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HIDDEN SYMMETRIES OF PARTIAL DIFFERENTIAL EQUATIONS

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Lie group symmetries of partial differential equations (PDE's) allow the construction of particular solutions for those equations. Each symmetry group allows the reduction of the dimensionality of the equations until they reach ordinary differential equations. If no symmetry groups exist then the method has, in the past, been abandonded for those equations. Here we demonstrate a new procedure for which PDE's allowing no symmetry groups can indeed be solved using symmetries. The method entails *expanding* the dimensionality using a simple group to a higher dimensional PDE which then allows multiple group reductions to obtain particular solutions. These solutions are then transformed back into the original variables and become group-invariant solutions of the original PDE which exhibited no group symmetries.

I. Introduction

The existence of symmetries of differential equations under Lie groups of transformations often allows those equations to be reduced to simpler equations. Specifically, a one parameter group can reduce (i) an nth order ordinary differential equation (ODE) to an (n-1)st order ODE, (ii) a first order ODE to quadrature, (ii) an nth order partial differential equation (PDE) with m independent variables to an nth order PDE with $m-1$ independent variables. Invariance therefore allows a reduction in order for ODE's and a reduction in the number of independent variables for PDE's.

In the case of the existence of more than one invariance group, a so-called multi-parameter group, several reductions can generally be made. These reductions are sequential, and the order in which they occur is important. The concepts of normal subgroups and ideals are critical to choosing the proper order in which to reduce.

The are presently several type of symmetries for differential equations in use. The simplest are the classical Lie point symmetries. After this are contact symmetries, where the coordinate functions contain first order derivatives. Generalized symmetries or Lie-Bäcklund symmetries occur when the coordinate functions contain finite but higher order derivatives. Potential symmetries are symmetries for a system of expanded equations involving the introduction of functions as new potential variables. Partial invariance occurs when only a submanifold of the solution space is invariant under the transformation. Discussions and examples of these can be found in the texts listed in Reference 1. In this paper we consider only the simplest of these symmetries, Lie point symmetries.

One of the major accomplishments of Sophus Lie was to identify that the properties of the global transformations of the group are completely and uniquely determined by the infinitesimal transformations around the identity transformation. This allows the nonlinear relations for the identification of invariance groups to be replaced by linear relations, greatly improving the accessibility of applications. Therefore, instead of dealing with global transformation equations, we use differential operators whose exponentiation generates the action of the group. These operators are called the group generators.

The collection of these differential operators forms the basis for the Lie algebra. There is a one-to-one correspondence between the Lie groups and the associated Lie algebras. For an r-parameter group there are r differential operators v_i, $i = 1, ..., r$, that generate the group action through exponentiation. The Lie bracket is the commutator of two operators, and the algebra is closed under this commutator: $[v_i, v_j] = C_{ij}^k v_k$, (sum over k) for all $i, j = 1, ..., r$.

Consider an r-parameter group G with generators v_i and associated Lie algebra $\mathcal{G}$. If a subset of the operators v_i is closed under commutation, *i.e.*, $[v_i, v_j] = C_{ij}^k v_k$ for all $i, j = 1, ..., s$, $s < r$ with $C_{ij}^k = 0$ for $k > s$, then these operators span an s-dimensional subalgebra $\mathcal{H}$ of $\mathcal{G}$, and they generate a corresponding

s-parameter subgroup H of G. Given a subalgebra $\mathcal{H}$ of $\mathcal{G}$, if the commutator of any element of $\mathcal{H}$ with any element outside of $\mathcal{H}$ goes back into $\mathcal{H}$, then $\mathcal{H}$ is an *ideal* of $\mathcal{G}$, and the subgroup H is a *normal subgroup* of G. For any subgroup H of G we can identify the collection of all operators v_i whose commutation with any element in $\mathcal{H}$ goes back into $\mathcal{H}$. This collection, which must contain all of H, generates the *normalizer* of H. That is, the normalizer $\mathrm{Nor}_G(H)$ of a subgroup H in G is generated by the algebra $\mathcal{N}_G(H) = \{v_j : [v_i, v_j] \in \mathcal{H}\ \forall v_i \in \mathcal{H}\ ,\ v_j \in \mathcal{G}\}$.

The theorem pertaining to sequential reductions can now be stated:

Theorem Consider a system of differential equations E invariant under a multiparameter group G with subgroup H. The system E/H obtained by reducing E with the subgroup H will be invariant under the quotient group $Q = \mathrm{Nor}_G(H)/H$.

