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TITLE: A GENERAL CLASS OF NONLINEAR BIFURCATION PROBLEMS FROM A POINT IN THE ESSENTIAL SPECTRUM: APPLICATION TO SHOCK WAVE SOLUTIONS OF KINETIC EQUATIONS

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1. Introduction and Background from Mechanics.

We investigate an abstract class of bifurcation problems from the essential spectrum of the associated Fréchet derivative. This class is a very general framework for the theory of one-dimensional, steady profile traveling shock wave solutions to a wide family of kinetic integro-differential equations from non-equilibrium statistical mechanics [1,2]. Such integro-differential equations usually admit the Navier-Stokes system of compressible gas dynamics or the M.H.D. systems in plasma dynamics as a singular limit [3-5], and exhibit similar viscous shock layer solutions [6,7].

The mathematical methods associated to systems of Partial Differential Equations must however be replaced by the following considerably more complex Bifurcation Theory setting, first outlined in [8-10] for special cases. We actually consider a hierarchy of bifurcation problems, starting with a simple (solved) bifurcation problem from a simple eigenvalue.

Let $\mathcal{G}(\mu,f)$ be a nonlinear mapping from a Banach space X, into a Banach space Y, parametrized by μ:

(1.1) $$\mathcal{G}(\mu,f) : R^1 \times X \to Y.$$

Consider

(1.2) $$\mathcal{G}(\mu,f) = 0,$$

such that

$$\mathcal{G}(\mu,0) \equiv 0, \forall \mu \in R^1.$$

We admit all the necessary hypotheses to insure bifurcation at $\mu = \mu^*$, from a simple isolated eigenvalue of the Fréchet derivative

$$\mathcal{G}_f(\mu^*,0).$$

Classical theory [11] insures that, in some neighborhood of $(\mu^*,0)$ in $R^1 \times X$, there exists a second branch $\omega(\mu)$:

$$\text{(1.3)} \qquad \begin{aligned} \mathcal{G}(\mu, \omega(\mu)) &= 0 \\ \omega(\mu^*) &= 0. \end{aligned}$$

Thus, the primary hypothesis is bifurcation from a simple eigenvalue for the operator $\mathcal{G}$. In concrete cases, the relative bifurcated and trivial branches correspond to different asymptotic steady states at the "tails" of the shock wave (space-independent subsonic and supersonic states related by Rankine-Hugoniot conditions; $\mu = \mu^*$ corresponds to the transonic regime).

We shall actually investigate the more involved extended operator equation, for $x \in R^1$, $-\infty < x < +\infty$:

$$\text{(1.4)} \qquad \begin{aligned} A(\mu) \frac{\partial f}{\partial x} - \mathcal{G}(\mu,f) &= 0, \text{ or} \\ \mathcal{J}(\mu,f) &= 0, \text{ where} \end{aligned}$$

$$\text{(1.5)} \qquad A(\mu) : R^1 \times X \to Y$$

is a linear operator from X into Y, parametrized by $\mu \in R^1$. f is now a vector valued function of $x \in R^1$, $-\infty < x < +\infty$, with values in the Banach space X. We may restrict ourselves to spaces of absolutely continuous functions. If $A(\mu) \equiv I$ and $x \equiv t$, (1.5) reduces to an evolution equation

$$\text{(1.6)} \qquad \frac{\partial f}{\partial t} - \mathcal{G}(\mu,f) = 0,$$

and one looks for solutions which are trajectories between critical points of (1.1), i.e., the trivial solution and the bifurcated solution $\omega(\mu)$. Such a problem (1.6) of trajectories joining two steady asymptotic states, has first been considered by B. Matkowsky, using matched asymptotic expansions [12,13]; it has been investigated in depth by

G. Iooss [14] and K. Kirchgässner [15], within the Navier-Stokes context (see also [16]).

However, in (1.4-5), fundamental properties of the physical context impose somewhat pathological conditions on $A(\mu)$:

Hypothesis.

1) Zero belongs to the continuous spectrum of $A(\mu)$, $\forall\mu$, i.e.:

$$\overline{R(A(\mu))} = Y \text{ and } A(\mu)f = 0 \Leftrightarrow f = 0.$$

2) $A(\mu)$ is neigher positive nor negative semi-definite, nor is it accretive. As a corollary $A(\mu)^{-1}$ does not exist, $\forall\mu$. Recall that

$$\mathcal{G}_f(\mu^*, 0)^{-1}$$

does not exist either at $\mu = \mu^*$. In fact the properties of $A(\mu)$ are such that an initial value problem for (1.4) is ill-posed. Attempts to straightforwardly extend methods developed for (1.6) lead to erroneous results.

We still look for critical trajectories of (1.4), between the trivial solution and $\omega(\mu)$. We investigate the possible existence of a branch $\Omega(\mu, x)$, solution of (1.4) such that:

1) $\Omega(\mu^*, x) \equiv 0$, but $\Omega(\mu, x) \neq 0$, $\mu \neq \mu^*$;

2a) $\Omega(\mu, -\infty) = 0$, $\Omega(\mu, +\infty) = \omega(\mu)$; or

2b) $\Omega(\mu, +\infty) = 0$, $\Omega(\mu, -\infty) = \omega(\mu)$,

for μ close to μ^*. In an appropriate Barach space of absolutely continuous functions normalized at $\pm\infty$, the hypothetical non-trivial branch $\Omega(\mu, x)$ corresronds to bifurcation from the essential spectrum of:

$$\mathcal{F}_f(\mu,0) = A(\mu)\frac{\partial}{\partial x} - \mathcal{G}_f(\mu,0) \quad . \tag{1.7}$$

Specifically at $\mu = \mu^*$, zero is a limit point of the spectrum (a non-isolated eigenvalue in the essential spectrum). The kernel is non-trivial, as it includes that of $\mathcal{G}_f(\mu^*,0)$. The non-isolated character stems from the individual essential specra of $A(\mu)$ and $\frac{\partial}{\partial x}$. (1.4) must be considered as a bona-fide problem of bifurcation from the essential spectrum. Finally, we shall demonstrate the non-trivial result that $\Omega(\mu, +\infty) = \omega(\mu)$ or $\Omega(\mu, -\infty) = \omega(\mu)$ (critical trajectory). Since will will emphasize the mathematical techniques, we briefly review the relevance of (1.4) to fluid and statistical mechanics.

