'^2} \qquad (2.3)$$

$$\frac{D\pm}{Dt'}P' \pm \gamma\rho'c'\,\frac{D\pm}{Dt'}u' = \frac{\gamma+1}{2\gamma}\,\rho'\delta^2 r' \qquad (2.4)$$

$$\frac{Do}{Dt'}\lambda = r', \qquad (2.5)$$

where

$$\frac{D\pm}{Dt'} = \frac{\partial}{\partial t'} + \left[D' - (u' \pm c')\right]\frac{\partial}{\partial \zeta'} \qquad (2.6a)$$

$$\frac{Do}{Dt'} = \frac{\partial}{\partial t'} + (D' - u')\frac{\partial}{\partial \zeta'} \qquad (2.6b)$$

$$\zeta = \int D(t)\,dt - x. \qquad (2.7)$$

In the above equations t' is the scaled time (kt), ζ' is the scaled distance coordinate in the shock frame $(\gamma+1)k\zeta/\gamma D^*$, u' is the scaled particle velocity in the laboratory frame, c' is the scaled sound speed, ρ' is the scaled density $\gamma/(\gamma+1)\rho_o$, and $D'(t')$ is the scaled detonation velocity. To simplify the notation, the primes will be dropped.

Equations (2.3), (2.4), (2.5), the initial condition of an impulsive piston, and the shock conditions serve to completely describe the problem we wish to consider. In the limit $\delta \to 0$ the solution is a simple wave known as a Taylor wave. We will show that for δ sufficiently small the solution is a reactive simple wave.

B. <u>Reactive Simple Wave</u>

Since we are considering a system for which δ is small, it is natural to seek a solution to the stated problem as a regular asymptotic expansion in δ

$$u = \frac{1}{\gamma} + u^{(0)} + \delta u^{(1)} + \delta^2 u^{(2)} + \cdots \qquad (2.8)$$

$$c = c^{(0)} + \delta c^{(1)} + \delta^2 c^{(2)} + \cdots, \text{ etc.} \qquad (2.9)$$

A straightforward calculation gives us $u^{(2)} \sim t$ for t large. Thus, for times greater than δ^{-2} Eq. (2.8) no longer gives us an asymptotic representation of the solution. Examining the governing differential equations, we find that the secularity in $u^{(2)}$ arises because the equations for the perturbations are linear. It follows that for long times they do not contain the nonlinear convective effects that bound reactive growth. Physically, we can understand this with the aid of the Master Equation (1). It states that the growth of the shock pressure is the difference between the rate of energy input of the reactions minus the rate of energy loss to the following flow. When the flow is sonic, as it is in our unperturbed flow, the loss rate is zero. Introducing δt as a time scale into the Master Equation leads to bounded solutions (1). This suggests that in addition to t, we should include δt as a time scale. Using the method of multiple time scales, we will show that a bounded solution can be found to our differential system (2).

We begin by formally integrating Eq. (2.3) and then using the results to rewrite Eqs. (2.4) as

$$\frac{D\pm}{Dt} R\pm = \frac{\gamma+1}{2\gamma} \frac{c}{\gamma(\gamma-1)} \left(\gamma\frac{\delta^2 r}{c^2}\right.$$
$$\left.\pm \frac{\partial}{\partial\zeta}\int\frac{\delta^2 r}{c^2}\, dt_0\right) + O(\delta^3), \qquad (2.10)$$

where $R_\pm$ are the Riemann variables

$$R_\pm = \frac{2}{\gamma-1} c \pm u \qquad (2.11)$$

and $\int dt_0$ denotes an integral taken along a particle path. Following in the spirit of the multiple time scale method, we introduce the time scales

$$y^{(i)} = \frac{\gamma+1}{2} \delta^i t \quad ; \quad i = 0,1,2, \cdots \qquad (2.12)$$

and assume that Eqs. (2.7) and (2.9) depend explicitly on all of these times as well as ζ. The characteristic derivatives become

$$\frac{2}{\gamma+1} \frac{D\pm}{Dt} = \frac{\partial}{\partial y^{(0)}} + [1-(u^{(0)}\pm c^{(0)})] \frac{\partial}{\partial\zeta}$$
$$+ \delta\left[\frac{\partial}{\partial y^{(1)}} - (u^{(1)}\pm c^{(1)})\frac{\partial}{\partial\zeta}\right] + \cdots . \qquad (2.13)$$

Focusing our attention on the negative Riemann variable, we find that at O(1) Eq. (2.10) is easily integrated. Since all the negative characteristics emanate from a region whose state is at most specified by the variables $y^{(1)}$, $y^{(2)}$, ...

$$R_-^{(0)} = g^{(0)}(y^{(1)}, y^{(2)}, \cdots). \qquad (2.14)$$

At $O(\delta)$, we find

$$R_-^{(1)} = -y^{(0)}\left(\frac{\partial g^{(0)}}{\partial y^{(1)}}\right) + g^{(1)}(y^{(1)}, y^{(2)}, \cdots). \qquad (2.15)$$

To avoid the secular behavior in Eq. (2.15), we set

$$R_-^{(0)} = g^{(0)}(y^{(2)}, \cdots). \qquad (2.16)$$

Proceeding to $O(\delta^2)$, we get

$$R_-^{(2)} = g^{(2)}(y^{(1)}, y^{(2)}, \cdots)$$
$$+ \frac{1}{\gamma^2(\gamma-1)}\iint\left[c\left(\frac{\gamma r}{c^2} + \frac{\partial}{\partial\zeta}\int\frac{r}{c^2}\, dt_0\right)\right]_{\zeta=f(y^{(0)})} dy^{(0)} . \qquad (2.17)$$

where

$$\frac{df}{dy^{(0)}} = \frac{2}{\gamma+1} (1-u^{(0)}+c^{(0)}), \qquad (2.18)$$

and we have taken

$$\frac{\partial g^{(0)}}{\partial y^{(1)}} + \frac{\partial g^{(1)}}{\partial y^{(1)}} = 0 \qquad (2.19)$$

to avoid the appearance of a secularity. Using these results, the positive Riemann variables can be written as

$$R_+^{(0)} = 2u^{(0)} + g^{(0)}(y^{(2)}, \cdots \qquad (2.20)$$

$$R_+^{(1)} = 2u^{(1)} - y^{(0)}\left(\frac{\partial g^{(0)}}{\partial y^{(1)}}\right) + g^{(1)}(y^{(2)}, \cdots), \qquad (2.21)$$

Requiring Eq. (2.21) to remain bounded, we are forced to set $(\partial g^{(0)}/\partial y^{(1)}) = 0$, which gives

$$R_-^{(0)} = g^{(0)}(y^{(2)}, \cdots), \; R_-^{(1)} = g^{(1)}(y^{(2)}, \cdots). \qquad (2.22)$$

Therefore, to $O(\delta)$ the flow is a simple wave for the scales $y^{(0)}$ and $y^{(1)}$.

Turning our attention to the equations governing the positive Riemann variables and arbitrarily setting $u^{(0)} = 0$ and $c^{(0)} = 1$, we find

$$\left(\frac{\partial}{\partial y^{(1)}} - u^{(1)}\frac{\partial}{\partial\zeta}\right)u^{(1)} = \frac{1}{2\gamma}, \begin{vmatrix} (1-\frac{1}{2}\zeta) & , \; 0 \le \zeta \le 2 \\ 0 & , \; 2 < \zeta \end{vmatrix}. \qquad (2.23)$$

Solving Eq. (2.23) subject to the relevant boundary and initial conditions, we find

$$u^{in} = \frac{1}{\gamma}\tanh\left(\frac{\xi^{in}}{4\gamma}\right) - \frac{\xi}{2\gamma}\coth\left(\frac{\xi^{in}}{4\gamma}\right), \quad 0 \le \xi \le 2$$

$$u^{out} = -\frac{\xi}{3m} + \frac{2}{3m}\left[1 + \gamma u^{in}\ln\left(-(\gamma u^{in})^{-1} + \sqrt{1+(\gamma u^{in})^{-2}}\right)\right], \quad 2 < \xi . \qquad (2.24)$$

Substituting Eq. (2.24) into Eq. (2.8) gives us an asymptotic representation of the solution which is valid over the entire physical region.

C. Discussion

The effects that a small fraction of resolved energy release can have on a detonation can best be appreciated by considering an example. We take

$$D^{*} = 8.8 \text{ mm/}\mu s, \quad \gamma = 3.0 \qquad (2.25)$$

$$\delta^{2} = 0.06 \quad , \quad k = 2\ \mu s^{-1} . \qquad (2.26)$$

Figures 2.1 and 2.2 compare the particle

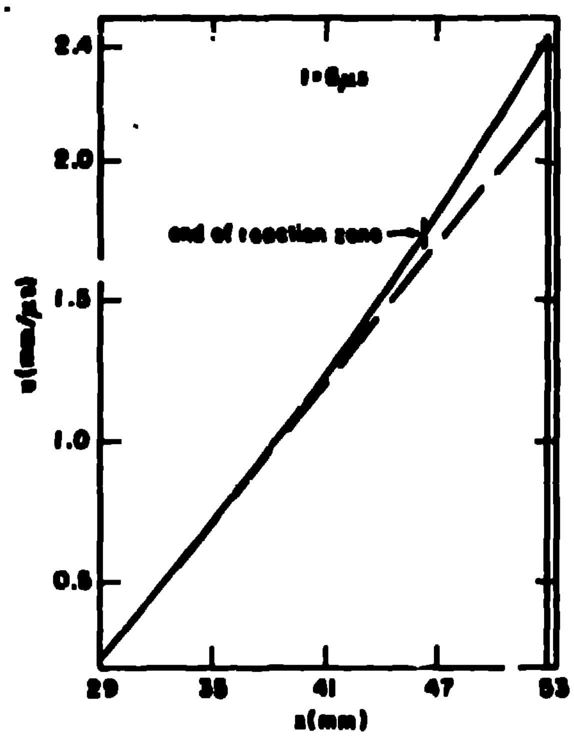

Fig. 2.1 - A comparison of particle velocity vs distance profiles for a Chapman-Jouguet detonation (--) and Eq. (2.24) (-).