The quotient group can be formed by simply removing one of the elements of H from $\mathrm{Nor}_G(H)$. A simple proof of this theorem can be found in Ovsiannikov[1982]. This theorem forms the basis for the choice of order for multiple reductions.

An r-parameter group for which a chain of normal subgroups can be constructed where each normal subgroup is one dimension lower is called *solvable*. A consequence of the above theorem is that if an r-parameter group is solvable, then r reductions can take place. Note that all two-parameter groups are solvable.

Given a system of differential equations invariant under an r-parameter group for which we wish to perform multiple reductions, the first reduction should use a subgroup with a nonzero quotient group Q, so further reductions are allowed. If we incorrectly choose a subgroup whose particular Q is empty, then generally no further reductions are possible and we are "stuck." Examples of the use of multiple reductions on PDE's can be found in Reference 2.

For two-parameter Abelian groups, where $[v_1, v_2] = 0$, both v_1 and v_2 represent normal subgroups, so one can reduce twice without regard to order. For a non-Abelian two-parameter group, which can always be written $[v_1, v_2] = v_2$ by suitable choice of basis, it is necessary to use first the normal subgroup (v_2) in order for the second reduction to be allowed. If the nonnormal subgroup (v_1) is used first, the reduced system generally loses the v_2 symmetry.

This brings up the following possibility: Suppose we are given a differential equation which has no symmetry groups. It is possible that this equation is one of these "stuck" equations, and can be solved by first recovering the higher dimensional equation, and then following the "correct" reduction path. This process is illustrated in Figure 1 for the example of a two-parameter non-Abelian group.

For ODE's, the expansion up one level increases the order of the system, but it remains ODE's. This procedure for ODE's was suggested by Olver[1986] and pursued in References 3 and 4. A collection of ODE's possessing these hidden symmetries was systematically developed for the case of the 8-parameter projective group in a plane. The parallel process for a single, specific PDE is demonstrated here. It is interesting to note that any ODE could be considered to be the "stuck" reduction of a PDE, so the process described here could also be attempted for ODE's.

II. Example

Consider the second order quasilinear PDE

$$(1+x^2)F_{xx} + 4\frac{y}{x}\left(F_{xy} + \frac{y}{x}F_{yy}\right) + (2x + \frac{1}{x})F_x + \frac{y}{x^2}(\alpha y + 4)F_y = 0. \tag{1}$$

This equation has two point symmetry invariance groups, whose generators are $v = \partial_F$ and $v = F\partial_F$, which represent translational and scaling invariance for the dependent variable. These symmetries are allowed since (1) contains F only in derivatives and is linear in those derivatives. Neither of these symmetries can be used to reduce the number of independent variables for this PDE since neither transforms the independent variables. Thus, from a point symmetry perspective, we are "stuck."

Following Figure 1, we consider this PDE as the result of a reduction by the "wrong" symmetry group of a higher dimensional PDE, and we look to reconstruct this former PDE and proceed with the proper reduction order. We arbitrarily choose a simple scaling group to enlarge the dimensionality of (1), and look for non-Abelian symmetry groups of the enlarged equation. If (1) is a result of the use of a scaling group from a higher dimension, this group generator can be written $v_1 = z\partial_z + aw\partial_w + bt\partial_t$, where z, w, and t are the independent variables in the higher dimension and a and b are free parameters. Since (1) is the reduced equation using this generator, the variables x, y and F are the group invariants of this generator and are the integration constants of the characteristic equations of the invariance condition $v_1\Psi = 0$:

$$\frac{dz}{z} = \frac{dw}{aw} = \frac{dt}{bt} = \frac{dG}{0}.$$

The solutions of these equations are

$$x = \frac{z^a}{w},\; y = \frac{z^b}{t} \text{ and } G = F(x,y). \tag{2}$$

To construct the expanded PDE in these new variables from (1), we need to replace the derivatives of F with respect to x and y with derivatives of G with respect to z, w and t. Simple calculations give

$$G_z = F_x x_z + F_y y_z = F_x\frac{az^{a-1}}{w} + F_y\frac{bz^{b-1}}{t}, \tag{3}$$

$$\begin{aligned} G_{zz} = {} & \frac{a(a-1)z^{a-2}}{w}F_x + \frac{b(b-1)z^{b-2}}{t}F_y \\ & + F_{xx}\frac{a^2z^{2a-2}}{w^2} + 2F_{xy}\frac{abz^{a+b-2}}{wt} + F_{yy}\frac{b^2z^{2b-2}}{t^2}, \end{aligned} \tag{4}$$

$$G_w = -\frac{z^a}{w^2}F_x, \tag{5}$$

$$G_{ww} = \frac{2z^a}{w^3}F_x + \frac{z^{2a}}{w^4}F_{xx}, \tag{6}$$

$$G_t = -\frac{z^b}{t^2}F_y. \tag{7}$$

We first replace F_{yy} in (1) using relation (4). It is then noticed that the term multiplying F_{xy} can be set to zero through the choice $b = 2a$. The term F_{xx} is replaced using (6), F_x replaced using (3), and finally, F_y with (7). This leaves

$$G_{ww} + \frac{1}{z^{2a-2}a^2}G_{zz} + \frac{1}{z^{2a-1}a^2}G_z - \alpha G_t = 0.$$