Steady profile shock waves in compressible fluid dynamics and magneto-hydrodynamics correspond to rather different mathematical theories according to the level and complexity of the fluid dynamical description. In order of increasing complexity, one has the well known hierarchy of equations, from the Euler level, to the compressible Navier-Stokes and the Magneto-hydrodynamic (M.H.D.) systems; and finally to the Boltzmann equation and the Kinetic integro-differential equations of collision-dominated plasmas. While viscosity terms are explicit in macroscopic Navier-Stokes equations, they are implicit in kinetic equations, where they result from explicit interparticle collision description on a microscopic scale. H. Grad [3-5] has carefully investigated the singular limit of the Boltzmann equation (for neutral gases) to the Navier-Stokes system when the mean free path between interparticle collisions (microscopic scale) becomes very small as compared to the macroscopic mean flow scale. His estimates do not cover, however, the shock case.

Hyperbolic systems are a standard tool for discontinuous shock solutions of Euler equations. Compressible Navier-Stokes systems exhibit viscous shock layers: in one dimension, Gilbard and Paolucci reduced them to a system of nonlinear autonomous Ordinary Differential Equations [6,7], and demonstrated that the shock layer is the unique trajectory between a node and a saddle point. For M.H.D. systems, such concepts have been extended by Conley and Smoller [17], using advanced tools of Topological Dynamics and Global Analysis. Yet none of the above mathematical methods apply to shock solutions of microscopic kinetic equations. Worse, it is well known that Partial Differential Equations approximations of the "13 moments" type break down at a finite Mach number ≈ 2 (non-existence of trajectories between critical points) [18]. The major problem is whether one can still consider the latter kinetic equations within the framework of critical orbits between critical states.

Moreover, there is plenty of experimental and numerical evidence for important microscopically originating effects observed in shock layers ruled by integrodifferential kinetic equations. Even in neutral gases, at small Mach numbers of 1.2 (weak shocks), a 40% deviation has been observed for the local ratio of the heat diffusion to the viscosity coefficients in the shock, as compared to predictions from the Navier-Stokes equation with Transport Coefficients calculated by the time-honoured Chapman-Enskog expansion [19]. This deviation is especially marked in the "hot (subsonic) tail" of the shock [20]. Previous numerical and experimental results have missed these important distortions by focusing only on the geometry of the sharp transition profile [21]. In (collision-dominated) Plasmas, small scale microinstabilities are as important as large scale (M.H.D.) macroinstabilities, and account for the difficulties in obtaining stable equilibria configurations.

The simplest kinetic equation - the Boltzmann operator for neutral rarefied gases, has been worked out in previous publications [8-10], and is summarized in the appendix. Technical estimates do, however, somewhat dissimulate the conceptual simplicity of the hierarchy of Bifurcation Problems.

Most of the pathology of the mathematical problem stems from the peculiar properties of the operator

$$A(\mu)\,\frac{\partial}{\partial x}\,.$$

This operator is nevertheless universally present in kinetic (statistical mechanics) equations. Usually called "the streaming operator", it represents transfer of very high velocity particles. The latter account for all deviations observed from Navier-Stokes. Physically, by traveling almost instantaneously in opposite upstream and downstream directions, these very high velocity particles cause a strong coupling between the asymptotic "tails" of the shock.

2. The Mathematical Problem: Principal Results.

We first consider a classical bifurcation setting for Problem I:

(2.1) $$\mathcal{G}(\mu, f) = 0$$

where $\mathcal{G}$ is a bounded nonlinear mapping from a Banach space X into a Banach space Y (usually graph-norm spaces):

$$\mathcal{G}(\mu, f) : R^1 \times X \to Y ,$$

(2.2) $$\mathcal{G}(\mu, 0) \equiv 0, \forall\mu .$$

$\mathcal{G}$ is assumed to be analytic, both in f and μ:

(2.3.a) $$\mathcal{G}(\mu,f) = \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{G}^{(n)} (0; (f)^n) ,$$

with the notations:

(2.3.b) $$\mathcal{G}^{(1)} (0; f) = T_\mu ,$$

(2.3.c) $$\frac{1}{2!} \mathcal{G}^{(2)} (0; f, f) = \Gamma_\mu (f, f) ,$$

(2.3.d) $$\sum_{n=2}^{\infty} \frac{1}{n!} \mathcal{G}^{(n)} (0; (f)^{(n)}) = \mathcal{N}(\mu, f) .$$

Hypothesis 0.

a) $\forall\mu$, T_μ is a Fredholm operator of index zero.

b) at $\mu = \mu^*$, dim ker $\{T_{\mu^*}\} = 1$ (zero is an isolated eigenvalue of T_{μ^*}).

c) $\mathcal{G}_{f,\mu}(0;\mu^*)h \notin R(T_{\mu^*})$, $\forall h \in \ker\{T_{\mu^*}\}$.

Conclusion. In some neighborhood of $(\mu^*, 0)$ in $R^1 \times X$, there exists a second branch $\omega(\mu)$:

(2.4) $$\mathcal{G}(\mu, \omega(\mu)) = 0 \text{ and } \omega(\mu^*) = 0 .$$

Hypothesis 0.d. The bifurcation is bilateral.

We now investigate Problem II. Let $x \in R^1$, $-\infty < x < +\infty$:

(2.5.a) $$A(\mu)\frac{\partial f}{\partial x} - \mathcal{G}(\mu,f) = 0,$$ equivalently

(2.5.b) $$\mathcal{F}(\mu,f) = 0,$$ where

(2.6) $$A(\mu) : R^1 \times X \to Y$$

is a bounded (in graph-norm) linear mapping from X into Y. $\mathcal{F}$ is a nonlinear mapping acting on spaces of vector-valued absolutely continuous functions:

(2.7) $$\mathcal{F} : R^1 \times AC^I[R^1 \to X] \to R^1 \times AC[R^1 \to Y].$$

The space AC^I is such that $\frac{\partial f}{\partial x} \in AC$. Generally, the AC norms are defined by:

$$|||f||| = \int_{-\infty}^{+\infty} \left\|\frac{\partial f}{\partial x}\right\|_{X,Y} dx .$$