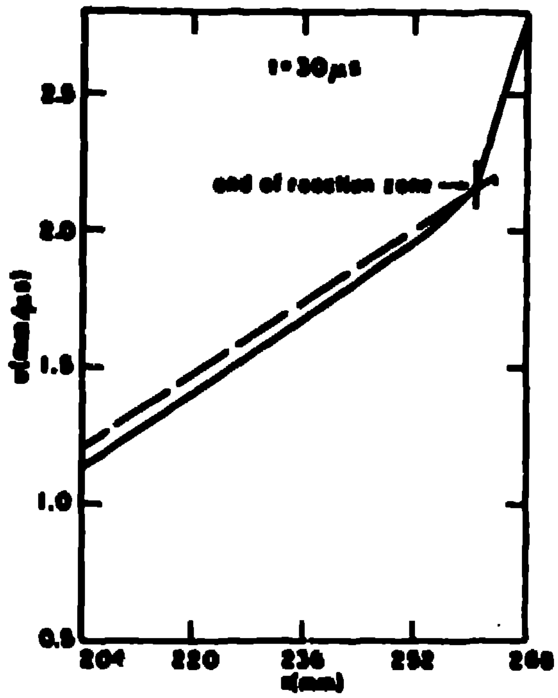

Fig. 2.2 - See Fig. 2.1.

velocity profiles to the reference Chapman-Jouguet detonation for 6 and 30 μs of run. Two features of the flow deserve special attention: (1) adding only 6% of the total energy via a slow reaction results in a 27% increase in the particle velocity at the shock, and (2) after 30 μs of run, the final steady-state has not been reached. Therefore, changes of $O(\delta^2)$ in the detonation energy (i.e., the Chapman-Jouguet state) produce changes of $O(\delta)$ in the shock state. Also, the rate at which the steady-state is approached is measured in units of $(\delta k)^{-1}$. Consequently, even if only a small amount of the available energy in a detonation is released relatively slowly, the deviations from the Chapman-Jouguet model will be large.

III. EDGE EFFECTS

A. Statement of the Problem

Consider a steady detonation of velocity D propagating in the positive x-direction. The explosive supporting the wave is taken to be semi-infinite with explosive occupying the half-space $y < 0$ and a vacuum for $y > 0$ (see Fig. 3.1).

If we assume a Chapman-Jouguet detonation, an observer riding with the shock would see a flat shock (y-axis), along which the flow would be exactly

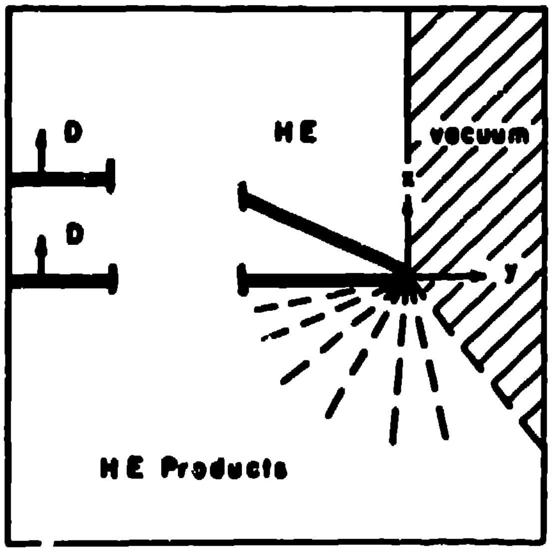

Fig. 3.1 - A schematic representation of the detonation edge effect. The shocks (both C-J and resolved) are double lines. The Prandtl-Meyer fan is represented by the dashed lines.

sonic. Behind the shock he would see a Prandtl-Meyer fan originating from a singularity at $y = 0$.

For the case of a resolved reaction zone, the flow at the intersection of the shock with the x-axis must again be locally described by the Prandtl-Meyer singularity. This requires that the flow at the shock be sonic at $y = 0$, which in turn, requires that the shock make an acute angle with the positive x-axis. Proceeding into the explosive along the shock, the effects of the singularity diminish, and leave in the limit an undisturbed one-dimensional flow with a subsonic shock which is parallel to the y-axis. In the intervening region $(-\infty < y < 0)$ the shock must smoothly connect these two limits with some convex form. The purpose of this section is to describe the structure of the region of reactive flow, including the determination of the shock shape, sonic locus, and limiting characteristic. For the general case, the analysis of this problem is difficult. However, we will again find that in the limit of small δ^2 (the resolved energy release fraction) a perturbation solution is possible. In this limit, progress becomes possible because we are dealing with a nearly sonic transonic flow.

B. <u>Preliminary Considerations</u>

Assuming that all transport processes can be neglected, the field equations for our steady two-dimensional plane flow (shock fixed coordinates) are

$$\underline{\nabla}\cdot(\rho\underline{u}) = 0 \qquad (3.1)$$

$$\underline{u}\cdot\underline{\nabla}\underline{u} = \frac{1}{\rho}\underline{\nabla}P \qquad (3.2)$$

$$\underline{u}\cdot\underline{\nabla}P - c^2\underline{u}\cdot\underline{\nabla}\rho = \frac{1}{E_p}q\delta^2 r \qquad (3.3)$$

$$\underline{u}\cdot\underline{\nabla}\lambda = r, \qquad (3.4)$$

where ρ is the density, $\underline{u}$ the particle velocity relative to the shock velocity, P is the pressure, c is the sound speed, E is the specific internal energy, q is the total energy release due to chemical reaction ($D^2 = 2(\gamma^2-1)q$), r is the rate of reaction and λ is the reaction progress variable ($\lambda = 1$ at the end of reaction). Since we wish to study the general features of the flow, we limit our discussion to the following constitutive relations: a polytropic equation of state

$$E = \frac{1}{\gamma-1}\frac{P}{\rho} + q\delta^2(1-\lambda) - q\ , \qquad (3.5)$$

where γ is the adiabatic exponent, and a state-independent square-root rate law

$$r = k(1-\lambda)^{1/2}, \qquad (3.6)$$

where k is a constant rate multiplier. After a straightforward transformation (see Serrin [3]), Eqs. (3.2) and (3.3) become

$$H + \frac{1}{2}|\underline{u}|^2 = \frac{1}{2}D^2, \qquad (3.7)$$

where $H = E + P/\rho$ and we have assumed the flow ahead of the shock is both homoenergetic and at zero pressure, and

$$\underline{u}\cdot\underline{\nabla}\left(\frac{\Omega}{\rho T}\right) = 0 \qquad (3.8a)$$

$$\underline{u}\cdot\underline{\nabla}\left(\frac{\Omega}{\rho}\right) = \frac{1}{\gamma}\left(\frac{\partial\rho^{-1}}{\partial x}\frac{\partial c^2}{\partial y} - \frac{\partial\rho^{-1}}{\partial y}\frac{\partial c^2}{\partial x}\right), \qquad (3.8b)$$

where T is the temperature and $\Omega = (\partial u_y/\partial x)-(\partial u_x/\partial y)$ is the vorticity which is directed into the plane of the paper. Using Eqs. (3.2), (3.3), and (3.7) we can rewrite Eq. (3.1) as

$$(c^2-u_x^2)\frac{\partial u_x}{\partial x} - 2u_x u_y\frac{\partial u_y}{\partial x} + (c^2-u_y^2)\frac{\partial u_y}{\partial y}$$
$$= (\gamma-1)q\delta^2 r - u_x u_y\Omega. \qquad (3.9)$$

Equations (3.6), (3.7), (3.8), and (3.9) serve as the working equations for our analysis. They are partial differential equations of mixed type. In regions of supersonic flow, they are of hyperbolic type; whereas in regions of subsonic flow they are of elliptic type. Formulating the boundary value problem for such a system requires some care. The Tricomi equation

$$\Xi_{xx} + x\Xi_{yy} = 0, \tag{3.10}$$

which is the simplest equation of mixed type, serves as a guide (4). It can be shown that if the potential Ξ is specified along some smooth curve in the subsonic region ($x > 0$), which originates and terminates on the sonic locus ($x = 0$), and also along a characteristic in the supersonic region ($x < 0$), which is joined to one of the end points of the boundary for $x > 0$, then Eq. (3.10) has a unique solution. Translated to the problem at hand, we are led to require that: (1.) along the shock (free boundary) both the normal and tangential jump conditions are satisfied (curve 1); (2.) at $y \to -\infty$ the streamline flow is that of the corresponding one-dimensional problem (curve 2); and (3.) along the cross characteristic encircling the Prandtl-Meyer singularity, the flow is that of an inert simple wave (curve 3). A schematic representation of the boundary is shown in Fig. 3.2. From a physical standpoint, applying these boundary conditions seems quite natural.

Of these, the shock conditions need some special consideration. We begin by defining the equation for the shock locus

$$x = -\psi(y) , \tag{3.11}$$

in terms of which the tangent , $\underline{t}$ and the

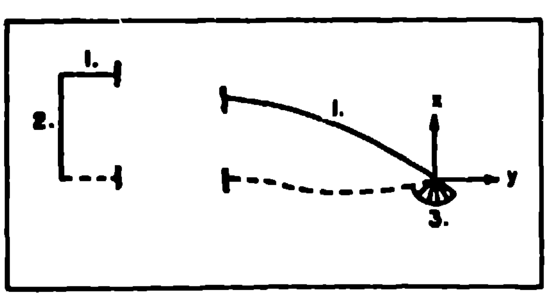

Fig. 3.2 - A schematic representation of the boundary curves for the edge effect on a reaction zone; (1.) the shock, (2.) streamline at infinity, (3.) cross characteristic. The dashed line represents the sonic locus.

normal, $\underline{n}$ to the shock surface are

$$\underline{t} = \left(-\frac{d\psi}{dy}\,\underline{i} + \underline{j}\right)\Big/\left(1 + \left(\frac{d\psi}{dy}\right)^2\right)^{1/2} \tag{3.12}$$

$$\underline{n} = \left(\underline{i} + \frac{d\psi}{dy}\,\underline{j}\right)\Big/\left(1 + \left(\frac{d\psi}{dy}\right)^2\right)^{1/2} . \tag{3.13}$$

The jump conditions across the shock require that the following relations hold:

$$\rho_+(\underline{u}\cdot\underline{n})_+ = \rho_o(\underline{u}\cdot\underline{n})_o \tag{3.14a}$$

$$(\underline{u}\cdot\underline{t})_+ = (\underline{u}\cdot\underline{t})_o \tag{3.14b}$$

$$P_+ + \rho_+(\underline{u}\cdot\underline{n})_+^2 = \rho_o(\underline{u}\cdot\underline{n})_o^2 . \tag{3.14c}$$

