LEGIBILITY NOTICE

A major purpose of the Technical Information Center is to provide the broadest dissemination possible of information contained in DOE's Research and Development Reports to business, industry, the academic community, and federal, state and local governments.

Although portions of this report are not reproducible, it is being made available in microfiche to facilitate the availability of those parts of the document which are legible.

3

LA-UR-88 -1742

CONF-880786--1

Received by OSTI

AUG 04 1988

Los Alamos National Laboratory is operated by the University of California for the United States Department of Energy under contract W-7405-ENG-36.

LA-UR--88-1742

DE88 014478

TITLE: **1-D Closure Models for Slender 3-D Viscoelastic Free Jets: von Kármán Flow Geometry and Elliptical Cross Section**

AUTHOR(S): *S. E. Bechtel, M. G. Forest, D. D. Holm, and K. J. Lin*

SUBMITTED TO: Proceedings of IUTAM meeting July 1988, Cincinnati, OH

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

By acceptance of this article, the publisher recognizes that the U.S. Government retains a nonexclusive, royalty-free license to publish or reproduce the published form of this contribution, or to allow others to do so, for U.S. Government purposes.

The Los Alamos National Laboratory requests that the publisher identify this article as work performed under the auspices of the U.S. Department of Energy.

Los Alamos National Laboratory
Los Alamos, New Mexico 87545

MASTER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

FORM NO 836 R4
ST. NO 2629 5/81

Darryl–

1-D Closure Models
For Slender 3-D Viscoelastic Free Jets:
von Kármán Flow Geometry and Elliptical Cross Section

S.E. Bechtel[1], M.G. Forest[2], D.D. Holm[3], and K.J. Lin[1]

[1] Department of Engineering Mechanics
The Ohio State University, Columbus, Ohio 43210

[2] Department of Mathematics
The Ohio State University, Columbus, Ohio 43210

[3] Center for Nonlinear Studies and Theoretical Division
Los Alamos National Laboratory, MS B284,
Los Alamos, NM 87545

In this paper we derive one space dimensional, reduced systems of equations (1-D closure models) for viscoelastic free jets. We begin with the three-dimensional system of conservation laws and a Maxwell-Jeffreys constitutive law for an incompressible viscoelastic fluid. First, we exhibit *exact truncations* to a finite, closed system of 1-D equations based on classical velocity assumptions of von Kármán [1]. Next, we demonstrate that the 3-D free surface boundary conditions overconstrain these truncated systems, so that only a very limited class of solutions exist. We then proceed to derive *approximate 1-D closure theories* through a slender jet asymptotic scaling, combined with appropriate definitions of velocity, pressure and stress unknowns. Our nonaxisymmetric 1-D slender jet models incorporate the physical effects of inertia, viscoelasticity (viscosity, relaxation and retardation), gravity, surface tension, and properties of the ambient fluid, and include shear stresses and time dependence. Previous special 1-D slender jet models correspond to the lowest order equations in the present asymptotic theory by an *a posteriori* suppression to leading order of some of these effects, and a reduction to axisymmetry.

Solutions of the lowest order system of equations in this asymptotic analysis are presented: For the special cases of elliptical inviscid and Newtonian free jets, subject to the effects of surface tension and gravity, our model predicts oscillation of the major axis of the free surface elliptical cross section between perpendicular directions with distance down the jet, and draw-down of the cross section, in agreement with observed behavior. In viscoelastic regimes, our model predicts swell of the elliptical extrudate and distortion of the elliptical extrudate cross section from the dimensions of the die aperture. Higher order corrections to these solutions can be examined, to test the validity of the lowest order asymptotic equations and obtain more detailed information on the jet behavior.

I. INTRODUCTION AND HISTORY

There are at least two motivations for one space dimensional (1-D) models of free, 3-D fluid jets. For engineering applications such as ink jet printing, polymer extrusion and fiber spinning, there is a need to reproduce and predict experimental jet phenomena with a simple and tractable system of equations. This has been a dominant theme in the history of the subject. Secondly, in light of the measured success of 1-D models in certain specific jet applications, it is natural to ask why the lower dimensional models are able to model 3-D phenomena. Can these 1-D models be derived in some approximate sense from the full 3-D free boundary value problem (b.v.p.)?

Our primary purpose in this paper is to answer this question. We: 1°) derive 1-D models for free jets *from* the 3-D free surface b.v.p.; and 2°) clarify the sense in which the 1-D models approximate and are consistent with the full 3-D b.v.p.. In answering this question, we find that existing 1-D models correspond to particular specifications of fluid and flow properties *within one comprehensive theory*.

The 1-D jet models are a truncation of the full 3-D system (which has infinite modes in three space and one time dimensions) *to a finite number of unknowns* (modes) *in one space* (axial coordinate) *and one time dimensions*.

Analogous truncations occur in all numerical simulations of 3-D fluids. For example, in spectral methods one chooses to truncate at some finite term in the Fourier mode expansion. In specific applications, often one exploits special properties and/or symmetries of the full 3-D b.v.p. to truncate modes and/or spatial dimensions. Two examples are the *exact truncation* to vortex sheet and vortex layer equations for 3-D Euler flows, and the von Kármán [1] velocity profile assumption for 3-D Newtonian flow between rotating concentric cylinders.

When the truncation scheme is *not* an exact reduction of the full system, an art arises as to the best way to "close the system" and produce the same number of equations as unknowns (*a closure model*). Rarely can or does one qualify the sense in which a truncated, non-exact closure model approximates the full system. The proof is usually by comparison with experiments. A novelty of the present application to 3-D jet flows is that *we deduce asymptotically valid, 1-D closure models from the full 3-D b.v.p.* The asymptotics is based on a slender jet geometry.

Throughout this paper we refer to the unknowns as *modal variables*, by analogy with amplitude variables in Fourier mode expansions. We then refer to the reduced equations that govern these unknowns as *modal equations*.

The remainder of this paper is organized as follows. In Section II we discuss an exact truncation for non-Newtonian *unbounded* flows to a finite closed system of 1-D modal equations. (This exact truncation will arise later as the "zeroth order" basis of our perturbation theory for bounded, free surface flows.) We then note that when a free surface is introduced, the 3-D interfacial boundary conditions overconstrain the previously closed system of exact equations, so that only trivial solutions exist.

