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LOS ALAMOS SCIENTIFIC LABORATORY

OF THE UNIVERSITY OF CALIFORNIA ○ LOS ALAMOS NEW MEXICO

ON THE DERIVATION OF DIFFERENCE EQUATIONS FOR HYDRODYNAMICS

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ON THE DERIVATION OF DIFFERENCE EQUATIONS FOR HYDRODYNAMICS

by

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ABSTRACT

This report discusses methods of obtaining difference equations to represent the conservation of mass, momentum, and energy--the conservation equations which represent the motion of a fluid. The basic principle on which this work is based is that these conservation equations can be written in forms which apply whether or not discontinuities are present when the equations are written in terms of integrals. When the integrals are replaced by approximations in terms of quadrature formulas, difference equations are obtained for the relevant functions.

1. INTRODUCTION

It is the purpose of this report to discuss methods of obtaining difference equations to represent the conservation of mass, momentum and energy—the conservation equations which represent the motion of a fluid. The basic principle on which this work is based is that these conservation equations can be written in forms which apply whether or not discontinuities are present when the equations are written in terms of integrals. When the integrals are replaced by approximations in terms of quadrature formulas, we obtain difference equations for the relevant functions.

The integrals may, of course, be written in terms of Eulerian or Lagrangian coordinates and the difference equations may be obtained in either case.

We shall be dealing with quantities of the form

(1.1) $$I(t) = \int_{V(t)} \rho\, f(x,t)\, dV = \int_{V(o)} \rho_o(\xi)\, f(\xi,t)\, dV_o$$

where the integral is a volume (surface or line) integral over the volume occupied at time t by particles originally in a volume V(o), where

(1.2) $$\frac{\rho}{\rho_o} = \det \left\| \frac{\partial x^i}{\partial \xi^j} \right\|$$

and

(1.3) $$x^i = x^i(\xi, t) \qquad i = 1, 2, 3, \ldots$$

$$(1.4) \qquad \xi^i = x^i(\xi, o)$$

are the "particle paths" followed by the particles originally at the positions $x^i = \xi^i$. The variables x^i are the Eulerian variables and the ξ^i are the Lagrangian ones.

If there is a discontinuity in the function ρf at a surface $\Sigma(t)$, given in parametric form as

$$(1.5) \qquad x^i = X^i(a, b, t)$$

or

$$\xi^i = \Xi^i(a, b, t)$$

we shall define

$$(1.6) \qquad I(t) = \int_{V_1(t)} \rho f \, dV + \int_{V_2(t)} \rho f \, dV \equiv \int_{V_1+V_2} \rho f \, dV$$

where each of the volumes V_1 and V_2 is bounded by a portion of the fixed boundary of V(t) and by the surface $\Sigma(t)$. These boundaries are chosen so that ρf is continuous inside of $V_1(t)$ and $V_2(t)$. Similar considerations may be applied in the Lagrangian case.

It follows from equation (1.6) that

$$(1.7) \qquad \frac{dI}{dt} = \int_{V_1+V_2} \frac{\partial(\rho f)}{\partial t} dV + \int_{S_1+S_2} \rho f u^i \lambda_i \, dS + \int_{\Sigma} [\rho f] \, v_n \, d\Sigma$$

where

$$(1.8) \qquad u^i = \frac{\partial x^i(\xi, t)}{\partial t} .$$

are the Eulerian components of the velocity of the fluid; for an arbitrary function h we write

$$\int_{S_1+S_2} h\,dS = \int_{S_1} h\,dS + \int_{S_2} h\,dS \tag{1.9}$$

where S_1 and S_2 are the boundaries of the volumes V_1 and V_2 minus the surface Σ; λ_i are the Cartesian components of the normal to the surface over which the integration is being carried out;

$$[\rho f] = (\rho f)_1 - (\rho f)_2 \tag{1.10}$$

the subscript denoting the value of the function on one side or the other of the surface Σ, and v_n is the normal component of the normal velocity of $\Sigma(t)$ in the direction of the normal to Σ drawn from V_1 to V_2.