Since the state ahead of the shock is quiescent in the laboratory frame, Eqs. (3.14) may be rewritten as

$$\frac{\rho_o}{\rho_+} = \frac{\gamma}{\gamma+1} - \frac{1}{\gamma+1}\sqrt{1-(1-\delta^2)\left(1+\left(\frac{d\psi}{dy}\right)^2\right)} \tag{3.15a}$$

$$u_{y+} = \left(u_{x+} + D\right)\frac{d\psi}{dy} \tag{3.15b}$$

$$u_{x+} = D\left[-\left(\frac{d\psi}{dy}\right)^2 - \frac{\gamma}{\gamma+1} + \frac{1}{\gamma+1}\sqrt{1-(1-\delta^2)\left(1+\left(\frac{d\psi}{dy}\right)^2\right)}\right]\Big/\left(1+\left(\frac{d\psi}{dy}\right)^2\right). \tag{3.15c}$$

Using the result of Hayes (5), the vorticity jump across the shock is

$$\Omega_+ = \frac{(1-\rho_o/\rho_+)^2\, D(d\psi/dy)(d^2\psi/dy^2)}{\rho_o/\rho_+\left[1+(d\psi/dy)^2\right]^2} , \tag{3.16}$$

where the flow ahead of the shock is irrotational. Equations (3.15) and (3.16) provide all the necessary boundary conditions along the shock. Unfortunately, the shape of the shock is not known *a priori*, so that at this point in the calculation, they are of only limited usefulness. The slope at the sonic point on the shock can be calculated by substituting Eqs. (3.15b) and (3.15c) into Bernoulli's law [Eq. (3.7)], and setting $c^2 = |\underline{u}|^2$. We obtain

$$\left(\frac{d\psi}{dy}\right)_{sonic} = \delta\,\frac{(\gamma^2-\delta^2)^{1/2}}{(\gamma+\delta^2)} , \tag{3.17}$$

where for later convenience we introduce

$$\frac{d\psi}{dy} = (1-\epsilon)\left(\frac{d\psi}{dy}\right)_{sonic} \quad . \tag{3.18}$$

Since we must require that the shock shape is a smooth convex function, Eq. (3.17) serves as an upper bound on the shock slope. Recalling that the energy release fraction is δ^2, Eq. (3.17) shows that the shock slope is sensitive to the amount of resolved energy release, particularly for small values of δ^2. From the vorticity jump condition we find that Ω_+ is proportional to the product $(d\psi/dy)(d^2\psi/dy^2)$. Assuming that the shock curvature, $d^2\psi/dy^2$, is also $O(\delta)$ then Ω_+ is $O(\delta^2)$. Since the product ρT decreases for a particle as it recedes from the shock, Eq. (3.8a) requires that the vorticity also decrease. Therefore, it seems likely that for δ sufficiently small we will be able to consider the flow as irrotational.

In the analysis of the flow equations, we will find it convenient to be in a coordinate system in which the flow at the edge ($y = 0$) is directed along a single coordinate axis. We select the direction of flow at the sonic point on the shock as our new x-axis (x_w) with the new y-axis (y_w) being perpendicular to it. Because the flow is locally a Prandtl-Meyer expansion, the sonic locus coincides with y_w at $y = 0$. In this system, the velocities are

$$u_{xw} = u_x \cos\omega - u_y \sin\omega \tag{3.19a}$$

$$u_{yw} = u_x \sin\omega + u_y \cos\omega \ , \tag{3.19b}$$

with the rotation angle given by

$$\sin\omega = \delta/\gamma \ . \tag{3.19c}$$

In these new coordinates, the shock velocity jump conditions are

$$\tilde{u}_{xw+} = \frac{(\gamma+\delta^2)}{\gamma\left[(\gamma+\delta^2)^2+\delta^2(\gamma^2-\delta^2)(1-\epsilon)^2\right]} \cdot \left[-\delta^2(1-\epsilon) - \frac{\delta^2(\gamma^2-\delta^2)(1-\epsilon)^2}{(\gamma+\delta^2)} + \delta\,\frac{(\gamma+\delta^2\epsilon)}{(\gamma+\delta^2)}\sqrt{(\gamma+\delta^2)^2-(1-\delta^2)(\gamma^2-\delta^2)(1-\epsilon)^2}\right] \tag{3.20}$$

$$\tilde{u}_{yw+} = \frac{\delta}{(\gamma^2-\delta^2)^{1/2}}\left[\frac{(\gamma+\delta^2)+(\gamma^2-\delta^2)(1-\epsilon)}{(\gamma+\delta^2)-\delta^2(1-\epsilon)}\right]\tilde{u}_{xw+} - \frac{\delta(\gamma+\delta^2)\epsilon}{(\gamma^2-\delta^2)^{1/2}\left[(\gamma+\delta^2)-\delta^2(1-\epsilon)\right]} \ , \tag{3.21}$$

where

$$\tilde{u}_{xw} \equiv 1 + u_{xw} \tag{3.22}$$

and the velocities in Eqs. (3.20), (3.21), and (3.22) have been scaled by $D(\gamma^2-\delta^2)^{1/2}/(\gamma+1)$. In the following sections, we will use the method of matched asymptotic expansions to find a solution to the stated problem in the limit of δ small. To simplify the notation, the subscript w and tildes will be dropped.

C. The Outer Problem

In applying a perturbation method to the solution of a problem, there are two essentially unique steps. The first is the determination of the form of the expansion of the dependent variable in terms of the small parameter. The second is the scaling by the small parameter of the independent variables. For the problem we are considering, the shock jump conditions argue strongly that an expansion of the dependent variables in integral powers of δ should be tried. Requiring that any solution that we generate include a sonic transition, the effects of reactivity, and streamline divergence, suggests that the independent variables be x and δy. Physically, we can understand the scaling of the independent variables as follows. Far from the edge we can expect the distance from the shock locus to the sonic locus to be near the undisturbed one-dimensional value which goes as x. Since the reaction zone is only slightly subsonic in the small δ limit, the flow at the shock changes from sonic at the edge to only slightly subsonic at great distances from the edge. Therefore, the sonic character of the flow is nearly the same everywhere so that there is little to differentiate the near from the far fields. The scale δy has this property.

We proceed with the perturbation solution by assuming that the dependent variables possess the following asymptotic expansions

$$u_x = \delta u_x^{(1)} + \delta^2 u_x^{(2)} + \cdots \tag{3.23}$$

$$u_y = \delta u_y^{(1)} + \delta^2 u_y^{(2)} + \cdots \tag{3.24}$$

$$c^2 = 1 + \delta(c^2)^{(1)} + \delta^2(c^2)^{(2)} + \cdots \tag{3.25}$$

$$\frac{\rho_o}{\rho} = \frac{\gamma}{\gamma+1} + \delta\left(\frac{\rho_o}{\rho}\right)^{(1)} + \delta^2\left(\frac{\rho_o}{\rho}\right)^{(2)} + \cdots \tag{3.26}$$

$$\lambda = \lambda^{(0)} + \delta\lambda^{(1)} + \delta^2\lambda^{(2)} + \cdots \tag{3.27}$$

$$\epsilon = \epsilon^{(0)} + \delta\epsilon^{(1)} + \delta^2\epsilon^{(2)} + \cdots , \tag{3.28}$$

where c^2 has been scaled by $\mathcal{D}^2(\gamma^2-\delta^2)/(\gamma+1)^2$. The independent variables are taken to be

$$x^* = \frac{(\gamma+1)k}{\mathcal{D}(\gamma^2-\delta^2)^{1/2}}\, x \tag{3.29}$$

$$\bar{y}^* = \delta\frac{(\gamma+1)k}{\mathcal{D}(\gamma^2-\delta^2)^{1/2}}\, y \ . \tag{3.30}$$

(The asterisks will be deleted from our notation). Since the reaction zone we are considering is of finite length, it is necessary to transform to λ, $\bar{y}$ as the independent variable set to insure that the amount of energy added in the reaction zone is compatible with the value of $\mathcal{D}$. The differential operators are thus replaced by

$$\frac{\partial}{\partial x} = \left(\frac{\partial\lambda^{(0)}}{\partial x} + \delta\frac{\partial\lambda^{(1)}}{\partial x} + \cdots\right)\frac{\partial}{\partial\lambda} \tag{3.31}$$

$$\frac{\partial}{\partial y} = \delta\frac{\partial}{\partial\bar{y}} + \delta\left(\frac{\partial\lambda^{(0)}}{\partial\bar{y}} + \delta\frac{\partial\lambda^{(1)}}{\partial\bar{y}} + \cdots\right)\frac{\partial}{\partial\lambda} \ . \tag{3.32}$$

Substituting Eqs. (3.27) and (3.6) into Eq. (3.4) and setting to zero the terms of $O(1)$ and $O(\delta)$, we get equations for $\lambda^{(0)}$ and $\lambda^{(1)}$

$\underline{O(1)}$
$$\frac{\partial\lambda^{(0)}}{\partial x} = -(1-\lambda^{(0)})^{1/2} \tag{3.33}$$

$\underline{O(\delta)}$
$$\frac{\partial\lambda^{(1)}}{\partial x} = u_x^{(1)}\frac{\partial\lambda^{(0)}}{\partial x} + \frac{1}{2}\lambda^{(1)}(1-\lambda^{(0)})^{-1/2} \ . \tag{3.34}$$

Equation (3.33) can easily be solved, yielding

$$\lambda^{(0)} = 1-\left[1-\frac{1}{2}(x_s-x)\right]^2 , \tag{3.35}$$

where $x_s(\bar{y})$ is the shock locus. Before we can integrate Eq. (3.34) $u_x^{(1)}$ must be found.