In Sections III–IV we show that an approximate 1-D closure theory can be salvaged in an appropriate scaling limit. In essence, we exploit the exact 1-D closure model of Section II in a perturbation expansion, with a slenderness ratio as the perturbation parameter.

There is a long history of approximate 1-D models for free Newtonian and viscoelastic jets, often referred to as the "thin filament" or "slenderness" approximation, or "nearly elongational" flows. The original formulation is due to Matovich & Pearson [2] in the study of fiber spinning. Many authors have since adopted their perturbation scheme, which is purely formal since the perturbation parameter is not identified in terms of any specific dimensionless flow or fluid parameter. This heuristic aspect of the theory clouds applications of the scheme since there is no physical scaling hypothesis. In general, the range of assumptions and validity of individual 1-D jet models is unknown. Also the existing models are presented and applied under a variety of *a priori* restrictions (e.g., in the absence of one or more of time dependence, shear stresses, gravity, inertial effects and surface tension), as dictated by the particular applications. All existing models are axisymmetric.

(A second group of 1-D jet models are based on *posited* self-consistent 1-D models (cf. [3]). The connections between the posited 1-D models and derivations from the 3-D free surface boundary value problem are discussed in [4].)

In Section IV we derive an asymptotically valid, 1-D theory of slender jet closure models. The theory is comprehensive, in that we begin with the full 3-D free surface boundary value problem, with the following physical effects incorporated: time dependence, shear stresses, inertial effects, viscoelasticity (viscosity, relaxation and retardation effects), gravity, surface tension and properties of the ambient fluid as they appear in the free surface interfacial conditions. In this way we develop the general context under which every 1-D jet closure model (with these physical effects, constitutive law and choice of modal variables) is deduced.

As is shown in Section V, existing 1-D jet theories correspond in this general framework to the lowest order equations in the asymptotic expansion, with *a posteriori* suppression to leading order of many of the physical effects. We thereby derive previous 1-D models from the 3-D free surface boundary value problem and clarify the sense of the 1-D closure model approximation.

In addition, we have deduced new, asymptotically valid, 1-D closure models for viscoelastic free jets. One particular new feature is the extension to elliptical free surface symmetry. Moreover, *higher order corrections* are available from this analytic framework, both from within a specific model and due to physical effects that are suppressed in the lowest order equations.

We begin with the equations of motion for an arbitrary, incompressible 3-D continuum:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = \rho\mathbf{g} + \mathrm{div}\,\mathbf{T},$$
$$\mathbf{T} = -p\mathbf{I} + \hat{\mathbf{T}} = \mathbf{T}^T, \quad div\,\mathbf{v} = 0. \qquad (I.1)$$

Here $\mathbf{v}$ is the velocity, $\hat{\mathbf{T}}$ is the determinate part of the stress tensor $\mathbf{T}$, p is the constraint pressure, ρ is the mass density (assumed constant), and $\rho\mathbf{g}$ is the gravitational body force. Equations (I.1a) and (I.1b) are balance laws for linear momentum and angular momentum, and (I.1c) is the incompressibility constraint.

A constitutive law must be adjoined to determine the stress $\hat{\mathbf{T}}$. In this paper we consider viscoelastic fluids and adopt a Maxwell-Jeffreys constitutive model:

$$\hat{\mathbf{T}} + \lambda_1 \frac{\mathcal{D}}{\mathcal{D}t}\hat{\mathbf{T}} = 2\eta\left(\mathbf{D} + \lambda_2 \frac{\mathcal{D}}{\mathcal{D}t}\mathbf{D}\right). \qquad (I.2)$$

The operator $\frac{\mathcal{D}}{\mathcal{D}t}$ must be suitably invariant; we choose a one-parameter family with rate parameter a,

$$\frac{\mathcal{D}}{\mathcal{D}t}(\bullet) = \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)(\bullet) + (\bullet)\mathbf{W}$$
$$- \mathbf{W}(\bullet) - a[(\bullet)\mathbf{D} + \mathbf{D}(\bullet)]. \qquad (I.3)$$

For the special values $a = 1, -1, 0$, the rate (I.3) is commonly referred to as upper convected, lower convected and corotational, respectively. The tensors $\mathbf{D}$ and $\mathbf{W}$ are the symmetric and skew parts of the velocity gradient.

II. EXACT CLOSURE MODELS

The assumed velocity profile that reveals a separation of variables and the choice of velocity modal variables in this theory begins with a generalization of the *von Kármán velocity ansatz* [1]:

$$\mathbf{v} = [x\Omega_1(z,t) - y\psi(z,t)]\mathbf{e}_1$$
$$+ [y\Omega_2(z,t) + x\psi(z,t)]\mathbf{e}_2 + v(z,t)\mathbf{e}_3. \qquad (II.1)$$

This is the most general linear polynomial in x and y which has reflection symmetry with respect to the (x,z) and (y,z) planes *; the original von Kármán ansatz assumes axisymmetry, i.e., symmetry about all planes containing the z axis, so that $\Omega_1 \equiv \Omega_2$. Here x, y, z denote the usual Cartesian coordinates, $\mathbf{e}_j$ $(j = 1, 2, 3)$ denote the corresponding base vectors and $\mathbf{e}_3$

* The truncated expansion (II.1) is presented for the purpose of exhibiting *exact* reductions of the 3-D problem. For the asymptotic scaling of Section VI, the velocity is assumed only to be expressible as a series in transverse coordinates x, y, which agrees with (II.1) to linear terms in the expansion.

represents the axial direction. Consistent with the assumption (II.1), we take the gravitational body force $\rho \mathbf{g}$ to be along $\mathbf{e}_3$.

In addition to the velocity ansatz (II.1) we assume that stress and pressure are also given by truncated power series in x, y. Substituting these expansions into the 3-D field equations (I.1), (I.2) and equating coefficients of powers of x, y yields the exact closure model of 28 equations in 28 unknown functions of z and t, involving five arbitrary functions of z under three constraints (see [5] for details).