In fact, we restrict ourselves to spaces such that $\frac{\partial f}{\partial x}$ is continuous. The absolutely continuous functions are normalized:

$$f \to 0 \text{ as } x \to -\infty \text{ \underline{or} } x \to +\infty .$$

Now, $\forall\mu$, $f \equiv 0$ is still a trivial solution. The question is whether there is still bifurcation for problem II, at $\mu = \mu^*$, such that:

$$\Omega(\mu,x) \in AC^I[R^1 \to X]$$

$$\Omega(\mu^*,x) = 0$$

$$\Omega(\mu,-\infty) = 0 \text{ \underline{or} } \Omega(\mu,+\infty) = 0 .$$

In the affirmative, one might speculate that

$$\Omega(\mu,+\infty) \equiv \omega(\mu) \text{ \underline{or} } \Omega(\mu,-\infty) \equiv \omega(\mu) .$$

This corresponds to bifurcation from the essential spectrum of:

(2.8) $$\mathcal{F}_f(\mu;0) = A(\mu)\frac{\partial}{\partial x} - T_\mu .$$

Specifically, at $\mu = \mu^*$, zero is a non-isolated point of the spectrum, with

$$\ker\{T_{\mu^*}\} \subset \ker\{\mathcal{F}_f(\mu^*;0)\} .$$

To insure the bifurcation, and as suggested by statistical mechanics, we need the further

Hypothesis 1. Zero belongs to the continuous spectrum of the linear operator $A(\mu)$, $\forall\mu$:

$$\overline{R(A(\mu))} = Y \text{ and } A(\mu)f = 0 \Leftrightarrow f = 0 .$$

Corollary. $(A(\mu))^{-1}$ is unbounded for every μ.

Remark that, for $\mu = \mu^*$, $T_{\mu^*}{}^{-1}$ does not exist either.

Hypothesis 1bis. $A(\mu)$ is neither positive nor negative definite, nor more generally accretive; moreover $(\lambda A(\mu) - T_\mu)^{-1}$ is not compact.

Thus one cannot construct any equivalent norm.

Hypothesis 2. The generalized spectrum of the operator

$$\lambda A(\mu) - T_\mu : X \to Y, \lambda \in C,$$

is included in two sectors, one in Re $\lambda < 0$, the other in Re $\lambda > 0$, uniformly in μ. (See Figure I). The generalized spectrum [27] is the set of λ such that $(\lambda A(\mu) - T_\mu)^{-1}$ does not exist, or is unbounded as a mapping from Y to X. Remark that $X \neq Y$, and $A(\mu) \neq I$.

From classical perturbation and invariance properties of Fredholm operators, we deduce from Hypotheses 0 and 1:

Theorem 2.1.

a) There exists a neighborhood of $\lambda = 0$ in C, where $\lambda A(\mu) - T_\mu$ is Fredholm of index zero, $\forall\mu$.

b) There exists a neighborhood of $(\mu^*, 0)$ in $R^1 \times C$, where $(\lambda A(\mu) - T_\mu)^{-1}$ has a simple pole in λ, corresponding to a simple generalized eigenvalue $\lambda_o(\mu)$:

$$\lambda_o(\mu) A(\mu) \hat{\varphi}_o(\mu) - T_\mu \hat{\varphi}_o(\mu) = 0 , \tag{2.9}$$

where $\hat{\varphi}_o(\mu) \in X$ is a generalized eigenfunction. |

Hypothesis 3. $\lambda_o(\mu)$ is real, and

$$\lambda_o(\mu) > 0 \text{ for } \mu > \mu^*,$$
$$\lambda_o(\mu) < 0 \text{ for } \mu < \mu^*.$$

This last hypothesis implies that the linearization (2.8) is <u>unstable</u> for $\mu > \mu^*$.

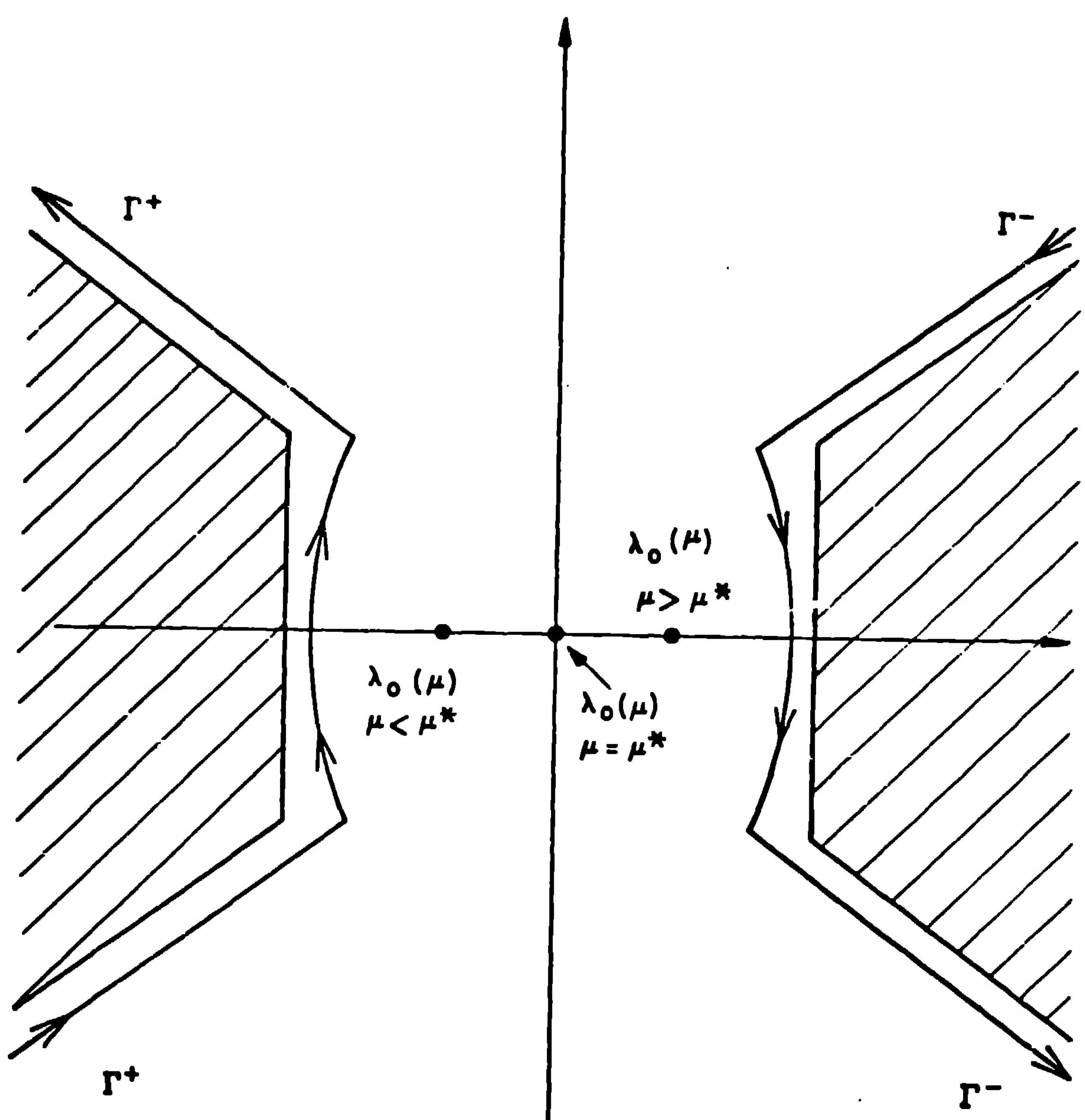

Figure I. The Spectrum of $(\lambda A(\mu) - T_\mu)$.