We begin the analysis of our system by first eliminating c^2 by applying Bernoulli's law [Eq. (3.7)]

$\underline{O(\delta)}$
$$(c^2)^{(1)} = (\gamma-1)u_x^{(1)} \tag{3.36}$$

$\underline{O(\delta^2)}$
$$(c^2)^{(2)} = (\gamma-1)\left[u_x^{(2)} - \frac{1}{2}(u_x^{(1)})^2 - \frac{1}{2}(u_y^{(1)})^2 + \frac{1}{2}\,\frac{\gamma+1}{\gamma-1}\,\frac{\lambda}{(\gamma^2-\delta^2)}\right] . \tag{3.37}$$

Using Eqs. (3.28) and (3.32) it follows that the vorticity jump is $O(\delta^3)$. From the definition of the vorticity it then follows that

$\underline{O(\delta)}$
$$\frac{\partial u_y^{(1)}}{\partial\lambda} = 0 \tag{3.38}$$

$\underline{O(\delta^2)}$
$$\frac{\partial\lambda^{(0)}}{\partial x}\frac{\partial u_y^{(2)}}{\partial\lambda} - \frac{\partial u_x^{(1)}}{\partial\bar{y}} - \frac{\partial\lambda^{(0)}}{\partial\bar{y}}\frac{\partial u_x^{(1)}}{\partial\lambda} = 0 \ . \tag{3.39}$$

Making use of Eqs. (3.33) through (3.39), we find that Eq. (3.5) becomes

$\underline{O(\delta^2)}$
$$u_x^{(1)}\frac{\partial u_x^{(1)}}{\partial\lambda} - \frac{1}{(\gamma+1)(1-\lambda)^{1/2}}\frac{\partial u_y^{(1)}}{\partial\bar{y}} = -\frac{1}{2\gamma^2} \tag{3.40}$$

$\underline{O(\delta^3)}$
$$\begin{aligned} &u_x^{(1)}\left(\frac{\partial\lambda^{(1)}}{\partial x}\frac{\partial u_x^{(1)}}{\partial\lambda} + \frac{\partial\lambda^{(0)}}{\partial x}\frac{\partial u_x^{(2)}}{\partial\lambda}\right) + \left[u_x^{(2)} - \frac{1}{2}(u_x^{(1)})^2 - \frac{1}{2}\left(\frac{\gamma-1}{\gamma+1}\right)(u_y^{(1)})^2 + \frac{\lambda}{2\gamma^2}\right]\frac{\partial\lambda^{(0)}}{\partial x}\frac{\partial u_x^{(1)}}{\partial\lambda} \\ &+ \frac{2}{\gamma+1}u_y^{(1)}\frac{\partial\lambda^{(0)}}{\partial x}\frac{\partial u_y^{(2)}}{\partial\lambda} + \frac{1}{\gamma+1}\left(\frac{\partial u_y^{(1)}}{\partial\bar{y}} + \frac{\partial\lambda^{(0)}}{\partial\bar{y}}\frac{\partial u_y^{(2)}}{\partial\lambda}\right) \\ &+ \left(\frac{\gamma-1}{\gamma+1}\right)u_x^{(1)}\frac{\partial u_y^{(1)}}{\partial\bar{y}} = 0 \end{aligned} \tag{3.41}$$

plus higher order terms. Expanding the shock conditions [Eqs. (3.20) and (3.21)], we get

$\underline{O(\delta)}$
$$u_{x+}^{(1)} = \frac{1}{\gamma}\sqrt{1-(1-\epsilon^{(0)})^2} \tag{3.42}$$

$$u_{y+}^{(1)} = -\frac{1}{\gamma}\,\epsilon^{(0)} \tag{3.43}$$

$\underline{O(\delta^2)}$
$$u_{x+}^{(1)}u_{x+}^{(2)} = -\frac{1}{\gamma}(1-\epsilon^{(0)})u_{y+}^{(2)} \tag{3.44}$$

$$u_{y+}^{(2)} = \left(\frac{\gamma+1}{\gamma} - \epsilon^{(0)}\right)u_{x+}^{(1)} - \frac{1}{\gamma}\epsilon^{(1)} . \tag{3.45}$$

The remaining boundary conditions require that the flow approach the one-dimensional limit as $\bar{y} \to -\infty$ and a Prandtl-Meyer singularity at $\bar{y} = 0$, $x \leq 0$.

The lowest order equations [Eqs. (3.38) and (3.40)] can be integrated without difficulty. Since $u_y^{(1)}$ is independent of λ, Eq. (3.40) can be treated as a first order ordinary differential equation (O.D.E.) in λ. As such it can satisfy only one boundary condition (shock condition) and the Prandtl-Meyer condition must be dropped. This then serves as the definition of the outer limit of the full problem:

Outer Problem - the system of O.D.E. [Eqs. (3.40), (3.41), etc.] and the shock boundary conditions which together describe the flow far from the Prandtl-Meyer singularity.

Solving Eq. (3.40), we get

$$(u_x^{(1)})^2 = (u_{x+}^{(1)})^2 + \frac{4}{\gamma+1}\frac{du_y^{(1)}}{dy}\left[1-(1-\lambda)^{1/2}\right] - \frac{1}{\gamma^2}\left[1-(1-\lambda)\right], \qquad (3.46)$$

where $u_{x+}^{(1)}$ and $u_y^{(1)}$ are unknown functions of $\varepsilon^{(1)}(\bar{y})$. Now, if Eq. (3.46) is to be an acceptable solution to our reactive flow problem, it must have the following properties: (1.) $u_x^{(1)}$ must be real, and (2.) $u_x^{(1)}$ must be equal to zero (sonic flow) at some point in the reaction zone. These can be considered as a generalized Chapman-Jouguet condition. Requiring this of Eq. (3.46) gives us a differential condition on $\varepsilon^{(1)}$

$$\frac{d\varepsilon^{(1)}}{dz} = 1-\sqrt{1-(1-\varepsilon^{(1)})^2}, \qquad (3.47)$$

where $z \equiv -(\gamma+1)\bar{y}/2\gamma$. Integrating Eq. (3.47) and requiring that $\varepsilon^{(1)}(0) = 0$ gives us an implicit expression for the shock slope

$$\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) - (1+\theta) = z, \qquad (3.48)$$

where $\cos\theta = 1-\varepsilon^{(1)}$. Transforming Eq. (3.18) into the edge sonic-line-fixed coordinates and integrating, we get a first approximation to the shock locus

$$x_s = \frac{2}{\gamma+1}\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) - (1+\theta) - \gamma\sin\theta - \gamma\ln(1-\sin\theta)\right]. \qquad (3.49)$$

Therefore, the first approximation to the outer velocity field is

$$u_x^{(1)} = \frac{1}{\gamma}\sin\theta - \frac{1}{\gamma}\left[1-(1-\lambda)^{1/2}\right] \qquad (3.50)$$

$$u_y^{(1)} = -\frac{1}{\gamma}(1-\cos\theta), \qquad (3.51)$$

where $\lambda = 0$ at the shock.

We find that the above solution has the following properties:

(1.) The solution merges into the one-dimensional flow as $\bar{y} \to -\infty$.

(2.) The distance (along the laboratory x-coordinate) from (0,0) to the lead point on the shock is infinite (for the square-root rate law).

(3.) The sonic line enters (0,0) with infinite slope instead of the required zero slope.

A plot of the shock locus and sonic locus is shown in Fig. 3.3. Therefore, we find that the outer solution agrees with both, the shock conditions and those at $\bar{y} \to -\infty$, but violates the conditions at the Prandtl-Meyer singularity. This is a substantial shortcoming.

Going on to the next order of the outer problem, matters become even worse. Solving Eqs. (3.34), (3.39), and (3.41) subject to Eq. (3.44), we find

$$u_x^{(2)} = u_{x+}^{(2)} + \frac{1}{2\gamma^2}\Big[(\gamma-2)(1-\sin\theta) - (\gamma+1) * (\csc\theta-1)\left(\csc^2\theta - \frac{2\gamma+1}{\gamma+1}\sin\theta\right)\Big]\left[1-(1-\lambda)^{1/2}\right] \qquad (3.52)$$

$$u_y^{(2)} = u_{y+}^{(2)} + \frac{\gamma+1}{\gamma^2}\left[(1-\sin\theta)\cot\theta - \frac{1+\gamma\cos\theta}{\gamma+1}\right] * \left[1-(1-\lambda)^{1/2}\right], \qquad (3.53)$$

where

$$u_{x+}^{(2)} = -u_{y+}^{(2)}\cot\theta \qquad (3.54)$$

$$u_{y+}^{(2)} = C(1-\sin\theta) + \frac{\gamma+1}{2\gamma^2}(1-\sin\theta)\ln\left(\tan\frac{\theta}{2}\right) + \frac{2}{\gamma}\cos\theta - \frac{5\gamma+3}{2\gamma^2}(1-\sin\theta)\theta + \frac{3}{2}\frac{\gamma+1}{\gamma^2}(1-\sin\theta)\cos\theta + \frac{1}{\gamma^2}, \qquad (3.55)$$

and C is an arbitrary constant. In the

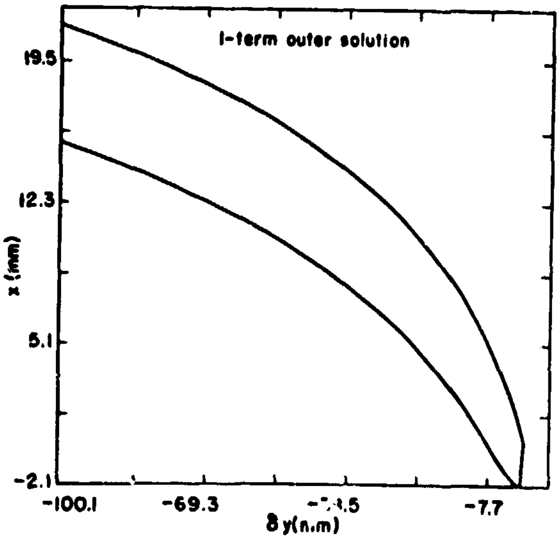

Fig. 3.3 - 1-term outer solution. The shock locus (upper curve) and the sonic locus (lower curve) in edge fixed coordinates. The parameter values are D = 8 mm/μs, γ = 3, and k = 2 μs^{-1}.

limit $\tilde{y} \to -\infty$ $(\theta \to \pi/2)$ the solution is well-behaved. However, near the edge $(\theta \to 0)$ Eqs. (3.52), (3.53), (3.54), and (3.55) all become singular. To resolve this difficulty, we must examine the vicinity of the Prandtl-Meyer singularity in some detail. In the next section, we formulate the inner limit of our problem and show how the singularities in the outer problem can be removed.