We see for the choice $a = 1$ that this equation becomes

$$G_{ww} + G_{zz} + \frac{1}{z}G_z = \alpha G_t, \tag{8}$$

which is the 2-D linear heat conduction equation in cylindrical coordinates. Therefore, Equation (1) is the result of reducing (8) with the symmetry $v_1 = z\partial_z + w\partial_w + 2t\partial_t$.

Equation (8) possesses many known symmetries besides v_1, including

$$v_2 = -\frac{2t}{\alpha}\partial_w + wG\partial_G \quad \text{and} \quad v_3 = wt\partial_w + zt\partial_z + t^2\partial_t - G\left[\frac{\alpha}{4}(w^2+z^2) + \frac{3}{2}t\right]\partial_G.$$

We see that $[v_1, v_2] = v_2$ and $[v_1, v_3] = 2v_3$, so both v_2 and v_3 satisfy the criterion we seek.

Following Figure 1, we use v_2 first and then v_1 to reduce (8) to a second order ODE. The invariants of v_2 are

$$a_1 = z, \quad a_2 = t, \quad \text{and} \quad a_3 = Ge^{\alpha w^2/(4t)}.$$

For the double reduction, we write the second generator in terms of the invariants of the first:

$$v_1 = f_1\partial_{a_1} + f_2\partial_{a_2} + f_3\partial_{a_3}.$$

The functions f_i are found through $f_i = v_1 a_i$, which yield $f_1 = a_1$, $f_2 = 2a_2$, and $f_3 = 0$. The invariants of both v_1 and v_2 are then the solutions of the characteristic equations

$$\frac{da_1}{a_1} = \frac{da_2}{2a_2} = \frac{da_3}{0},$$

which are

$$s = \frac{a_1^2}{a_2} = \frac{z^2}{t} \quad \text{and} \quad H(s) = Ge^{\alpha w^2/(4t)}.$$

Calculating the derivatives of G with respect to z, w and t in terms of the new variables H and s and substituting into (8), the equation is reduced to the ODE

$$4sH'' + H'(4 + \alpha s) - \frac{\alpha}{2}H = 0,$$

which has the general solution

$$H(s) = c_1 H_1 + c_2 H_2,$$
$$H_1 = \mathrm{e}^{-3\alpha s/4}\mathrm{L}_{1/2}\left(-\frac{\alpha s}{4}\right)$$
$$H_2 = \mathrm{e}^{-2\alpha s}\mathrm{U}\left[\frac{3}{2}, 1, \frac{\alpha s}{4}\right],$$

where L is the Laguerre polynomial and U is the Kummer Hypergeometric function. These two solutions are then written in terms of the original variables F, x and y and become

$$F_1(x,y) = \mathrm{e}^{-\alpha y/(4x^2)}\mathrm{L}_{1/2}\left(-\frac{\alpha y}{4}\right)$$

and

$$F_2(x,y) = \mathrm{e}^{-\alpha y(x^2+1)/(4x^2)}\mathrm{U}\left[\frac{3}{2}, 1, \frac{\alpha y}{4}\right].$$

These solutions are then two particular solutions of the partial differential equation (1).

III. Discussion

This specific example shows the construction of particular solutions to a PDE that possesses no nontrivial point symmetries through a new technique of symmetry reduction. It involves expanding the equation to a potentially more complicated one whose solutions may be found. The symmetry reductions of this expanded equation must include the one used to obtain it from the original PDE, otherwise the solutions will not carry over to the original system.

A parallel technique using expansion of order for ODE's has been developed in a systematic fashion in Reference 4. We note that similar systematic development can be pursued with PDE's, although the number of possible groups is now much larger (even limiting oneself to projective groups).

REFERENCES

1. P. J. Olver, *Application of Lie Groups to Differential Equations*, Springer, NY 1986; L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, NY 1982; N. H. Ibragimov, *Transformation Groups Applied to Mathematical Physics*, D. Reidel, Dordrecht, Holland 1985; G. W. Bluman and

S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, NY 1989.