We now assume that the fluid is bounded by an elliptical free surface, given by

$$\frac{x^2}{\phi_1^2(z,t)} + \frac{y^2}{\phi_2^2(z,t)} = 1. \tag{II.2}$$

Each cross section $z = z_0$ is an ellipse with semi-axis lengths ϕ_1, ϕ_2, which deform in z and t. See Figure 1. The surface unknowns $\phi_1(z,t)$, $\phi_2(z,t)$ are additional modal variables. To complete the 3-D viscoelastic free surface boundary value problem, we adjoin to (I.1), (I.2) the interfacial boundary conditions:

1) *The kinematic boundary conditions*: The free surface is convected with the fluid. From the velocity ansatz (II.1) and free surface ansatz (II.2), this condition yields:

$$\phi_{\alpha,t} + v\phi_{\alpha,z} = \phi_\alpha \zeta_\alpha, \ \alpha = 1, 2, \quad (\phi_1^2 - \phi_2^2)\psi = 0. \tag{II.3}$$

The second condition is very restrictive: *either* there is no swirl ($\psi = 0$), *or* the swirling flow must be axisymmetric ($\phi_1 = \phi_2$ and $\zeta_1 = \zeta_2$). *For the remainder of this paper we restrict to the case of no swirl,* $\psi \equiv 0$.

2) *The kinetic boundary conditions*: Shear stresses are assumed continuous across the fluid/ambient interface, whereas the normal stress is discontinuous. The jump in normal stress across the free surface is assumed to be balanced by the constant *surface tension* σ times the *free surface mean curvature* κ. These conditions state:

$$\mathbf{t}_f - \mathbf{t}_a = -\sigma \kappa \mathbf{n}, \tag{II.4}$$

where $\mathbf{t}_f$ and $\mathbf{t}_a$ are the boundary stress vectors in the jet and ambient material, respectively, and $\mathbf{n}$ is the unit outward normal to the interface. For the free surface (II.2) the mean curvature κ is given by

$$\begin{aligned}\kappa(\theta, z, t) = &-[(\phi_1^2 \sin^2\theta + \phi_2^2 \cos^2\theta)(\phi_{1,zz}\phi_2 \cos^2\theta \\ &+ \phi_{2,zz}\phi_1 \sin^2\theta) + 2(\phi_1\phi_{2,z} - \phi_2\phi_{1,z})(\phi_1\phi_{1,z} \\ &- \phi_2\phi_{2,z})\cos^2\theta \sin^2\theta - \phi_1\phi_2(\phi_{1,z}^2 \cos^2\theta \\ &+ \phi_{2,z}^2 \sin^2\theta + 1)][(\phi_1\phi_{2,z}\sin^2\theta + \phi_2\phi_{1,z}\cos^2\theta)^2 \\ &+ \phi_1^2 \sin^2\theta + \phi_2^2 \cos^2\theta]^{-3/2}\end{aligned} \tag{II.5}$$

We further assume the ambient material exerts a constant pressure p_a:

$$\mathbf{t}_a = -p_a \mathbf{n}. \tag{II.6}$$

Given these free surface boundary conditions, any closure model for free jets derived from the 3-D theory which is based on the elliptic von Kármán ansatz (II.1) (thus far, all are based on the axisymmetric special case) must respect these boundary conditions. For our exact closure model, the kinetic boundary conditions (II.4) *overdetermine* the system of modal equations, so that only a very limited class of solutions exist. This is because the power series definition of stress and pressure modal variables forces the stress and pressure variables into the boundary conditions. See [5] for details. We now return to the general situation where there is no exact power series truncation, and reassess the choice of modal variables.

III. INTEGRATED MOMENTUM AND CONSTITUTIVE EQUATIONS : SELECTION OF STRESS AND PRESSURE MODAL VARIABLES

To derive 1-D jet models from the 3-D theory with the necessary flexibility to describe interesting behavior, such as non-axisymmetric free jets and jets with swell, we retain the power series assumption (II.1) on $\mathbf{v}$, but choose stress and pressure unknowns to be integrals over the jet cross section. This approach is taken by [2, 6].

This leads us to two important points. First, our power series ansatz for $\mathbf{v}$ limits the ability of this theory to meet velocity boundary conditions, such as no slip. Since boundary values of velocity are explicit combinations of the velocity modal variables (i.e., the coefficients in the power series expansion) and free surface modal variables ϕ_1 and ϕ_2, the imposition of a condition on velocity at the boundary would constrain the velocity within the cross section. However, our second point is that, historically, the reason for the use of area-averaged stress and pressure variables (rather than pointwise, power series expansions) is precisely to not limit the ability to meet stress boundary conditions for free jets. With the power series expressions for stress and pressure there is not enough flexibility to meet the stress boundary conditions, for the same reason the $\mathbf{v}$ ansatz fails to meet flow boundary conditions. The boundary values of stress and pressure are explicit combinations of the stress and pressure modal variables (i.e., the coefficients in the power series expansions) and free surface modal variables ϕ_1 and ϕ_2. Therefore the boundary conditions (II.3) and (II.4) are coupled to the modal equations as severe constraints on the class of solutions of the modal equations, and hence limit the ability to model interesting free jet phenomena.

The first step is to compute certain cross sectional area integrations and moment integrations of the components of the conservation of momentum equations (I.1a), *evaluated on the velocity ansatz (II.1). We make no a priori stress and pressure modal ansatz.*

We compute the following integrations over the area A bounded by the ellipse, at fixed z, given by (II.2):

$$\iint_A \{x \cdot (\mathrm{I.1a})_1\} dA, \quad \iint_A \{y \cdot (\mathrm{I.1a})_2\} dA, \quad \iint_A (\mathrm{I.1a})_3 dA, \qquad \text{(III.1)}$$

where, for instance, $(\mathrm{I.1a})_1$ indicates the component of the vector equation (I.1a) along e_1. One uses the divergence theorem, integration by parts, and Leibniz' rule for differentiation of integrals, and *all boundary terms involving p and $\hat{T}_{ij}$ either cancel, or what remains is precisely the linear combination that appears in the interfacial kinetic boundary conditions* (II.4). Thus, one "incorporates" the boundary conditions (II.4) into the integrated momentum equations; in other words, one replaces the boundary values of stress and pressure by the mean curvature, surface tension, and ambient pressure variables. The resulting exact equations (given in [5]) involve the following integrated stress and pressure variables:

$$\begin{aligned} A_{11} &\equiv \iint_A \hat{T}_{11} dA, \quad A_{22} \equiv \iint_A \hat{T}_{22} dA, \\ A_{33} &\equiv \iint_A \hat{T}_{33} dA, \quad A_{131} \equiv \iint_A \hat{T}_{13} x dA, \qquad \text{(III.2)} \\ A_{232} &\equiv \iint_A \hat{T}_{23} y dA, \quad P \equiv \iint_A (p - p_a) dA. \end{aligned}$$