D. The Inner Problem

The problem encountered at the end of the last section is similar in principle to that treated by Cole (6). There as here the singularity arises because the O.D.E. being studied has singular coefficients. However, in our case the resolution of the difficulty proceeds somewhat differently. The principal shortcoming of the outer limit is that the O.D.E.'s which are obtained are capable of handling only a very restricted class of transonic flows. Since the reactivity is of secondary importance near the edge (relative to the Prandtl-Meyer singularity), let us neglect it and the vorticity for the moment and obtain the kernel transonic partial differential operator contained in Eq. (3.9). The object of this exercise is to obtain a partial differential equation (P.D.E.) which is capable of satisfying all of the applicable boundary conditions near the edge. We proceed by introducing a potential and scaled independent variables

$$\Phi = -x + \delta^m \phi(\tilde{x},\tilde{y}) \qquad (3.56)$$

$$\tilde{x} = x\delta^{-\nu} \ , \quad \tilde{y} = y\delta^{-\mu} \ . \qquad (3.57)$$

Using Eqs. (3.56) and (3.57) to calculate the velocities and Bernoulli's law to eliminate c^2, the dominant terms in Eq. (3.9) yield the equation

$$(\gamma+1)\frac{\partial\phi}{\partial\tilde{x}}\frac{\partial^2\phi}{\partial\tilde{x}^2} + \frac{\partial^2\phi}{\partial\tilde{y}^2} = 0, \qquad (3.58)$$

where we have the constraint

$$3(m-\nu) = 2(m-\mu) \ . \qquad (3.59)$$

Equation (3.58) is the model equation for transonic flow(7). It is capable of describing the flow in the neighborhood of a Prandtl-Meyer singularity imbedded in a mixed flow. In fact, the specification of a unique solution of Eq. (3.58) requires that ϕ be given along the shock, a curve connecting the shock to the sonic locus, as well as the Prandtl-Meyer condition. To determine the parameters m, ν, and μ, we require that the orders of magnitude of the velocities calculated from Eq. (3.58) match the dominant singularities found in the outer solution. This can be thought of as satisfying the boundary conditions along a curve connecting the shock locus to the sonic locus in an order of magnitude sense. The most singular terms in $u_x^{(2)}$ and $u_y^{(2)}$ are

$$\delta^2 u_x^{(2)} = O(\delta^{1/2+\nu-3\mu/2})\ \frac{\tilde{x}}{(-\tilde{y})^{3/2}} \qquad (3.60)$$

$$\delta^2 u_y^{(2)} = O(\delta^{1/2+\nu-\mu/2})\ \frac{\tilde{x}}{(-\tilde{y})^{1/2}} \ , \qquad (3.61)$$

so that we get the conditions

$$\frac{1}{2}+\nu-\frac{3}{2}\mu = m-\nu \qquad (3.62)$$

$$\frac{3}{2}+\nu-\frac{1}{2}\mu = m-\mu \ . \qquad (3.63)$$

Solving Eqs. (3.59), (3.62), and (3.63), we get

$$\phi = O(\delta^{5/3}), \quad \tilde{x} = x/\delta^{1/3}, \quad \tilde{y} = \delta^{1/3}y \qquad (3.64)$$

$$u_x = O(\delta^{4/3}), \quad u_y = O(\delta^2). \qquad (3.65)$$

In terms of the variables of Eq. (3.64) the remaining singular terms in the outer $u_x^{(2)}$ and $u_y^{(2)}$ are

$$\delta^2 u_x^{(2)}: \qquad \delta^{5/3}\frac{\ln(-\tilde{y})}{(-\tilde{y})^{1/2}} + \frac{2}{3}\delta^{5/3}\frac{\ln\delta}{(-\tilde{y})^{1/2}} \qquad (3.66)$$

$$\delta^2 u_y^{(2)}: \qquad \delta^2\ln(-\tilde{y}) + \frac{2}{3}\delta^2\ln\delta \ . \qquad (3.67)$$

When expressed in terms of Eq. (3.64), the dominant terms in the outer $u_x^{(1)}$ and $u_y^{(1)}$ are $O(\delta^{4/3})$ and $O(\delta^{5/3})$ respectively.

Using this information as a guide, we assume that the flow potential can be expressed as the following asymptotic sequence

$$\phi = \delta^{4/3}\phi^{(4/3)} + \delta^{5/3}\ln\delta\ \phi^{(l5/3)} + \delta^{5/3}\phi^{(5/3)}$$
$$+ \delta^2\ln\delta\ \phi^{(l2)} + \delta^2\phi^{(2)} + \delta^{7/3}\ln\delta\ \phi^{(l7/3)}$$
$$+ \delta^{7/3}\phi^{(7/3)} + \cdots \ , \qquad (3.68)$$

which is valid for velocities up to at least $O(\delta^{7/3})$ since Eq. (3.16) gives

$$\Omega_+ = O(\delta^{7/3})\ \frac{d(1-\varepsilon)^2}{d\tilde{y}} . \qquad (3.69)$$

For this set of dependent and independent scales, the reaction progress variable must be

$$\lambda = \delta^{1/3}\lambda^{(1/3)} + \delta^{2/3}\lambda^{(2/3)} + \cdots, \quad (3.70)$$

where $\lambda^{(1/3)}$ and $\lambda^{(2/3)}$ satisfy

$$\frac{\partial\lambda^{(1/3)}}{\partial\bar{x}} = -1 \quad (3.71)$$

$$\frac{\partial\lambda^{(2/3)}}{\partial\bar{x}} = \frac{1}{2}\lambda^{(1/3)}, \quad (3.72)$$

so that

$$\lambda^{(1/3)} = \bar{x}_s - \bar{x} \quad (3.73)$$

$$\lambda^{(2/3)} = -\frac{1}{4}(\bar{x}_s - \bar{x})^2 . \quad (3.74)$$

Introducing $\tilde{\lambda} \equiv \lambda/\delta^{1/3}$ and $\tilde{y}$ as the independent variables, Eqs. (3.7) and (3.9) yield the following set of equations for ϕ:

$O(\delta^{5/3})$

$$\frac{\partial}{\partial\tilde{\lambda}}\left(\frac{\partial\phi^{(4/3)}}{\partial\tilde{\lambda}}\right)^2 = 0 \quad (3.75)$$

$O(\delta^{2}\ln\delta)$

$$\frac{\partial}{\partial\tilde{\lambda}}\left(\frac{\partial\phi^{(4/3)}}{\partial\tilde{\lambda}}\frac{\partial\phi^{(\ell 5/3)}}{\partial\tilde{\lambda}}\right) = 0 \quad (3.76)$$

$O(\delta^{2})$

$$-(\gamma+1)\frac{\partial}{\partial\tilde{\lambda}}\left(\frac{\partial\phi^{(4/3)}}{\partial\tilde{\lambda}}\frac{\partial\phi^{(5/3)}}{\partial\tilde{\lambda}}\right) + \frac{1}{2}(\gamma+1)\left(\frac{\partial\phi^{(4/3)}}{\partial\tilde{\lambda}}\right)^2 + \frac{\partial^2\phi^{(4/3)}}{\partial\tilde{y}^2} = \frac{(\gamma+1)}{2\gamma^2} \quad (3.77)$$

$O(\delta^{7/3}(\ln\delta)^2)$

$$\frac{\partial}{\partial\tilde{\lambda}}\left(\frac{\partial\phi^{(\ell 5/3)}}{\partial\tilde{\lambda}}\right)^2 = 0 \quad (3.78)$$

$O(\delta^{7/3}\ln\delta)$

$$-(\gamma+1)\frac{\partial}{\partial\tilde{\lambda}}\left(\frac{\partial\phi^{(4/3)}}{\partial\tilde{\lambda}}\frac{\partial\phi^{(\ell 2)}}{\partial\tilde{\lambda}} + \frac{\partial\phi^{(\ell 5/3)}}{\partial\tilde{\lambda}}\frac{\partial\phi^{(5/3)}}{\partial\tilde{\lambda}}\right) + \frac{1}{2}(\gamma+1)\left(\frac{\partial\phi^{(\ell 5/3)}}{\partial\tilde{\lambda}}\frac{\partial\phi^{(4/3)}}{\partial\tilde{\lambda}}\right) + \frac{\partial^2\phi^{(\ell 5/3)}}{\partial\tilde{y}^2} = 0 \quad (3.79)$$

$O(\delta^{7/3})$

$$-(\gamma+1)\frac{\partial}{\partial\tilde{\lambda}}\left[\frac{1}{2}\left(\frac{\partial\phi^{(5/3)}}{\partial\tilde{\lambda}}\right)^2 + \frac{\partial\phi^{(4/3)}}{\partial\tilde{\lambda}}\frac{\partial\phi^{(2)}}{\partial\tilde{\lambda}} - \tilde{\lambda}\frac{\partial\phi^{(4/3)}}{\partial\tilde{\lambda}}\frac{\partial\phi^{(5/3)}}{\partial\tilde{\lambda}} - \frac{1}{2}\frac{\partial\phi^{(4/3)}}{\partial\tilde{\lambda}}\frac{\partial\phi^{(5/3)}}{\partial\tilde{\lambda}}\right] - \frac{1}{2}(\gamma+1)\tilde{\lambda}\left(\frac{\partial\phi^{(4/3)}}{\partial\tilde{\lambda}}\right)^2 + \frac{\partial^2\phi^{(5/3)}}{\partial\tilde{y}^2} - \tilde{\lambda}\frac{d\tilde{x}_s}{d\tilde{y}}\frac{\partial^2\phi^{(4/3)}}{\partial\tilde{y}\partial\tilde{\lambda}} = -\frac{(\gamma+1)}{4\gamma^2}\tilde{\lambda}. \quad (3.80)$$

Since the higher order equations are increasingly more complex, we will not consider them here. Doing so will not affect the first approximation to the uniformly valid solution.