From Hypotheses 0-3, we demonstrate the fundamental

<u>Theorem 2.2</u>. There exists a neighborhood $\mathcal{D}$ of $(\mu^*,0)$ in $R^1 \times AC[R^1 \to X]$, where there exists another branch solution of problem II, $\Omega(\mu,x)$, unique up to a translation, with:

$$
\begin{aligned}
\text{a)}\quad \mu > \mu^* &: \Omega(\mu,-\infty) \equiv 0 \\
&\ \Omega(\mu,+\infty) \equiv \omega(\mu) \\
\text{b)}\quad \mu < \mu^* &: \Omega(\mu,+\infty) \equiv 0 \\
&\ \Omega(\mu,-\infty) \equiv \omega(\mu) \ ,
\end{aligned}
$$

and $\Omega(\mu^*, x) \equiv 0$ at $\mu = \mu^*$. $\omega(\mu)$ was defined in (2.4). Specifically, $\Omega(\mu, x)$ belongs to a closed subspace of $AC[R^1 \to X]$ such that, denoting by C^α the standard space of Hölder continuous functions of index α:

$$\frac{\partial\Omega}{\partial x} \in C^\alpha[R^1 \to X] \cap L^1[R^1 \to X]$$

$$\frac{\partial^2}{\partial x^2} A(\mu)\Omega \in C^\alpha[R^1 \to Y] \cap L^1[R^1 \to Y] \ ,$$

(with appropriate asymptotic decay conditions at $x = \pm\,\infty$, specified in later sections). Here, $0 < \alpha < 1$.|

As a word of caution, note that $\frac{\partial^2}{\partial x^2}\Omega$ does not exist. Recall that $(A(\mu))^{-1}$ does not exist, $\forall\mu$.

The pathology introduced by the operator $A(\mu)$ re-specifically new methods for the bifurcation Problem II. The general line is to attempt to rescue the time-honored Lyapunov-Schmidt decomposition, at the following cost:

1) The generalized Lyapunov-Schmidt decomposition requires infinite dimensional projection operators. These are constructed with the help of a generalized Operational Calculus, characterized by <u>non-commutativity</u> properties.

2) The first generalized Lyapunov-Schmidt equation is closely related to the essential spectrum and represents the "fast particles contribution". It is solved with the help of a generalized operator inverse; the latter is constructed with generalized holomorphic semi-groups which do not admit any infinitesimal generator.

3) The second generalized Lyapunov-Schmidt equation

is not a mapping on finite-dimensional spaces. Rather it is a Functional-Differential equation in the sole variable $x \in R^1$, $-\infty < x < +\infty$, and global (non-local) in character; the initial value problem is ill-posed. Moreover, this equation in itself is again a bifurcation problem from a purely continuous spectrum.

We outline the mathematical techniques in the next sections. Full details will appear in [22] and elsewhere.

3. A Generalized Operational Calculus, and the Derivation of the Generalized Lyapunov-Schmidt Equations.

Let

$$R(\lambda,\mu) = (\lambda\, A(\mu) - T_\mu)^{-1} \tag{3.1}$$

where T_μ is defined as the Frechet derivative of $\mathcal{G}(\mu, f)$ at $f = 0$ (2.3.b).

In order to construct appropriate projections associated to the isolated pole $\lambda_o(\mu)$, one cannot use the classical operational calculus based on Dunford Integrals of $R(\lambda,\mu)$, since the standard resolvent identity

$$R(\lambda) - R(\lambda') = (\lambda' - \lambda)\, R(\lambda)\, R(\lambda')$$

is false (non commutativity of $A(\mu)$ and T_μ). It must be replaced by the following correct identities:

$$A\, R(\lambda) - A\, R(\lambda') = (\lambda' - \lambda)\, A\, R(\lambda)\, A\, R(\lambda') \tag{3.2.a}$$

$$R(\lambda)\, A - R(\lambda')A = (\lambda' - \lambda)\, R(\lambda)A\, R(\lambda')A \tag{3.2.b}$$

Based on (3.2), a Generalized Operational Calculus is constructed, characterized by anticommutativity properties.

Proposition 3.1. There exists two families of projection operators:

$$E_{av}\,(\lambda_o(\mu)) : Y \to Y \quad \text{(range)}$$

$$E_{ap}\,(\lambda_o(\mu)) : X \to X \quad \text{(domain)}$$

associated to the generalized eigenvalue $\lambda_o(\mu)$, such that:

$$E_{av}(\lambda_o)\; A(\mu) = A(\mu)\; E_{ap}(\lambda_o)$$

where

$$P_{av}(\lambda_0(\cdot)) = \frac{1}{2\pi i} \int_{C(\lambda_0)} A(\cdot)\, R(\lambda, \cdot)\, d\lambda$$

$$P_{ap}(\lambda_0(\cdot)) = \frac{1}{2\pi i} \int_{C(\lambda_0)} R(\lambda, \cdot)\, A(\cdot)\, d\lambda$$

and [illegible] are [illegible] around λ_0 in the complex λ-plane.

[illegible]

[illegible] [?] introduced a somewhat [illegible] in a [illegible]-eigenvalue [illegible] [illegible] dim $N(T - \lambda K)$ = codim $N(T - \lambda K)$ [illegible], where $N(T - \lambda K)$ = span λ_0. This [illegible] strengthened by the [illegible] and [illegible] condition for λ to be [illegible] an isolated eigenvalue corresponding [illegible]. The generalized operators [illegible] with the [illegible] properties

a) [illegible] P_{av} [illegible]

b) [illegible] P_{av} [illegible]

[illegible] is easily [illegible] to the case of a [illegible] algebraically K simple [illegible].