2. S. V. Coggeshall and J. Meyer-ter-Vehn, "Group-invariant solutions and optimal systems for multidimensional hydrodynamics," J. Math. Physics **33**, 3585 (1992).

3. B. Abraham-Shrauner, "Hidden Symmetries of Nonlinear Ordinary Differential Equations," Trans. Amer. Math. Society Summer Seminar *Exploiting Symmetry in Applied and Numerical Analysis*, Fort Collins, Co (1992).

4. B. Abraham-Shrauner and A. Guo, "Hidden Symmetries Associated with the Projective Group of Nonlinear First-Order Ordinary Differential Equations," J. Phys. A: Math. Gen., 25, 5597 (1992).

S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, NY 1989.

2. S. V. Coggeshall and J. Meyer-ter-Vehn, "Group-invariant solutions and optimal systems for multidimensional hydrodynamics," J. Math. Physics 33, 3585 (1992).

3. B. Abraham-Shrauner, "Hidden Symmetries of Nonlinear Ordinary Differential Equations," Trans. Amer. Math. Society Summer Seminar *Exploiting Symmetry in Applied and Numerical Analysis*, Fort Collins, Co (1992).

4. B. Abraham-Shrauner and A. Guo, "Hidden Symmetries Associated with the Projective Group of Nonlinear First-Order Ordinary Differential Equations," J. Phys. A: Math. Gen., 25, 5597 (1992).

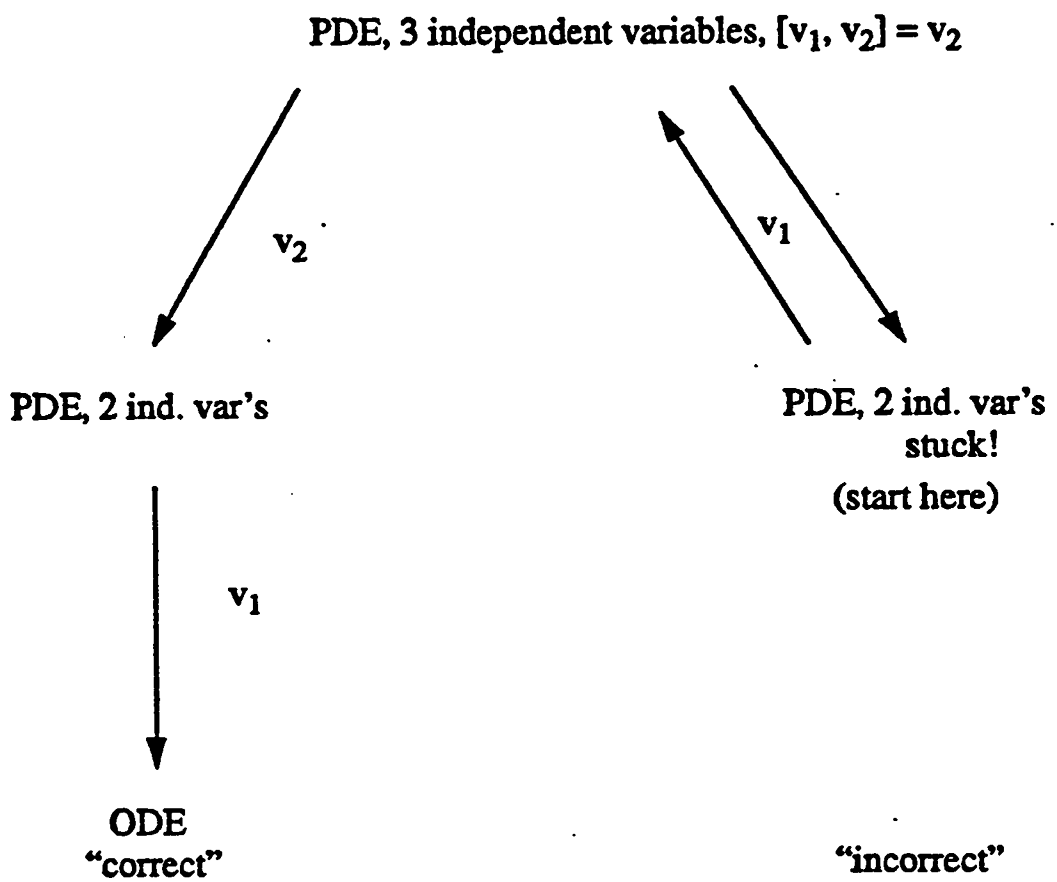

Figure 1.

Correct and incorrect reduction paths using a 2-parameter non-Abelian group.