Let us now obtain the boundary conditions for Eqs. (3.75) - (3.80). Given the potential of Eq. (3.68) and the shock condition of Eq. (3.21), we take

$$\varepsilon = \delta^{2/3}\varepsilon^{(2/3)} + \delta\ln\delta\,\varepsilon^{(\ell 1)} + \delta\varepsilon^{(1)} + \delta^{4/3}\ln\delta\,\varepsilon^{(\ell 4/3)} + \cdots. \quad (3.81)$$

Substituting Eq. (3.81) and the velocities into Eqs. (3.20) and (3.21), we get the shock boundary conditions

$$O(\delta^{5/3}) \quad \left(\frac{\partial\phi^{(4/3)}}{\partial\tilde{\lambda}}\right)_+ = 0 \quad (3.82)$$

$$O(\delta^{7/3}(\ln\delta)^2) \quad \left(\frac{\partial\phi^{(\ell 5/3)}}{\partial\tilde{\lambda}}\right)_+ = 0 \quad (3.83)$$

$$O(\delta^{7/3}) \quad \left(\frac{\partial\phi^{(5/3)}}{\partial\tilde{\lambda}}\right)_+^2 = -\frac{2}{\gamma}\left(\frac{\partial\phi^{(4/3)}}{\partial\tilde{y}}\right)_+ \quad (3.84)$$

$$O(\delta^{8/3}\ln\delta) \quad \left(\frac{\partial\phi^{(\ell 2)}}{\partial\tilde{y}}\right)_+ = 0 \quad (3.85)$$

$$O(\delta^{8/3}) \quad \left(\frac{\partial\phi^{(5/3)}}{\partial\tilde{\lambda}}\right)_+\left(\frac{\partial\phi^{(2)}}{\partial\tilde{\lambda}}\right)_+ = -\frac{1}{\gamma}\left(\frac{\partial\phi^{(5/3)}}{\partial\tilde{y}}\right)_+ \quad ([illegible]), \quad (3.86)$$

where the terms in the shock slope are

$$\varepsilon^{(2/3)} = -\gamma\left(\frac{\partial\phi^{(4/3)}}{\partial\tilde{y}}\right)_+, \quad \varepsilon^{(\ell 1)} = 0 \quad (3.87)$$

$$\varepsilon^{(1)} = -\gamma\left(\frac{\partial\phi^{(5/3)}}{\partial\tilde{y}}\right)_+, \quad \varepsilon^{(\ell 4/3)} = -\gamma\left(\frac{\partial\phi^{(\ell 2)}}{\partial\tilde{y}}\right)_+.$$

Far from the edge (i.e., $-\tilde{y}$ large), the flow calculated for the inner problem must match that of the outer problem. Calculating the outer potential from Eqs. (3.50), (3.51), (3.52), and (3.53) and then taking the inner limit of this outer potential gives us the match potential for the inner problem

$$\phi = \delta^{4/3}\frac{\gamma+1}{4\gamma^2}\tilde{y}^2 + \delta^{5/3}\left\{-\frac{1}{\gamma}\left(\frac{\gamma+1}{\gamma}\right)^{1/2}(-\tilde{y})^{1/2}\tilde{\lambda} + \frac{1}{4\gamma}\tilde{\lambda}^2 - \frac{\gamma+1}{4\gamma^2}(-\tilde{y})\ln(-\tilde{y}) - \frac{2}{15}\frac{1}{\gamma}\left(\frac{\gamma+1}{\gamma}\right)^{1/2}(-\tilde{y})^{3/2}\right\}$$

$$+ \delta^2\left\{\frac{1}{3}\frac{\gamma+1}{\gamma^2}(-\tilde{y})\tilde{\lambda} - \frac{1}{4\gamma}\left(\frac{\gamma+1}{\gamma}\right)^{1/2}(-\tilde{y})^{1/2}\tilde{\lambda}^2 + \frac{1}{8\gamma}\tilde{\lambda}^3 + \frac{1}{6\gamma}\left(\frac{\gamma+1}{\gamma}\right)^{1/2}(-\tilde{y})^{3/2}[\ln(-\tilde{y}) + 1]\right.$$

$$+\frac{1}{36\gamma}\left(\frac{\gamma+1}{\gamma}\right)^2(-\bar{y})^3\Bigg\}. \qquad (3.88)$$

where we have set the arbitrary constant in Eq. (3.55) equal to

$$C = -\frac{\gamma+1}{4\gamma^2}\left[\ln\left(\frac{\gamma+1}{4\gamma}\,\delta^{2/3}\right) - 1\right] - \frac{7\gamma+5}{2\gamma^2}\,.$$

Lastly we have the Prandtl-Meyer condition at $\bar{y} = 0^-$, $\bar{x} = 0^-$. Thus, the inner limit of the full problem is:

> Inner Problem – the system of P.D.E. [Eqs. (3.75), (3.76), (3.77), (3.78), (3.79), (3.80), etc.], the shock boundary conditions [Eqs. (3.82), (3.83), (3.84), (3.85), (3.86), (3.87), etc.], the match into the outer problem [Eq. (3.88)], and the Prandtl-Meyer singularity which together describe the flow near the edge.

Finding the solution of Eqs.(3.75), (3.76), (3.77), (3.78), and (3.79) subject to the appropriate boundary conditions is straightforward. We obtain

$$\phi^{(4/3)} = \frac{\gamma+1}{4\gamma^2}\,\bar{y}^2 \qquad (3.89)$$

$$\psi^{(4/3)} = 0 \quad . \qquad (3.90)$$

Using these results to simplify Eq. (3.30), we get

$$-(\gamma+1)\frac{\partial\phi^{(4/3)}}{\partial\bar{x}}\frac{\partial^2\phi^{(4/3)}}{\partial\bar{x}^2} + \frac{\partial^2\phi^{(4/3)}}{\partial\bar{y}^2} = -\frac{\gamma+1}{4\gamma^2}\,\bar{\lambda}\,. \qquad (3.91)$$

Equation (3.91) is an inhomogeneous transonic P.D.E.. Finding an analytic solution to it subject to the shock, match, and Prandtl-Meyer boundary conditions is not a simple matter. Since the reactivity is not a dominant effect near the edge, we will first examine the homogeneous form of Eq. (3.91). The simplest approach is to seek a similarity solution to Eq. (3.91). The drawback with this method is that one may not be able to satisfy all of the boundary conditions.

The most general similarity solution to the homogeneous form of Eq. (3.91) is that solution which is invariant under an infinitesimal one-parameter Lie group of transformations. We find

$$\phi_h = -(-\bar{y}+b_2)^{3\eta-2}G(s) + (-\bar{y}+b_2)B_1 + B_2 \qquad (3.92)$$

$$\frac{\partial\phi_h}{\partial\bar{x}} = (\gamma+1)^{-1/3}(-\bar{y}+b_2)^{2\eta-2}G'(s) \qquad (3.93)$$

$$s = \frac{-\bar{\lambda}(\gamma+1)^{-1/3} + b_1}{(-\bar{y}+b_2)^{\eta}}\,, \qquad (3.94)$$

where η, b_1, b_2, B_1, and B_2 are constants and $G(s)$ satisfies the O.D.E.

$$(G'-\eta^2 s^2)G'' + 5\eta(\eta-1)sG'$$
$$- 3(\eta-1)(3\eta-2)G = 0\,. \qquad (3.95)$$

(See Bluman and Cole (8).) Setting B_1, B_2, b_2 to zero, $\eta = 5/4$, and assuming that $b_1 = O(\delta^{1/3})$, we find that Eq.(3.93) satisfies Eq. (3.84) to within a distance $O(\delta^{1/3})$ of the shock. Analyzing the singular points of Eq. (3.95) we find a Prandtl-Meyer singularity at $\bar{x} = 0^+$, $\bar{y} = 0^-$, (i.e., $s \to -\infty$) when $b_1 = 0$ at the edge. Taking b_1 to be a function of $\bar{y}$ which behaves like $(\gamma+1)^{-1/3}\bar{x}_s$ near the edge and never exceeding $O(\delta^{1/3})$ far from the edge, the sonic line leaves the singularity along the $-\bar{y}$ – axis as required. Since $b_1 = O(\delta^{1/3})$ the error made in Eq. (3.91) is of higher order and will be recovered as the higher order equations are considered. Fortunately, for $\eta = 5/4$ the solution to Eq. (3.95) can be found in closed form (9,10)

$$\phi_h = \frac{9}{56}\,\alpha^3(-\bar{y})^{7/4}\left(-\frac{3}{5}\xi\right)^{-4/3}\left(\xi^2+\frac{14}{3}\xi+\frac{7}{27}\right), \qquad (3.96)$$

where α is an arbitrary scaling constant and

$$\left(\frac{3}{5}\xi+1\right)\left(-\frac{3}{5}\xi\right)^{-1/3} = -2\alpha s\,. \qquad (3.97)$$

Therefore, we find that an analytic solution of the homogeneous form of Eq. (3.91) can be found that satisfies both the shock and the Prandtl-Meyer boundary conditions.

Finding a solution to the inhomogeneous form of Eq. (3.91) is more difficult. One possibility is to express $\phi^{(4/3)}$ as an infinite power series in $(-\bar{y})$

$$\phi^{(4/3)} = \phi_h + \sum_{\nu}(-\bar{y})^{\nu}f_{\nu}(\xi)\,, \qquad (3.98)$$

with the ν's being selected so that the inhomogeneity in Eq. (3.91) is accounted for. Proceeding in this fashion, we find that the $f_\nu(\xi)$'s satisfy an inhomogeneous hypergeometric equation whose homogeneous solutions are terminating series in ξ. Therefore, as a practical matter, the $f_\nu(\xi)$'s are obtainable. The sonic locus computed from the inner solution, $\phi^{(5/3)} = \phi_h + (-\tilde{y})^{1\,1/6} f_{1\,1/6}(\xi)$, is shown in Fig. 3.5. Now, if Eq. (3.98) is to be a useful inner solution, it must be valid for $\tilde{y} \to -\infty$. Clearly, any finite sum does not have this property. As an alternative, let us consider the expression

$$\phi^{(5/3)} \approx \psi \equiv \phi_h + \frac{1}{4\gamma}\tilde{\lambda}^2 - \frac{2}{15}\frac{\gamma+1}{\gamma}\alpha^2(\gamma+1)^{-1/3}(-\tilde{y})^{5/2} \tag{3.99}$$

as a possible approximate solution valid for $-\infty \leq \tilde{y} \leq 0$. The motivation for selecting Eq. (3.99) is that it yields the same $\tilde{x}$ - velocity component as the match potential of Eq. (3.88) and in addition, satisfies the shock and Prandtl-Meyer conditions. Substituting Eq. (3.99) into Eq. (3.91), we find that ψ satisfies

$$-(\gamma+1)\frac{\partial\psi}{\partial\tilde{\lambda}}\frac{\partial^2\psi}{\partial\tilde{\lambda}^2} + \frac{\partial^2\psi}{\partial\tilde{y}^2} = -\frac{\gamma+1}{4\gamma^2}\tilde{\lambda} - R, \tag{3.100}$$

where the remainder is

$$R = \frac{\gamma+1}{2\gamma}\alpha^{-2}(\gamma+1)^{-1/3}(-\tilde{y})^{1/2} \cdot \left[1+\frac{3}{4}\left(-\frac{3}{5}\xi\right)^{-1/6}\left(\frac{7}{5}\xi+1\right)\right]. \tag{3.101}$$