[illegible], we have the [illegible] $P_{ap}(\lambda_0(\cdot))$ and $P_{av}(\lambda_0(\cdot))$ [illegible] and the [illegible] of both [illegible], where the latter in the [illegible] defined in (2.8). [illegible] with the [illegible] a [illegible] for [illegible] P_{av} [illegible] $P_{ap}(u)$ [illegible]

[illegible] $P_{av}(u)$ [illegible] $P_{ap}(u)$ [illegible]

Following the general trend of the Lyapunov-Schmidt method, we decompose the solution f in the space X as:

(3.4.a) $f = E_{ap}(\mu) f + (1 - E_{ap}(\mu)) f$

(3.4.b) $= c(x)\hat{\varphi}_o(\mu) + w(\mu, x)$, where

(3.4.c) $c(x)\hat{\varphi}_o(\mu) = E_{ap}(\mu) f$,

(3.4.d) $w(\mu, x) = (1 - E_{ap}(\mu)) f$,

and $\hat{\varphi}_o(\mu) \in X$ is the generalized eigenfunction associated to $\lambda_o(\mu)$ and defined in (2.9). In fact:

$$c(x) \in A\,C^I\,[R^1]$$

$$w(\mu, x) \in A\,C\,[R^1 \to (I - E_{ap})X]$$

Applying the projection $(I - E_{av}(\mu))$ to Problem II (2.5), using the anticommutation properties of Proposition (3.1) and the reduction properties of Corollary (3.3), we obtain the following generalized Lyapunov-Schmidt equations, which we shall call L1 and L2:

(3.5) (L1)

$$A(\mu)\frac{\partial w}{\partial x} - T_\mu w = (I - E_{av}(\mu))\,\mathcal{N}(\mu; c(x)\hat{\varphi}_o + w)$$

where $\mathcal{N}$, defined in (2.3.d) is the remainder of the analytic expansion of $\mathcal{G}(\mu,f)$ at $f = 0$, excluding the first order Frechet derivative T_μ, but <u>including</u> the second order derivative $\Gamma_\mu(f,f)$.

Should the major problem of constructing a pseudo-inverse for the left hand side of (3.5) be solved, the implicit function theorem in Banach Spaces would yield w as a functional of c(x):

(3.6) $w\,(c(x)) : AC^I[R^1] \to AC\,[R^1 \to (I - E_{ap})X]$.

To obtain the second Lyapunov-Schmidt equation, we apply the projection $E_{av}(\mu)$ to (2.5), use the anticommutation properties of Proposition (3.1) and assume the result (3.6). To simplify L2, we also need

<u>Theorem 3.4.</u> For a simple isolated eigenvalue $\lambda_o(\mu)$, there

exists $\eta(\mu) \in Y^*$ and $\psi(\mu) \in X^*$, such that:

$$A^*(\mu)\ \eta(\mu) = \psi(\mu)\ ,$$

$$E_{ap}(\mu)\ f = \hat{\varphi}_o(\mu)\ \langle\psi(\mu),f\rangle/\langle\psi(\mu),\hat{\varphi}_o(\mu)\rangle,$$

$$E_{av}(\mu)\ f = A(\mu)\ \hat{\varphi}_o(\mu)\ \langle\eta(\mu),f\rangle/\langle\psi(\mu),\hat{\varphi}_o(\mu)\rangle\ .|$$

With the previous result, L2 simplifies as:

$$\text{(L2)} \qquad \frac{dc}{dx} = \lambda_o(\mu)\ c(x) - k(\mu)\ (c(x))^2$$

$$\text{(3.7.a)} \qquad + \mathcal{H}\{c(x),w\},$$

where

$$k(\mu) = - \langle\eta(\mu),\Gamma_\mu(\hat{\varphi}_o,\hat{\varphi}_o)\rangle_{Y^*}$$

$$\text{(3.7.b)} \qquad k(\mu) \neq 0,\ k(\mu) = O(1),$$

$$\text{(3.8.a)} \quad \mathcal{H}\{c(x),w\} = \langle\eta(\mu),\ \mathcal{N}(\mu;\ c(x)\hat{\varphi}_o + w)\rangle_{Y^*}$$

$$- \langle\eta(\mu),\Gamma_\mu\ (c(x)\hat{\varphi}_o,c(x)\hat{\varphi}_o)\rangle_{Y^*}$$

and the normalization $\langle\psi(\mu),\hat{\varphi}_o(\mu)\rangle_{X^*} = 1$. ($\mathcal{N}$ and Γ have been defined in (2.3.c-d)).

A priori $\mathcal{H}$ is a mapping on $c(x)$ and w:

$$\text{(3.8.b)} \qquad \mathcal{H}:\ AC^I\ [R^1] \times AC\ [R^1 \to X] \to AC\ [R^1]\ ;$$

but with the implicit functional $w\{c(x)\}$ in (3.6):

$$\mathcal{H}:\ AC^I\ [R^1] \to AC\ [R^1],$$

where $w\{c\}$ depends globally upon $c(x)$, $-\infty < x < +\infty$. So in fact, L2 (3.7) is a functional differential equation for $c(x)$, global in nature. The initial value problem is a nonsense, as initial data ought to be specified $\forall x$, $-\infty < x < +\infty$! We remark that the differential part (including the quadratic term) of the functional differential equation L2 is in fact Landau's Equation [24]. The exact corrective term to Landau's model is, interestingly enough, neither polynomial, nor differential, but a non-local mapping $\mathcal{H}$.

We now define exactly the functional subspaces $\mathcal{S}_x$, $\mathcal{S}_y$, and $\mathcal{S}_c$ appropriate for the investigation of L1-L2:

$$\mathcal{S}_x \subset AC[R^1 \to (1-E_{ap})X] \qquad (3.9)$$
$$\mathcal{S}_y \subset AC[R^1 \to (1-E_{av})Y]$$

$f \in \mathcal{S}_x$ if:

a) $\frac{\partial f}{\partial x} \in C^{\alpha}[R^1 \to (I-E_{ap})X] \cap L^1[R^1 \to (1-E_{ap})X]$

b) $\frac{\partial^2}{\partial x^2} Af \in C^{\alpha}[R^1 \to (I-E_{ap})X] \cap L^1[R^1 \to (I-E_{ap})X]$

c) $\exp(-2\lambda_o x)\, \frac{\partial f}{\partial x}$ and $\exp(-2\lambda_o x)\, \frac{\partial^2}{\partial x^2} Af \in L^{\infty}[R^1]$ for $x<0$

(asymptotic decay at $x = -\infty$)

d) $\exp((\lambda_o-\varepsilon)x)\frac{\partial f}{\partial x}$ and $\exp((\lambda_o-\varepsilon)x)\frac{\partial^2}{\partial x^2} Af \in L^{\infty}[R^1]$ for $x>0$

(asymptotic decay at $x = +\infty$; $\varepsilon > 0$).