We now compute area and moment integrations of the constitutive equations (I.2) to obtain equations for the resultants A_{11}, A_{22}, A_{33}, A_{131}, A_{232}. The necessary integrations are

$$\iint_A (I.2)_{11} dA, \iint_A (I.2)_{22} dA, \iint_A (I.2)_{33} dA,$$
$$\iint_A x(I.2)_{13} dA, \iint_A y(I.2)_{23} dA. \qquad (III.3)$$

Once again, in a calculation involving integration by parts and Leibniz' rule, all of the boundary terms cancel in each of these integrated equations.

Thus, we have the good fortune of boundary values of stress not entering into the stress resultant equations and integrated conservatiuon equations. *The integration technique has decoupled the boundary value unknowns* $p|_\partial$, $\hat{T}_{ij}|_\partial$ *from the principal modal variables* v, Ω_1, Ω_2, ϕ_1, ϕ_2, $\bar{p}$, A_{11}, A_{22}, A_{33}, A_{131}, A_{232}. Information on the boundary value unknowns $p|_\partial$, $\hat{T}_{ij}|_\partial$ can be obtained *a posteriori* from the free surface stress boundary conditions (II.4) and the solution of the modal equations. Since for the Maxwell-Jeffreys model the boundary value unknowns are independent of the principal modal variables, the stress boundary conditions (II.4) do not constitute constraints on the modal variables, as they did in the previous approach involving power series expansions for stress and pressure. *This is the crucial advantage of using the integrated stress and pressure modal variables, as opposed to the coefficients in power series expansions.*

However, the exact equations for A_{11}, A_{22}, A_{33}, A_{131}, A_{232} couple additional, higher moment stress resultants, A_{1111}, A_{1212}, A_{3311}, A_{2222}, A_{3322}, where, for instance

$$A_{3311} \equiv \iint_A \hat{T}_{33} x^2 \, dA.$$

We are thus led to the classic closure difficulty, where next we seek equations for these second moment area averages, which couples new stress resultants, and so on. As expected, there is no exact closure.

(We note that the closure difficulty exists only if $\lambda_1 \neq 0$ in the Maxwell-Jeffreys constitutive model. If $\lambda_1 = 0$, i.e., for the special cases of an inviscid fluid ($\eta = \lambda_1 = \lambda_2 = 0$), Newtonian fluid ($\lambda_1 = \lambda_2 = 0$) and second order fluid ($\lambda_1 = 0$), the 1-D model is closed; however, the model is *overconstrained* in these degenerate cases by the kinetic free surface boundary condition (II.4), so that only very limited classes of solutions exist. We comment in passing that the same asymptotic analysis which will be found in the following sections to produce closure in the general case of $\lambda_1 \neq 0$ also relieves the overdeterminism of the degenerate cases with $\lambda_1 = 0$. A complete treatment can be found in [7].)

IV. ASYMPTOTIC CLOSURE: SLENDERNESS SCALING ON THE INTEGRATED MOMENTUM AND CONSTITUTIVE EQUATIONS

The next step is to restrict the exact integrated equations to a "slenderness" regime, by introducing a scaling analysis which is consistent with the elliptical von Kármán velocity ansatz (II.1). This scaling is modeled after that of Schultz & Davis [8] in their study of axisymmetric Newtonian jets. First we nondimensionalize the coordinates (x, y, z, t) and the modal velocity variables (Ω_1, Ω_2, v). Let r_0 = a typical length scale in the jet cross section, and L_a = a typical length scale in the axial direction. The scaling hypothesis is:

$$x = \hat{x} r_0, \; y = \hat{y} r_0, \; z = \hat{z} L_a, \; t = \hat{t} t_0, \qquad (IV.1a)$$

and the small parameter ϵ is the ratio of length scales,

$$\epsilon = \frac{r_0}{L_a} << 1. \qquad (IV.1b)$$

Thus, the approximation is that a typical radial scale is much shorter than a typical axial scale, and therefore is called the slenderness scaling.

The free surface and velocity modal variables are nondimensionalized as

$$\phi_\alpha = \bar{\phi}_\alpha r_0, \quad \zeta_\alpha = \bar{\zeta}_\alpha \frac{1}{t_0}, \quad v = \bar{v} v_a, \qquad \text{(IV.1c)}$$

where v_a is a characteristic axial velocity. To preserve the incompressibility condition and kinematic boundary conditions upon scaling, the characteristic velocity, length, and time scales must be related as

$$v_a = \frac{L_a}{t_0} = \frac{r_0}{\epsilon t_0}. \qquad \text{(IV.1d)}$$

Then

$$\begin{aligned} v^{(x)} &= \epsilon v_a \bar{x} \bar{\zeta}_1 + O(\epsilon^3), \\ v^{(y)} &= \epsilon v_a \bar{y} \bar{\zeta}_2 + O(\epsilon^3), \\ v^{(z)} &= v_a \bar{v} + O(\epsilon^2), \end{aligned} \qquad \text{(IV.1e)}$$

so that the slenderness approximation in combination with the von Kármán velocity profile is equivalent to a slowly varying axial versus radial velocity ansatz.

REMARK. The scaled velocity formula (IV.1e) *includes higher order corrections*, $O(\epsilon^2)$, *to the von Kármán ansatz* (II.1). These correspond to higher order polynomial terms, $O(x^2, y^2, xy)$, in a general power series expansion for v. Consistency demands that we return to the previous integrated momentum equations, incompressibility constraint, kinematic boundary conditions, and integrated constitutive law, and add corrections which result from the integration with the higher order velocity expansion. These additional terms do not enter the lowest order closure models, but yield higher order corrections in the asymptotics. The precise form of these terms, along with the resolved question of consistency to higher order in the perturbation expansion, appears in [9].