Although it would be difficult to get a rigorous error bound on $|\phi^{(5/3)}-\psi|$, we can get some estimates of the degree to which ψ satisfies Eq. (3.91) in a global sense. Writing Eq. (3.100) in divergence form and then integrating over some closed region $\mathcal{a}$ in $\tilde{\lambda},\tilde{y}$, we get

$$\oint \underline{m}\cdot\left[-\frac{1}{}\,\frac{\gamma+1}{2}\left(\frac{\partial\psi}{\partial\tilde{\lambda}}\right)^2 + \frac{1}{}\,\frac{\partial\psi}{\partial\tilde{y}}\right]d\ell = -\iint_{\mathcal{a}}\left[\frac{\gamma+1}{4\gamma^2}\tilde{\lambda} + R\right]d\tilde{\lambda}d\tilde{y}, \tag{3.102}$$

where $\underline{m}$ is the outward normal to the boundary of $\mathcal{a}$. We first consider the region $\mathcal{a}_1$ near the Prandtl-Meyer singularity (see Fig. 3.4). There we readily find that the source due to the Prandtl-Meyer singularity, S_{pm}

$$S_{pm} = O\left(\frac{\sin^4\theta}{\cos^3\theta}\right), \tag{3.103}$$

is stronger than the effective reactive source

$$-\iint_{\mathcal{a}_1}\left[\frac{\gamma+1}{4\gamma^2}\tilde{\lambda} + R\right]d\tilde{\lambda}d\tilde{y} = O\left(\tilde{\lambda}^2\,\frac{\sin\theta}{\cos\theta}\right), \tag{3.104}$$

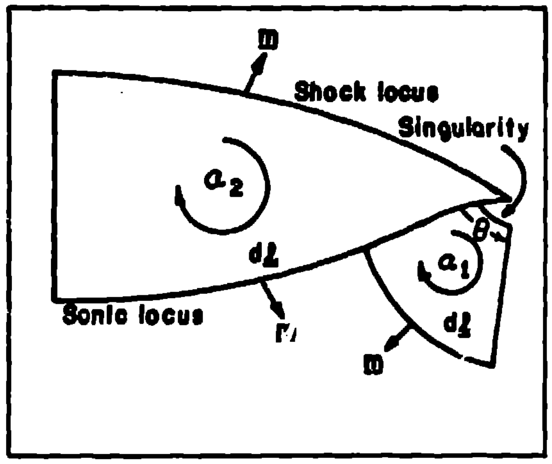

Fig. 3.4 - Regions of the flow over which the global accuracy of the flow ψ is examined. $\mathcal{a}_1$ is the neighborhood of the P-M singularity. $\mathcal{a}_2$ is the region of subsonic flow.

in the region of the singularity. Focusing our attention on region $\mathcal{a}_2$, we find that

$$\iint_{\mathcal{a}_2} R\, d\tilde{\lambda}d\tilde{y} = 0\ , \tag{3.105}$$

when the lower boundary (sonic locus) is taken as either the sonic locus for the near-field homogeneous flow given in Eq. (3.96) ($\xi = -1/3$) or the sonic locus for the far-field flow given in Eq. (3.88). For any other lower boundary of region $\mathcal{a}_2$

$$\iint_{\mathcal{a}_2}\tilde{\lambda}\, d\tilde{\lambda}d\tilde{y} = O((-\tilde{y})^{7/2}) \tag{3.106}$$

$$\iint_{\mathcal{a}_2} R\, d\tilde{\lambda}d\tilde{y} = O((-\tilde{y})^{1\,1/6}). \tag{3.107}$$

Therefore, ψ represents a reasonable approximation to Eq. (3.91) in a global sense. Comparing the inner sonic locus calculated via Eq. (3.98) (one term past the homogeneous solution) to that calculated via Eq. (3.99), we find little difference in the range $-0.8 < \tilde{y} < 0$ (see Fig. 3.5). Thus we conclude that Eq. (3.99) provides a reasonable approximation to the velocity $\partial\phi^{(5/3)}/\partial\tilde{\lambda}$. Since $\partial\phi^{(5/3)}/\partial\tilde{\lambda}$ contributes to the velocity at $O(\delta^{4/3})$, and whereas $\partial\phi^{(5/3)}/\partial\tilde{y}$ contributes at $O(\delta^{5/3})$, it follows that the boundary terms in Eq. (3.88) that have been omitted will not influence the solution up to and including $O(\delta^{4/3})$ in the velocity. If a solution valid to $O(\delta^{5/3})$ in the velocities is desired, the equation

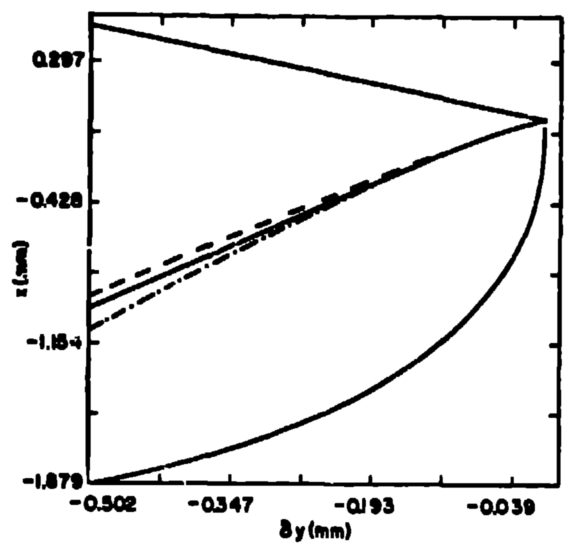

Fig. 3.5 - A comparison of the inner sonic locus as calculated with the flow of Eq. (3.96) (—•—), Eq. (3.98) (—), and Eq. (3.99) (– --) displayed in edge fixed coordinates. In all cases $b_1 = (\gamma+1)^{-1/3}\bar{x}_s$. The 1-term outer shock locus (upper curve) and sonic locus (lower curve) appear as references. The parameter values are D = 8 mm/μs, γ = 3, k = 2 μs^{-1}, δ = 0.1.

governing $\phi^{(2)}$ must be found (a simple matter), and Eq. (3.99) must be discarded in favor of a $\phi^{(4/3)}$ that satisfies all of the match conditions in Eq. (3.88).

Summarizing, we find that the inner velocities [i.e., Eqs. (3.89), (3.90), and (3.99)] have been calculated up to and including $O(\delta^{4/3})$. To this order, the shock locus is given by the 1-term outer solution [Eq. (3.49)]. However, probably the most important result is qualitative rather than quantitative. We find that the features of the inner flow depend on $\delta^{2/3}y$ and thus can penetrate well into the explosive.

E. The Composite Solution

The outer and inner solutions found in the previous sections are valid over only restricted regions in y. To get a uniformly valid asymptotic expansion of the solution, the two limiting solutions must be matched in a region of overlapping validity. Following Van Dyke's matching procedure, we express the outer solution in inner variables, the inner solution in outer variables, and then match at each order in δ (11). The composite expansion is then formed as the inner expansion plus the outer expansion minus the terms that are common to both in the overlap region. Retaining terms up to and including $O(\delta^{4/3})$ in the velocities, the composite velocity expansions are

$$u_x = \delta \frac{1}{\gamma}\left\{\sin\theta - \left[1-(1-\lambda)^{1/2}\right]\right\} - \delta^{4/3} \frac{1}{\gamma}\left(\frac{\gamma+1}{\gamma}\right)^{1/2} (-\bar{y})^{1/2}\ \frac{3}{4}\left(\xi+\frac{1}{3}\right)\left(-\frac{3}{5}\xi\right)^{-2/3} - \delta\ \frac{1}{\gamma}\left(\frac{\gamma+1}{\gamma}\right)^{1/2}(-\bar{y})^{1/2} \qquad (3.108)$$

$$u_y = -\ \delta\ \frac{1}{\gamma}(1-\cos\theta), \qquad (3.109)$$

where the match requires that

$$\alpha = \frac{\gamma^{2/3}}{(\gamma+1)^{1/3}}\ . \qquad (3.110)$$

To this order the shock locus is given by Eq. (3.49).

Equations (3.108), (3.109) and the required auxiliary equations consitute a full solution to $O(\delta^{4/3})$ of the boundary value problem posed in part A. of this section. Using it we will now determine some of the salient features of the flow. In all of the examples, we will take the function b_1 appearing in the similarity variable to be

$$b_1 = \delta^{1/3}\ \frac{k(\gamma+1)}{D\gamma^2}^{1/3}\tanh\left(-\frac{D\gamma}{k(\gamma+1)}\ \bar{y}\right)\ . \qquad (3.111)$$

Figures 3.6 and 3.7 show a comparison of the outer and composite solutions in the far and near fields respectively. To $O(\delta^{4/3})$, the shock loci for the two solutions are identical. The sonic loci, however, are quite dissimilar. Unlike the outer solution, the composite solution satisfies the conditions at the Prandtl-Meyer singularity (see Fig. 3.7). Perhaps the most striking feature of th composite flow is the range over which edge, through the inner soluti ., influences the flow. Figure 3.8 shows that the influence propagates in ≈ 50 reaction zone lengths. Considering that the inner scale is $\delta^{2/3}y$ and that $\delta^{2/3}$ changes by only a factor of two for $0.01 \le \delta^2 \le 1$, the range of influence of the inner solution (in real space) is nearly the same for large and small values of δ. However, since the outer scale is δy, the inner solution becomes relatively more important as δ is increased. Figure 3.9 shows that increasing δ from 0.1 to 0.33 makes the inner solution relatively more important.