$f \in \mathcal{S}_y$ if the conditions a, c, d, (excluding b and any conditions for $\frac{\partial^2}{\partial x^2} Af$) are satisfied, with X replaced by Y, and E_{ap} by E_{av}. C^{α} is the usual Hölder space of index α, $0 < \alpha < 1$.

$c(x) \in \mathcal{S}_c$ if:

a) $\frac{dc}{dx} \in C^{o}[R^1] \cap L^1[R^1]$ and

b) $\exp(-\lambda_o x)\, \frac{dc}{dx} \in L^{\infty}[R^1]$, for $x < 0$,

c) $\exp((\lambda_o-\varepsilon)x)\, \frac{dc}{dx} \in L^{\infty}[R^1]$, for $x > 0$.

$c(x) \in \mathcal{S}_c^I$ if:

a) $c(x) \in \mathcal{S}_c$ and

b) $\frac{dc}{dx} \in \mathcal{S}_c$.

4. Methods of Solution for the Lyapunov-Schmidt and the Functional Differential Equations.

Consider the first Lyapunov equation (3.5) as a mapping $\mathcal{A}$:

$$\mathcal{A}(c,w) : \mathcal{S}_c^I \times \mathcal{S}_x \to \mathcal{S}_y\,, \qquad (4.1.a)$$

$$\mathcal{A}(0,0) = 0, \qquad (4.1.b)$$

$$\mathcal{A}(c,w) = A(\mu)\frac{\partial w}{\partial x} - T_\mu w$$

$$-(I-E_{av})\ \mathcal{N}(\mu;\ c(x)\ \hat{\varphi}_o + w) = 0 . \tag{4.1.c}$$

Theorem 4.1. $\mathcal{A}_w(0,0)$ is an isomorphism of $\mathcal{S}_x$ onto $\mathcal{S}_y$.|

Corollary 4.2. In some neighborhood of (0,0) in $\mathcal{S}_c^I \times \mathcal{S}_x$, there exists a unique continuous mapping:

$$w\{c(w),\mu\} : \mathcal{S}_c^I \to \mathcal{S}_x$$

such that:

$$\mathcal{A}(c(x),\ w\{c(x),\mu\}) \equiv 0.|$$

Theorem 4.1 hinges upon the existence of $\mathcal{A}_w(0,0)^{-1}$; let:

$$\mathcal{A}_w(0,0)f = A(\mu)\frac{\partial f}{\partial x} - T_\mu f = S(x) \tag{4.2}$$

To solve for f in (4.2), we construct generalized holomorphic semi-groups. The mapping $\mathcal{A}_w(0,0)$ acts from $(I-E_{ap})X$ into $(I-E_{av})Y$, cf. the reduction diagram (3.3). In particular the reduced operator $\lambda A(\mu) - T_\mu$ is invertible in a neighborhood of $\lambda_o(\mu)$ (deletion of the eigenvalue $\lambda_o(\mu)$). The essential spectrum (cf. figure I) remains only, which allows for the definition of Dunford Path Integrals along it. Let Γ^+, Γ^- be such paths along respectively the left and right side essential spectra (cf. figure I).

Proposition 4.3. If $S(x) \in \mathcal{S}_y$ in (4.2), then the solution $f \in \mathcal{S}_x$ of (4.2) is given by:

$$f(x) = \int_{-\infty}^{x} U^+(x-y)\,[S(y)-S(x)]\,dy + \int_{+\infty}^{x} U^-(x-y)\,[S(y)-S(x)]\,dy - T_\mu^{-1}\,S(x), \tag{4.3}$$

(T_μ^{-1} is the pseudo-inverse, which now does exist even at $\mu = \mu^*$);

$$U^\pm(x) = \frac{1}{2\pi i}\int_{\Gamma^+,\Gamma^-} e^{\lambda x}(\lambda A(\mu)-T_\mu)^{-1}\,d\lambda; \tag{4.4}$$

$\|U^\pm\| = O(\frac{1}{|x|}),\ |x| \to 0;$

although $U^\pm$ are not semi-groups, $A\,U^\pm$ and $U^\pm A$ are:

$$A\,U^\pm(x+y) = A\,U^\pm(x)\ A\,U^\pm(y), \tag{4.5}$$

for $x > 0,\ y > 0$ or $x < 0,\ y < 0$;

(4.6) $$\frac{\partial AU^{\pm}}{\partial x} f = T_{\mu} U^{\pm} f ,$$

(4.7) $$\frac{\partial U^{\pm} A}{\partial x} f = U^{\pm} T_{\mu} f \; ; \text{ but}$$

(4.8) $$\| T U^{\pm} \| = O(\tfrac{1}{x}), \; |x| \to 0,$$

and the holomorphic semi-groups (4.5) have no infinitesimal generator. |

A technical hypothesis needed for Proposition (4.1), and suggested by statistical mechanics is:

Hypothesis 4. Let $R(\lambda,\mu) = (\lambda A(\mu) - T_{\mu})^{-1}$; then as $|\lambda| \to \infty$, within the resolvent set:

(4.a) $$\| R(\lambda,\mu) \| = O(\frac{1}{|\lambda|^{\alpha}}), \; 0 < \alpha < 1$$

or:

(4.b) $$\| R(\lambda,\mu) \| = O(1).$$

Proof of Proposition (4.1) is more complicated with hypothesis 4.b. Specifically, $A\, U^{\pm}(0^{\pm})$ exist in the case of hypothesis 4.a (although limits are projections, but not the identity!), but are undefined under hypothesis 4.b.