Next we scale the 3-D pressure and stress components as

$$\begin{aligned} p(x, y, z, t) &= \bar{p}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) \frac{f}{r_0^2}, \\ \hat{T}_{ij}(x, y, z, t) &= \hat{\bar{T}}_{ij}(\bar{x}, \bar{y}, \bar{z}, \bar{t}) \frac{f}{r_0^2}, \\ &(i, j = 1, 2, 3), \end{aligned} \qquad \text{(IV.1f)}$$

where f is a characteristic force scale. With scalings (IV.1) we have nondimensionalized the 3-D free surface b.v.p. The integrated equations of the previous section can also be nondimensionalized; the scaling of the resultants in the integrated equations follows from the 3-D scalings (IV.1), e.g.,

$$\begin{aligned} A_{11}(z, t) &\equiv \int\int_A \hat{T}_{11}\, dx\, dy \\ &= \int\int_{\bar{A}} \frac{f}{r_0^2} \hat{\bar{T}}_{11}\, r_0^2\, d\bar{x}\, d\bar{y} = f \bar{A}_{11}(\bar{z}, \bar{t}). \end{aligned}$$

To complete the asymptotics, we must also expand the nondimensionalized dependent variables v, ζ_1, ζ_2, ϕ_1, ϕ_2, $\bar{p}$, $A_{j\alpha}$, $A_{j\alpha\beta}$ in powers of ϵ, e.g.,

$$\begin{aligned} v &= v^{(0)} + \epsilon v^{(1)} + \dots, \quad \bar{p} = \bar{p}^{(0)} + \epsilon \bar{p}^{(1)} + \dots, \\ \phi_1 &= \phi_1^{(0)} + \epsilon \phi_1^{(1)} + \dots . \end{aligned} \qquad \text{(IV.2)}$$

We now list the nondimensional asymptotic integrated equations, *retaining only the lowest power of* ϵ *within each* *physical term*. Thus, specifying the physical properties will alter the relative order of these lowest terms we keep, but these will be the leading order contributions of each physical property no matter which physical properties dominate. (All dependent variables in (IV.3) are the leading order contributions; we omit the superscript (0).)

$$B(A_{11} - \bar{p}) = \frac{1}{W}\phi_1\phi_2\chi_a^{(0)} - \frac{\epsilon^2}{4}\phi_1^3\phi_2(\Omega_{1,t} + v\Omega_{1,s} + \Omega_1^2),$$

$$B(A_{22} - \bar{p}) = \frac{1}{W}\phi_1\phi_2\chi_b^{(0)} - \frac{\epsilon^2}{4}\phi_2^3\phi_1(\Omega_{2,t} + v\Omega_{2,s} + \Omega_2^2),$$

$$B(A_{33,s} - \bar{p}_{,s}) = \frac{-1}{F}\phi_1\phi_2 + \frac{1}{W}(\phi_{1,s}\phi_2\chi_a^{(0)} + \phi_{2,s}\phi_1\chi_b^{(0)}) + \phi_1\phi_2(v_{,t} + vv_{,s}),$$

$$v_{,s} + \Omega_1 + \Omega_2 = 0,$$

$$\phi_{\alpha,t} + v\phi_{\alpha,s} = \phi_\alpha\Omega_\alpha, \quad \alpha = 1, 2$$

$$A_{11} + \Lambda_1[A_{11,t} + vA_{11,s} - ((2a + 1)\Omega_1 + \Omega_2)A_{11}] = 2Z\phi_1\phi_2[\Omega_1 + \Lambda_2\{\Omega_{1,t} + v\Omega_{1,s} - 2a\Omega_1^2\}],$$

$$A_{22} + \Lambda_1[A_{22,t} + vA_{22,s} - ((2a + 1)\Omega_2 + \Omega_1)A_{22}] = 2Z\phi_1\phi_2[\Omega_2 + \Lambda_2\{\Omega_{2,t} + v\Omega_{2,s} - 2a\Omega_2^2\}],$$

$$A_{33} + \Lambda_1[A_{33,t} + vA_{33,s} - (\Omega_1 + \Omega_2 + 2av_{,s})A_{33}] = 2Z\phi_1\phi_2[v_{,s} + \Lambda_2\{v_{,st} + vv_{,ss} - 2av_{,s}^2\}],$$

$$A_{131} + \Lambda_1[A_{131,t} + vA_{131,s} - (2\Omega_1 + \Omega_2 + a(\Omega_1 + v_{,s}))A_{131}] = \frac{\epsilon Z}{4}\phi_1^3\phi_2[\Omega_{1,s} + \Lambda_2\{\Omega_{1,st} + v\Omega_{1,ss} + \Omega_{1,s}(2(1 - a)\Omega_1 - (2a + 1)v_{,s})\}], \tag{IV.3}$$

$$A_{232} + \Lambda_1[A_{232,t} + vA_{232,s} - (2\Omega_2 + \Omega_1 + a(\Omega_2 + v_{,s}))A_{232}] = \frac{\epsilon Z}{4}\phi_2^3\phi_1[\Omega_{2,s} + \Lambda_2\{\Omega_{2,st} + v\Omega_{2,ss} + \Omega_{2,s}(2(1 - a)\Omega_2 - (2a + 1)v_{,s})\}],$$

where

$$\chi_a^{(0)} = -\frac{\phi_1\phi_2}{\pi}\int_0^{2\pi} \frac{\cos^2\theta\, d\theta}{(\phi_1^2\sin^2\theta + \phi_2^2\cos^2\theta)^{3/2}},$$

$$\chi_b^{(0)} = -\frac{\phi_1\phi_2}{\pi}\int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{(\phi_1^2\sin^2\theta + \phi_2^2\cos^2\theta)^{3/2}}.$$

and

$$B = \frac{f t_0^2}{\pi\rho r_0^2 L_*} = \frac{f}{\pi\rho r_0^2 v_*^2} = \frac{\text{viscoelastic and constraint pressure effects}}{\text{inertial effects}},$$

$$\frac{1}{F} = \frac{g t_0^2}{L_*} = \frac{\text{gravity effects}}{\text{inertial effects}},$$

$$\frac{1}{W} = \frac{\sigma t_0^2}{\rho r_0 L_*^2} = \frac{\sigma}{\rho r_0 v_*^2} = \frac{\text{surface tension (capillary) effects}}{\text{inertial effects}},$$

$$Z = \frac{\pi\eta r_0^2}{t_0 f} = \frac{\pi\eta r_0^2 v_*}{L_* f},$$

$$\Lambda_1 = \frac{\lambda_1}{t_0}, \quad \Lambda_2 = \frac{\lambda_2}{t_0}.$$

Equations (IV.3) derive from, respectively, the integrations of components of balance of linear momentum indicated by (III.1), the incompressibility constraint (I.1c), the kinematic free surface conditions (II.3a,b), and the integrations of the Maxwell-Jeffreys constitutive model indicated by (III.3).