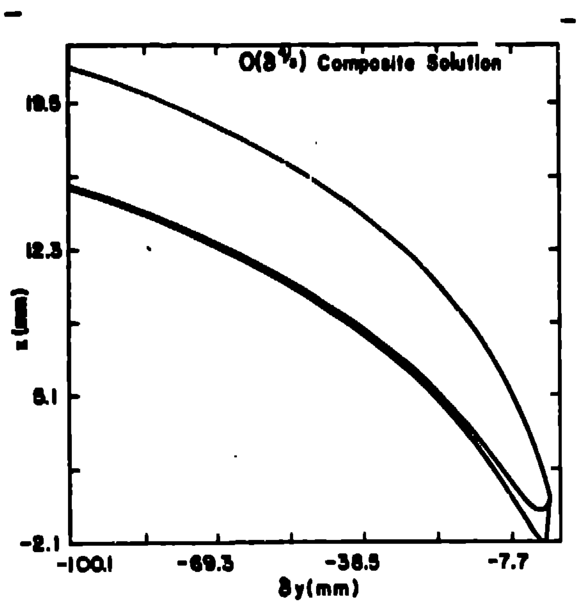

Fig. 3.6 - $O(\delta^{4/3})$ composite solution. The shock locus (upper curve), composite sonic locus (middle curve), and outer sonic locus (lower curve) in edge fixed coordinates. The parameter values are D = 8 mm/μs, γ = 3, k = 2 μs^{-1}, and δ = 0.1.

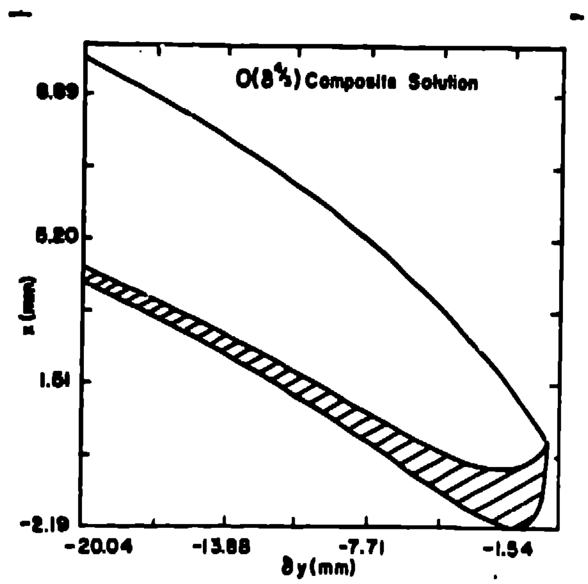

Fig. 3.8 - $O(\delta^{4/3})$ composite solution. See Fig. 3.6.

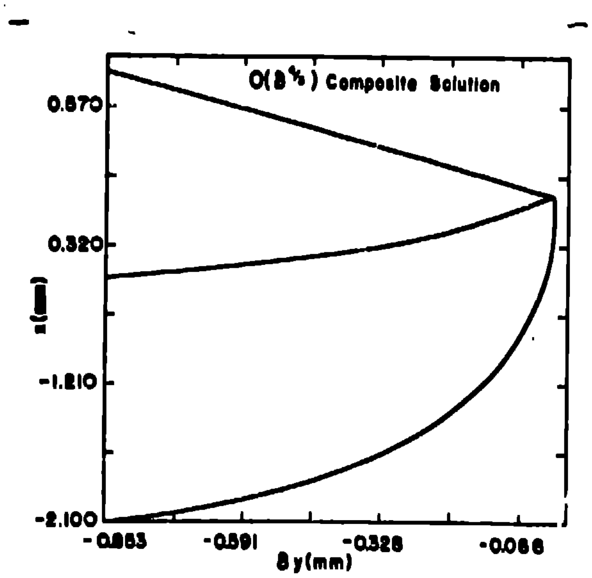

Fig. 3.7 - $O(\delta^{4/3})$ composite solution. See Fig. 3.6.

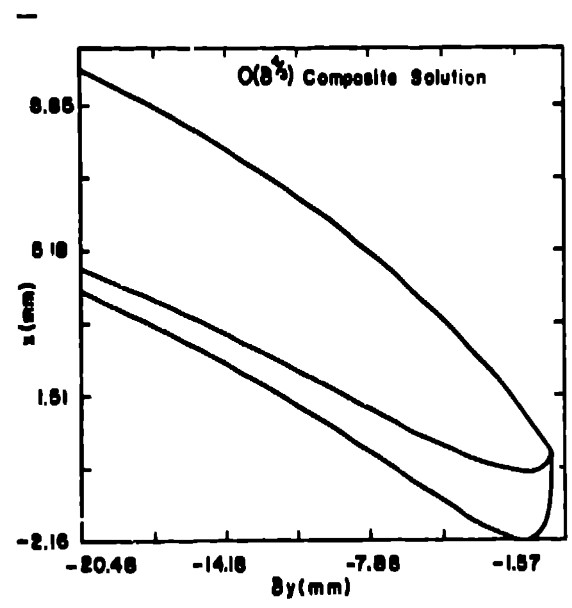

Fig. 3.9 - $O(\delta^{4/3})$ composite solution. The parameter values are D = 8 mm/us, γ = 3, k = 2 μs^{-1}, and δ = 0.33.

Although the sonic locus is not a flow property which is physically as apparent as the shock locus, it is of greater importance in determining the flow. This is because only the chemical energy released in the subsonic region is effective in driving the detonation. It is for this reason that proper satisfaction of the Prandtl-Meyer condition is crucial to any calculation. As an example, consider the case of an explosive charge of finite size. Including the energy released in the shaded area of Fig. 3.8 (unavailable energy) in the calculation of the detonation velocity would lead to a substantial error. More important perhaps is the effect that the form of the inner solution has on the problem of confinement. Considering the family of characteristics emanating from the Prandtl-Meyer singularity, we find a characteristic (the limiting characteristic) which is just tangent to the sonic locus (see Fig. 3.10). All the characteristics leaving the singularity downstream of the limiting characteristic never contact the sonic locus. Therefore, information about confinement traveling along them cannot influence the structure of the subsonic flow. Put another way, the limiting characteristic defines the critical degree of confinement below which the confinement has no influence on the structure of the subsonic flow.

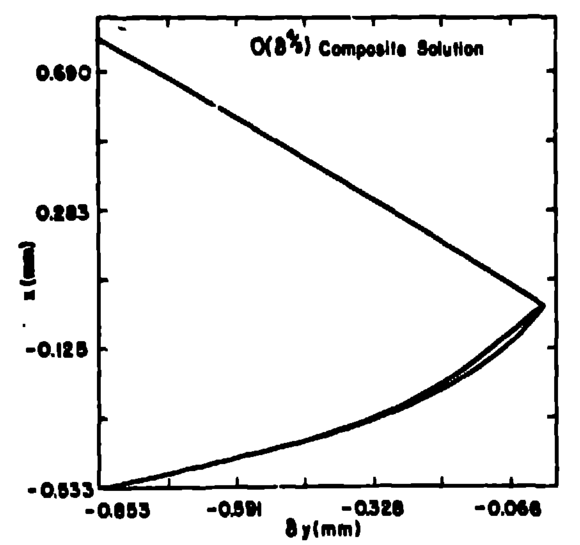

Fig. 3.10 - $O(\delta^{1/3})$ composite solution. The shock locus (upper curve), composite sonic locus (middle curve) and limiting characteristic (lower curve) in edge fixed coordinates. The parameter values are $D = 8$ mm/μs, $\gamma = 3$, $k = 2\ \mu s^{-1}$, and $\delta = 0.1$.

For the example considered in Fig. 3.10 the critical confinement angle (i.e., the angle that the wall makes with the the x-axis) for a sufficiently smooth wall is 1.94°. For a system with the parameter values given in Fig. 3.9, (i.e., increasing δ to 0.33) the critical confinement angle is 6.63°. In both cases, these angles are equal to the streamline angle at the sonic point on the shock. Therefore, we find that the resolved-δ^2 portion of the reaction zone for a system with an edge proceeds as an essentially unconfined detonation unless the confinement is heavy (i.e., aluminum or heavier).

F. Summary

Reviewing the results of this section, we find: (1.) Far from the edge (outer region) the flow is governed by O.D.E.'s (with independent variables x, δy) and the shock boundary conditions. The outer problem determines the shock locus to $O(\delta^{1/3})$ for $-\infty < y \le 0$ and the sonic locus in the very far field. (2.) Near the edge (inner region) the flow is governed by P.D.E.'s (with the independent variables $x/\delta^{1/3}$, $\delta^{1/3}y$) and the boundary conditions along the shock locus, at the Prandtl-Meyer singularity, and the match into the outer solution. The inner problem, which is strongly influenced by the singularity, has a long range influence on the sonic locus and properties which depend on it. (3.) The critical confinement angle is equal to the angle that the streamlines at the sonic point on the shock make with the edge.

ACKNOWLEDGEMENTS

I would like to thank W.C. Davis and Ray Engelke for stimulating discussions.

*This work was performed under the auspices of the United States Energy Research and Development Administration.

REFERENCES

1. J. B. Bdzil and W. C. Davis, Los Alamos Report, LA-5926-MS, p. 14 (1975).
2. A. H. Nayfeh, Perturbation Methods, p. 228, Wiley, New York (1973).
3. James Serrin, The Encyclopedia of Physics, Vol. VIII/1, p. 125, S. Flügge, Ed. Springer Verlag, Berlin (1959).
4. K. G. Guderley, The Theory of Transonic Flow, p. 112, Pergamon Press, Oxford (1962).

5. W. D. Hayes, J. Fluid Mech. 2, 595-600 (1957).

6. J. D. Cole, Perturbation Methods in Applied Mathematics, p. 21, Blaisdell, Waltham, Massachusetts (1968).

7. Lipman Bers, Mathematical Aspects of Subsonic and Transonic Gas Dynamics, p. 21, Wiley, New York (1958).

8. G. W. Bluman and J. D. Cole, Similarity Methods for Differential Equations, p. 143, Springer-Verlag, New York (1974).

9. R. Vaglio-Laurin, J. Fluid Mech. 9, 81-103 (1960).

10. S. V. Falkovich and I. A. Chernov, J. Appl. Math. Mech. 28, 342-347 (1964).

11. Milton Van Dyke, Perturbation Methods in Fluid Mechanics, p. 89, Academic, New York (1964).