We now investigate the functional differential equation L2 (3.7-8):

$$L2 : \mathcal{S}_c^I \to \mathcal{S}_c$$

(4.9) $$\frac{dc}{dx} = \lambda_o(\mu)\, c(x) - k(\mu)\, (c(x))^2 + \mathcal{H}(\mu, c, w(c)).$$

$\forall \mu$, $c \equiv 0$ is a trivial solution (as $\mathcal{H}$ is multilinear in c and w(c)). (4.9) is again a full-sized bifurcation problem from a continuous spectrum, at $\mu = \mu^*$;

$$\lambda_o(\mu^*) = 0, \; k(\mu^*) = O(1), \; \mu = \mu^*.$$

At $\mu = \mu^*$, the Frechet derivative of (4.9) reduces to $\frac{dc}{dx}$. The latter's spectrum, in spaces of absolutely continuous functions, is a purely continuous spectrum containing the full left or right half complex plane, including the imaginary axis (depending on normalization of the AC spaces).

New techniques are needed for (4.9). We first make the following remarks; in a neighborhood of $\mu = \mu^*$:

$$\lambda_o(\mu) = O(\mu-\mu^*) ,$$

$$\|c(x)\| = O(\mu-\mu^*) ,$$

(4.10) $$f = c(x)\, \hat{\varphi}_o(\mu) + O(\mu-\mu^*)^2$$

$$= \frac{\lambda_o(\mu)}{k(\mu)} \frac{\exp(\lambda_o x)}{\exp(\lambda_o x)+1} \hat{\varphi}_o(\mu) + O(\mu-\mu^*)^2 ;$$

the lowest order Landau differential operator approximation (4.10) is accurate only to $O(\mu-\mu^*)$. The exact $\mathcal{H}(\mu,c,w)$ contribution appears at $O(\mu-\mu^*)^2$ and corresponds to deviations from the "Navier-Stokes" solution (so called since the Landau equation (4.9) without the functional $\mathcal{H}$ admits the universal hyperbolic tangent Taylor weak shock profile for one-dimensional Navier-Stokes systems).

The key concept is to consider (4.9) not as a bifurcation from $c(x) = 0$, but as a branching from the Landau-Taylor profile

(4.11) $$f = \frac{\lambda_o(\mu)}{k(\mu)} \frac{\exp(\lambda_o x)}{1+\exp(\lambda_o x)} \hat{\varphi}_o(\mu) .$$

To do so, we introduce a change of function, a change of variable and a change of parameter in (4.9):

(4.12.a) $$\tau = \lambda_o(\mu) \Leftrightarrow \mu = \mu(\tau) ,$$

(4.12.b) $$y = \tau x = \lambda_o x ,$$

(4.12.c) $$c(x) = \tau \frac{1}{k(\mu)} \frac{e^y}{e^y+1} (1+\theta(y)) .$$

L2 becomes a functional-differential equation for $\theta(y)$, on $-\infty < y < +\infty$, parametrized by τ:

(4.13.a) $$\frac{d\theta}{dy} + \frac{e^y}{e^y+1}\theta = - \frac{e^y}{e^y+1} \theta^2 + \tau\, k(\tau)(1+e^{-y}) \tilde{\mathcal{H}}\{\theta\},$$

(4.13.b) $$\theta \equiv 0 \text{ at } \tau = 0 ;$$

let L3 be the operator defined by (4.13), then:

(4.13.c) $$L3 : \mathcal{S}_c^{\,I} \to \mathcal{S}_c ;$$

$\tilde{\mathcal{H}}(\theta(y))$ is identical to $\mathcal{H}(c(x))$ (3.8), after substitution of (4.12.a-b-c).

Whereas we look for $\theta \to 0$ as $\tau \to 0^+$ (branching from Landau's solution (4.11)), the former trivial branch $c(x) \equiv 0$ now becomes $\theta(y) \equiv -1$, $\forall\tau$. We have effectively achieved <u>separation of branches</u>. This is confirmed by:

<u>Theorem 4.4</u>. Let $\mathcal{L}(\theta)$ be the Fréchet derivative of L3 at $\theta = 0$:

$$\mathcal{L}(\theta) = \frac{d\theta}{dy} + \frac{e^y}{e^y+1}\,\theta \; ; \tag{4.14}$$

then $\mathcal{L}^{-1}$ is a bounded mapping from $\mathcal{S}_c$ onto $\mathcal{S}_c^{\,I}$, $\forall\tau > 0$.|

<u>Remark</u>: $\mathcal{L}^{-1}$ is an integral operator on $-\infty < y < +\infty$, which is in general unbounded on spaces of integrable functions. This required a much more complicated theory in [10]. If we do take into account the asymptotic decay conditions included in $\mathcal{S}_c$, $\mathcal{S}_c^{\,I}$ (3.9):

$$\exp(-y)\,\frac{d\theta}{dy} \text{ and } \exp(-y)\,\frac{d^2\theta}{dy^2} \in L^\infty(R^1)$$

for $y < 0$, and

$$\exp((1-\varepsilon)y)\,\frac{d\theta}{dy} \text{ and } \exp((1-\varepsilon)y)\,\frac{d^2\theta}{dy^2} \in L^\infty[R^1]$$

for $y \geq 0$, $\varepsilon > 0$, then $\mathcal{L}^{-1}$ is bounded from $\mathcal{S}_c$ onto $\mathcal{S}_c^{\,I}$. These decay conditions are, of course, suggested by the behavior of the derivatives of Landau's solution (4.11) at $y = \pm\infty$. To conclude:

<u>Corollary 4.5</u>. In some neighborhood of $\tau = 0$ in R^1, there exists a unique mapping

$$\tau \to \theta\{\tau\}$$
$$R^1 \to \mathcal{S}_c^{\,I},$$

such that $\theta\{\tau\}$ is the unique solution of (4.13) with $\theta\{0\} \equiv 0$.|

To demonstrate the corollary, we use the implicit function theorem applied to (4.13.a) considered as a mapping from: $R^1 \times \mathcal{S}_c^{\,I} \to \mathcal{S}_c$.

Finally from $\hat{\varepsilon}(\tau)$, we reconstruct

$$c(\tau) = \frac{\tau}{k}\,\frac{\exp(\tau x)}{1+\exp(\tau x)}\,(1+\hat{\varepsilon}(\tau))$$

$$f = c(\tau)\,\hat{\varphi}_o(\tau) + w\{c(\tau)\}\ .$$

The solution is actually unique up to a translation, since we have chosen an arbitrary (normalized) origin is obtaining the Landau profile (4.11) solution of:

$$\text{(4.15)} \qquad \frac{dc}{dx} = \lambda_o(\mu)\, c(x) - k(\mu)(c(x))^2 \ .$$

The asymptotic behavior of f at $x = \pm\infty$ shows that deviations from the "Navier-Stokes" component c(x), caused by w{c}, appear $O(\tau^2)$ in the "hot tail" of the shock. Roughly speaking, c(y) decays O(exp(y)) as $y \to -\infty$, whereas w(y) decays O(exp(2y)). As $y \to +\infty$, both c(y) and w(y) decay $O(\exp-(1-\varepsilon)y)$.