The non-dimensional parameters F, W, $(BZ)^{-1}$ and Λ_1 are recognized as the Froude, Weber, Reynolds, and Weissenberg numbers respectively. Z, Λ_1, Λ_2 are the non-dimensional zero strain rate viscosity, relaxation time and retardation time, respectively, of the fluid.

We emphasize that equations (IV.3) depict the balance among all the physical effects that are incorporated. In the slenderness approximation, $0 < \epsilon << 1$, these equations yield the ability to perform theoretical experiments in which the relative physical effects are adjusted through the non-dimensional parameters. That is, $\{B, \frac{1}{W}, \frac{1}{F}, \Lambda_1, \Lambda_2, Z\}$, which measure the various properties of the free jet, are scaled in powers of the slenderness ratio ϵ,

$$B = B_0\epsilon^b, \quad \frac{1}{W} = \frac{1}{W_0}\epsilon^w, \quad \frac{1}{F} = \frac{1}{F_0}\epsilon^f,$$
$$\Lambda_j = \Lambda_{j0}\epsilon^{\lambda_j}, \quad Z = Z_0\epsilon^z. \qquad \text{(IV.4)}$$

We assume $B_0, \ldots, Z_0$ are $O(1)$, and vary the relative properties of the fluid by choosing the integer exponents in (IV.4).

We define the choice of integer exponents in (IV.4) as the *regime* of free jet behavior, as this choice reflects the relative magnitudes of competing physical effects.

We are now in a position to exhibit lowest order 1-D jet closure models. We specify a particular jet regime through a choice of integer exponents in (IV.4) and obtain equations from (IV.3) to arbitrary order in ϵ. There is clearly a tremendous amount of latitude in exploring all the specialized closure models which derive from our general construction.

As will be shown in the next section, existing 1-D theories correspond to the axisymmetric, steady forms of the lowest order equations with certain physical effects suppressed to higher order. Before connecting with the existing models, however, we first illustrate with three more general (nonaxisymmetric and time dependent) regimes.

As one example of a 1-D closure model for a particular jet regime, consider the case where all of the parameters in the set $\{B, \frac{1}{W}, \frac{1}{F}, \Lambda_1, \Lambda_2, Z\}$ are $O(\epsilon^0)$, i.e., we choose all exponents in (IV.4) to be zero. The lowest order equations in the asymptotic expansion are then

$$B(A_{11} - \bar{p}) = \frac{1}{W}\phi_1\phi_2\chi_e^{(0)},$$

$$B(A_{22} - \bar{p}) = \frac{1}{W}\phi_1\phi_2\chi_e^{(0)},$$

$$B(A_{33,s} - \bar{p}_{,s}) = -\frac{1}{F}\phi_1\phi_2 + \frac{1}{W}(\phi_{1,s}\phi_2\chi_e^{(0)}$$
$$+ \phi_{2,s}\phi_1\chi_e^{(0)}) + \phi_1\phi_2(v_{,t} + vv_{,s}),$$

$$v_{,s} + \Omega_1 + \Omega_2 = 0$$

$$\phi_{1,t} + v\phi_{1,s} = \phi_1\Omega_1, \quad \phi_{2,t} + v\phi_{2,s} = \phi_2\Omega_2, \qquad \text{(IV.5)}$$

$$A_{11} + \Lambda_1[A_{11,t} + vA_{11,s} - ((2a+1)\Omega_1 + \Omega_2)A_{11}]$$
$$= 2Z\phi_1\phi_2[\Omega_1 + \Lambda_2(\Omega_{1,t} + v\Omega_{1,s} - 2a\zeta_1^2)],$$

$$A_{22} + \Lambda_1[A_{22,t} + vA_{22,z} - ((2a+1)\varsigma_2 + \varsigma_1)A_{22}] = 2Z\phi_1\phi_2[\varsigma_2 + \Lambda_2(\varsigma_{2,t} + v\varsigma_{2,z} - 2a\varsigma_2^2)]\,,$$

$$A_{33} + \Lambda_1[A_{33,t} + vA_{33,z} - (\varsigma_1 + \varsigma_2 + 2av_{,z})A_{33}] = 2Z\phi_1\phi_2[v_{,z} + \Lambda_2(v_{,zt} + vv_{,zz} - 2av_{,z}^2)]\,.$$

In this regime, inertial effects, surface tension and gravity are all leading order in the axial direction (see equation (IV.5c)). *Note that this demands inertial effects to be higher order in the transverse directions* (see equations (IV.5a,b). Viscosity, relaxation and retardation effects are all leading order in the constitutive model in this regime.

In this special regime, the lowest order equations are a closed set of *nine* equations for the *nine* modal variables $\phi_1^{(0)}, \phi_2^{(0)}, v^{(0)}, \varsigma_1^{(0)}, \varsigma_2^{(0)}, \bar{p}^{(0)}, A_{11}^{(0)}, A_{22}^{(0)}, A_{33}^{(0)}$. The shear stress resultants $A_{131}^{(0)}$, $A_{232}^{(0)}$ decouple to lowest order from equations (IV.5), and appear in the problem for the first order corrections $\phi_1^{(1)}, \phi_2^{(1)}, v^{(1)}$, etc. (see [9]). The behavior predicted by (IV.5) for one set of steady nozzle conditions and parameters is shown in Figure 2. Note that these solutions predict swell of the elliptical extrudate (increase of the product $\phi_1\phi_2$ from the value 1 at the nozzle) *and distortion* (change of the aspect ratio ϕ_2/ϕ_1). Additional cases and complete discussions of the behavior in this regime can be found in [4, 10, 11].

As an example of a 1-D closure model for a different jet regime, consider the case where Λ_1 is $O(\epsilon^0)$, and $B, \frac{1}{W}, \frac{1}{F}, \Lambda_2$, Z are $O(\epsilon^2)$. For this regime the lowest order equations in the asymptotic expansion are

$$B(A_{11} - \bar{p}) = \frac{1}{W}\phi_1\phi_2\chi_*^{(0)} - \frac{1}{4}\phi_1^3\phi_2(\varsigma_{1,t} + v\varsigma_{1,z} + \varsigma_1^2)\,,$$

$$B(A_{22} - \bar{p}) = \frac{1}{W}\phi_1\phi_2\chi_*^{(0)} - \frac{1}{4}\phi_2^3\phi_1(\varsigma_{2,t} + v\varsigma_{2,z} + \varsigma_2^2)\,,$$

$$0 = \phi_1\phi_2(v_{,t} + vv_{,z})\,, \qquad \text{(IV.6)}$$

$$v_{,z} + \varsigma_1 + \varsigma_2 = 0\,,$$

$$\phi_{1,t} + v\phi_{1,z} = \phi_1\varsigma_1\,, \quad \phi_{2,t} + v\phi_{2,z} = \phi_2\varsigma_2\,,$$

$$A_{11} + \Lambda_1[A_{11,t} + vA_{11,z} - ((2a+1)\varsigma_1 + \varsigma_2)A_{11}] = 0\,,$$

$$A_{22} + \Lambda_1[A_{22,t} + vA_{22,z} - ((2a+1)\varsigma_2 + \varsigma_1)A_{22}] = 0\,.$$

In this regime inertial effects are important for motion within the jet cross section (see IV.6a,b), and gravity is neglected to leading order. *These choices demand that momentum is conserved in the axial direction* (equation IV.6c). Only relaxation effects are included to leading order in the constitutive model, equations (IV.6f,g).

In this regime, the lowest order equations are a closed set of *eight* equations for the *eight* modal variables $\phi_1^{(0)}, \phi_2^{(0)}, v^{(0)}$, $\varsigma_1^{(0)}, \varsigma_2^{(0)}, \bar{p}^{(0)}, A_{11}^{(0)}, A_{22}^{(0)}$. The axial stress resultant $A_{33}^{(0)}$, and shear stress resultants $A_{131}^{(0)}$, $A_{232}^{(0)}$ decouple from this lowest order problem.

The behavior predicted by the closed equations (IV.6) differs significantly from the behavior predicted by equations (IV.5), a reflection of the disparate parameter specifications. The behavior predicted for one set of parameters and steady nozzle conditions is shown in Figure 3. Note that these solutions predict oscillation in z of the major axis of the free surface elliptical cross section between the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions. Additional cases and discussion can be found in [10]. In particular, the special cases of elliptical inviscid and Newtonian free jets, subject only to surface tension and gravity, are considered. Our model predicts oscillation of the major axis of the free surface cross section between perpendicular directions and draw down of the cross section, in agreement with observed behavior

As a third example, consider the particular jet regime where $B, \frac{1}{W}, \frac{1}{F}$ are $O(\epsilon^3)$. Then, from (IV.3), we obtain the lowest order equations:

$$
\begin{aligned}
&0 = \phi_1^3\phi_2(\varsigma_{1,t} + v\varsigma_{1,s} + \varsigma_1^2), \quad 0 = \phi_2^3\phi_1(\varsigma_{2,t} + v\varsigma_{2,s} + \varsigma_2^2), \\
&0 = \phi_1\phi_2(v_{,t} + vv_{,s}), \quad v_{,s} + \varsigma_1 + \varsigma_2 = 0, \qquad \text{(IV.7)} \\
&\phi_{1,t} + v\phi_{1,s} = \phi_1\varsigma_1, \quad \phi_{2,t} + v\phi_{2,s} = \phi_2\varsigma_2 .
\end{aligned}
$$

In this regime, only inertial effects are leading order. Here the lowest order equations are a set of *six* equations for the *five* unknowns $\phi_1^{(0)}, \phi_2^{(0)}, v^{(0)}, \varsigma_1^{(0)}, \varsigma_2^{(0)}$, which are easily shown to be overconstrained, and, in fact, incompatible. With this regime we have demonstrated another important result of this analysis: the ability to determine what properites of slender viscoelastic free jets combine to produce *consistent* 1-D closure, and which do not.

Many other specialized closure models are clearly available. We refer to [3, 4, 10, 11] for applications which have already derived from this work. Additional applications are planned.

In [8] we exhibit and analyze the first order corrections to the lowest order equations. These higher order equations allow us to test the predictions of the lowest order models, to determine if neglected effects become important, and to obtain more detailed information about the 3-D flow.

V. CONTACT WITH EXISTING 1 – D THEORIES

To illustrate the comprehensive nature of the above analysis, we now indicate how several widely referenced 1-D models for Newtonian and viscoelastic free jets are obtained by specification of particular jet regimes, and by reduction to the steady, axisymmetric forms. We list the order of magnitude of the parameters $\boldsymbol{B}$, $\boldsymbol{W}$, $\boldsymbol{F}$, $\boldsymbol{Z}$, Λ_1, Λ_2 in the slenderness ratio which produce exemplary existing models from our system (IV.3) as the lowest order equations.

The axisymmetric, steady form of the lowest order equations with the parameters $\boldsymbol{B}$, $\boldsymbol{W}$, $\boldsymbol{F}$, $\boldsymbol{Z}$ all $O(\epsilon^0)$ and the parameters Λ_1, Λ_2 both $O(\epsilon)$ is the Newtonian thin filament model, equation (34) from [2]. In this regime, Newtonian viscosity, surface tension and gravity are leading order, with the elastic and second order viscosity effects suppressed to higher order. (Recall the Reynolds number $R = (\boldsymbol{B}\boldsymbol{Z})^{-1}$.)

The axisymmetric, steady form of the lowest order equations with the parameters $\boldsymbol{B}$, $\boldsymbol{W}$, $\boldsymbol{F}$, $\boldsymbol{Z}$, Λ_2 all specified as $O(\epsilon^0)$, Λ_1 specified as $O(\epsilon)$, and the rate parameter a taken as -1 (lower convected rate) is the second order, non-Newtonian thin filament model, equation (53) in [2].