To conclude, we remark that the concept of modified Landau's equation has also been introduced by N. N. Janenko [25,26]: he has added higher order polynomial terms in c(x) to (4.15), in order to study the transition to turbulence in incompressible Navier-Stokes flows. Here, at the kinetic level, we have a corrective global functional operator $\mathcal{H}$.

A natural extension of Problems I-II is:

$$\text{(4.16)} \qquad \frac{\partial f}{\partial t} + A(\mu)\,\frac{\partial f}{\partial x} - \mathcal{G}(\mu,f) = 0\ ;$$

in this respect, we have the

Conjecture. For $\mu < \mu^*$, $\tau < 0$, the second branch $\Omega(\mu,x)$ is unstable in time; it is stable for $\mu > \mu^*$, $\tau > 0$. (This corresponds to well-known Entropy Conditions across the shock for Navier-Stokes). Also more general wave solutions of (4.16) may be investigated, including Burgers-like waves. Work is in progress on these questions.

APPENDIX

We summarize technical results of [8-10]. The Boltzmann equation [27] rules the evolution of a local velocity particle distribution $F(\vec{c})$, with the local velocity vector

(A.1) $$\vec{c} = (c_1, c_2, c_3);\ c = |\vec{c}| \ .$$

The space-independent Boltzmann operator:

(A.2) $$Q[F, F] = 0$$

is a bilinear integral operator in $L^2(R^3)$ It acts only upon the velocity vector $\vec{c}$. Classically:

(A.3) $$Q[F, F] = 0 \Rightarrow F \equiv \omega(\mu, \vec{c}) \ ,$$

where ω is a maxwellian (gaussian) distribution:

$$\omega(\mu, \vec{c}) = \frac{\rho}{(2\pi RT)3/2} \exp\left\{ - \frac{(c_1-\mu)^2+c_2{}^2+c_3{}^2}{2RT} \right\} \ ,$$

where ρ is the density, μ the mean velocity (directed along the x-axis) and T the temperature. These macroscopic quantities which appear in the Navier-Stokes equations, are simply related to weighted averages of $F(\vec{c}, x)$:

$$\rho = \int F(\vec{c},x)d\vec{c},\ \rho\mu = \int c_1\ F(\vec{c},x)d\vec{c} \ ,$$
$$3\ \rho\ R\ T = \int (\vec{c}-\vec{\mu})^2\ F(\vec{c},x)d\vec{c} \ ,$$

where R is the perfect gas constant.

In one space dimension, the space dependent Boltzmann equation for the velocity distribution

(A.4) $$F(c_1, c, x, t),\ c = |\vec{c}| \ ,$$
$$x \in R^1,\ -\infty < x < +\infty \ ,$$

becomes:

(A.5) $$\frac{\partial F}{\partial t} + c_1\frac{\partial F}{\partial x} = Q[F,F]$$

The second term on the left side is the one dimensional version of the ubiquitous "streaming operator"

$$\vec{c}.\ \nabla_x\ F \ .$$

We look for traveling waves of the type

$$F(c_1, c, x + \mu t) \ .$$

A viscous shock is defined as a nonlinear transition profile between two asymptotic ($x = \pm\infty$) Maxwellians; one with mean velocity μ^+ subsonic; the other with μ^- supersonic. It must be noted that the same Rankine-Hugoniot conditions as for Navier-Stokes uniquely relate μ^+, ρ^+, T^+ and μ^-, ρ^-, T^-. After renormalization [9,10]:

(A.6) $$(\mu+c_1)\frac{\partial f}{\partial x} = L_\mu f + \Gamma_\mu[f,f] ,$$

where L_μ is the Fréchet derivative of Q, and Γ_μ, an appropriate second order derivative; together with the normalization:

(A.7) $$f(c_1, c, -\infty) = 0 \text{ or } f(c_1, c, +\infty) = 0 .$$

(A.6) is investigated in a space AC of absolutely continuous functions:

$$AC[R^1 \to X] \to L^1_{loc}[R^1 \to Y]$$

(normalized at $\pm\infty$, cf. (A.7)) and X,Y are appropriate graph-norm Banach spaces defined uniquely on the velocity variable. The following is then demonstrated:

Proposition A.1. In appropriate spaces X,Y (implicitly incorporating the Rankine-Hugoniot conditions),

(A.2bis) $$Q[f,f] = L_\mu f + \Gamma_\mu[f,f]$$

is a bifurcation problem from a simple isolated eigenvalue of L_{μ^*} at the critical sonic value of $\mu = \mu^*$. The two branches correspond to a subsonic and a supersonic Maxwellian, identical at $\mu = \mu^*$.|

Looking for a critical trajectory joining the two asymptotic bifurcated subsonic and supersonic maxwellians, we consider (A.6) as a bifurcation problem from the essential spectrum, superimposed upon the simple bifurcation problem (A.2bis). In (A.6), $f \equiv 0$ is indeed a trivial solution $\forall\mu$. The essential spectrum is evident from the identification:

(A.8) $$A(\mu) \equiv (\mu+c_1)I ,$$

which does not possess an inverse in $L^2(R^3)$, since

$$-\infty < c_1 < +\infty$$

(cf. $c_1 = -\mu$). The hypothesis required by the abstract setting are completed through the:

Proposition A.2. The generalized eigenvalue problem

$$\lambda(\mu+c_1)\varphi - L_\mu\varphi = 0$$

has a real, simple, isolated eigenvalue $\lambda_o(\mu)$, in the spaces X and Y:

$$\lambda_o(\mu) < 0, \ \mu < \mu^*$$

$$\lambda_o(\mu) > 0, \ \mu > \mu^*. |$$

Similar results were obtained by H. Weyl in 1949 [28], for the Navier-Stokes equations linearized about sub- or supersonic equilibria. Finally, the "streaming operator" $A(\mu)$ defined in (A.8), though responsible for the pathology of the problem, is universally present in kinetic (statistical mechanics) equations. It represents transfer of very high velocity particles, and generates the essential spectrum of kinetic operators.

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