The 1-D viscoelastic model in [12] (equations (7.31) – (7.33)) is obtained as the axisymmetric, steady form of the lowest order equations from (IV.3) with the parameters $\boldsymbol{B}$, $\boldsymbol{Z}$, Λ_1 specified as $O(\frac{1}{\epsilon})$, the parameters $\boldsymbol{W}$, $\boldsymbol{F}$, Λ_2 $O(\epsilon^0)$, the rate parameter a taken as 1 (upper convected rate), and the choice of notation

$$A_{11} = P\phi^2, \quad A_{33} = T\phi^2 .$$

Recalling the definitions

$$
\begin{aligned}
A_{11} &\equiv \iint_{\text{cross section}} \hat{T}_{11}\,dA, \\
A_{22} &\equiv \iint_{\text{cross section}} \hat{T}_{22}\,dA, \\
A_{33} &\equiv \iint_{\text{cross section}} \hat{T}_{33}\,dA,
\end{aligned}
$$

we see that $T(s)$ is the average normal stress over the jet cross section in the axial direction e_3, and $P(s)$ is the average normal stress in the transverse directions. In this regime the leading order effects are viscosity and elasticity, with inertia in the axial direction, surface tension, gravity and retardation time effects suppressed. The axisymmetric, steady form of this regime with upper convected rates is also the steady form, equations (16) and (17), of the viscoelastic model in [13] with the power law viscosity parameter n in their model set equal to 1, the model of [6] (equations (18), (19)), with the ratio ν of stress differences in their constitutive model taken as zero, and the model of [14], with a spectrum of one relaxation time.

The 1-D model in [15] for a free jet of an Oldroyd fluid B is obtained as the axisymmetric, steady form of the lowest order equations from (VI.7) with the parameters B, Z, Λ_1, Λ_2 all $O(\frac{1}{\epsilon})$, the parameters W, F both $O(1)$, and the rate parameter a taken as 1 (upper convected rates).

VI. CONCLUDING REMARKS

We have satisfied the goals set in the abstract and introduction. Beginning with the full 3-D viscoelastic free boundary value problem, we have derived, by slenderness asymptotics, a comprehensive framework of 1-D closure models for slender, free viscoelastic jets. The physical effects of inertia, gravity, viscosity, elasticity, surface tension, curvature, and the free surface boundary conditions involving surface tension, ambient pressure and the curvature of the free surface, are represented in the 1-D modal equations, and most importantly, these effects appear as they derive from the full 3-D free surface boundary value problem. These resultant 1-D equations have the flexibility to vary the relative strengths of the physical properties of the fluid and interface. Existing 1-D theories correspond to special cases within our general framework.

ACKNOWLEDGEMENTS: M.G.F. wishes to acknowledge one very influential conversation with Harvey Segur, the hospitality of the Center for Nonlinear Studies, Los Alamos, where much of this work was performed, and support from the National Science Foundation, Grant No. DMS 8411002. The research of S.E.B. and K.J.L. was supported by the National Science Foundation Engineering Research Center for Net Shape Manufacturing at the Ohio State University under Grant CDR-8522662. M.G.F. and S.E.B. wish to acknowlege helpful discussions with D.D. Joseph. All authors thank Mac Hyman for encouragement at various stages of this work.

REFERENCES

1. Kármán, T. von, Über Laminare und Turbulente Reibung. *Z. angew. Math. Mech.* 1 (1921), 233–252.

2. Matovich, M.A. & Pearson, J.R.A. Spinning a Molten Threadline, *Ind. Eng. Chem. Fundam.* 8 (1969), 512–520.

3. Bechtel, S.E., Bogy, D.B., & Forest, M.G. A One-Dimensional Theory for Viscoelastic Fluid Jets, With Application to Extrudate Swell and Draw-Down Under Gravity, *J. Non-Newtonian Fluid Mech.* 21 (1986), 273–308.

4. Bechtel, S.E., Forest, M.G., & Lin, K.J. Effective Stress Rates of Viscoelastic Free Jets, *J. Non-Newtonian Fluid Mech.* 26 (1987), 1–41.

5. Bechtel, S.E., Forest, M.G., Lin, K.J., & Holm, D.D. 1-D Closure Models for 3-D Incompressible Viscoelastic Free Jets: von Kármán Flow Geometry and Elliptical Cross Section, *J. Fluid Mech.*, to appear.

6. Denn, M.M., Petrie, C.J.S., & Avenas, P. Mechanics of Steady Spinning of a Viscoelastic Liquid, *AIChE J.* 21 (1975), 791–799.

7. Bechtel, S.E. & Forest, M.G. On Free Surface Boundary Conditions in Closure Models for Viscoelastic Jets, to be submitted to *J. Fluid Mech.*

8. Schultz, W.W. & Davis, S.H. One-Dimensional Liquid Fibers, *J. Rheol.* 26 (1982), 331–345.

9. Bechtel, S.E., Forest, M.G., & Lin, K.J. 1-D Closure Models for 3-D Incompressible Viscoelastic Free Jets: Higher Order Flow Perturbations, to be submitted to *J. Fluid Mech.*

10. Bechtel, S.E., Forest M.G., & Lin, K.J. On the Behavior of Viscoelastic Free Jets With Elliptical Cross Section, *J. Non-Newtonian Fluid Mech.*, 27 (1988), 87–126.

11. Bechtel, S.E., Forest M.G., & Hyman, J.M. Effective Relaxation and Retardation Times for Viscoelastic Free Jets, to be submitted to *J. Non-Newtonian Fluid Mech.*

12. Tanner, R.I. *Engineering Rheology*, Claredon Press, Oxford 1985.

13. Fisher, R.J. & Denn, M.M. A Theory of Isothermal Melt Spinning and Draw Resonance, *AIChEJ.* 22 (1976), 236–246.

14. Denn, M.M. & Marrucci, G. Effect of a Relaxation Time Spectrum on Mechanics of Polymer Melt Spinning, *J. Non-Newtonian Fluid Mech.* 2 (1977), 159–167.

15. Gupta, R.K., Puszynski, J. & Sridhar, T. Steady Spinning of the Oldroyd Fluid B, I: Theory, *J. Non-Newtonian Fluid Mech.* 21 (1986), 99–113.