The usual differential equations of hydrodynamics and the Rankine-Hugoniot equations follow from equation (1.7) applied to volumes V(t) arising from an arbitrary choice of the volume V(o) for specific choices of the function f.

In terms of Lagrange coordinates we have

$$\frac{dI}{dt} = \int_{V_1(o)+V_2(o)} \rho_o \frac{\partial f}{\partial t} dV_o + \int_{\Sigma_o} [\rho_o f]\, v_{on}\, d\Sigma_o \tag{1.11}$$

where $V_1(o)$ and $V_2(o)$ are subvolumes of V(o) defined in a manner analogous to that by which V_1 and V_2 are defined.

2. THE CONSERVATION OF MATTER

The equation describing this conservation law is contained in the statement

$$\frac{d}{dt}\int_{V(t)} \rho \, dV = 0 \tag{2.1}$$

This leads to the equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u^i)}{\partial x^i} = 0 \tag{2.2}$$

holding in regions where no discontinuity occurs and to the statement that

$$\rho_1(u_1^i - v^i)\,\lambda_i = \rho_2(u_2 - v^i)\,\lambda_i \equiv m \tag{2.3}$$

across discontinuities moving with a velocity vector v^i, and with a normal vector with components λ_i.

If we integrate equation (2.2) over a fixed volume in space containing no discontinuities, we have

$$\int \frac{\partial \rho}{\partial t} dV = -\int \frac{\partial(\rho u^i)}{\partial x^i} dV = -\int (\rho u^i)\,\lambda_j \, dS \tag{2.4}$$

This may be written as

$$\frac{d}{dt}\int \rho \, dV = -\int (\rho u^j)\,\lambda_j \, dS \tag{2.5}$$

where the surface integral is applied to the surface bounding the fixed volume in space.

Equation (2.5) with the same definition of the surface integral applies to volumes within which there is a discontinuity present. The justification for this statement is the following: Integrate equation (2.3) over the volumes $W_1(t)$ and $W_2(t)$ into which the fixed volume referred to in equation (2.5) is subdivided by the discontinuity surface $\Sigma(t)$. Then adding these we obtain

(2.6) $$\int_{W_1(t)} \frac{\partial\rho}{\partial t} dV + \int_{W_2(t)} \frac{\partial\rho}{\partial t} dV = -\int (\rho u^j)\,\lambda_j\, dS - \int_{\Sigma} [\rho u^i]\;\lambda_{\Sigma i}\, d\Sigma$$

where $\lambda_{\Sigma i}$ is the normal to the surface discontinuity drawn in the direction from the region $W_1(t)$ to $W_2(t)$, and the first surface integral on the right-hand side is taken over the fixed boundary of the fixed volume $W_1(t) + W_2(t)$. It is a consequence of equation (2.3) that

$$[\rho u^i]\;\lambda_{\Sigma i} = [\rho]\; v_n$$

where v_n is the normal velocity of the boundary. Hence, equation (2.6) may be written as

$$\int_{W_1(t)} \frac{\partial\rho}{\partial t} dV + \int_{W_2(t)} \frac{\partial\rho}{\partial t} dV + \int [\rho]\; v_n\, d\Sigma = -\int (\rho u^i)\,\lambda_j\, dS$$

However, it follows from arguments similar to those used in deriving equation (1.7) that this equation is just equation (2.5) with the definition of the right-hand side used there.

If in equation (2.5) we approximate the integrals involved by using quadrature formulas, we obtain a differential difference form of equation (2.5) which applies to regions where there are discontinuities present as well as those to which there are no discontinuities present. Of course, a given quadrature formula will not be equally accurate in both types of regions.

In the one dimensional case equation (2.5) becomes

(2.7) $$\frac{d}{dt}\int_{x}^{x+\Delta x} \rho\, dx = -(\rho u)_{x+\Delta x} + (\rho u)_x$$

where the linear integral is rigorously computed as described above if a discontinuity occurs in the interval x to x + Δx.

In the two dimensional case we may write equation (2.5) as

$$\frac{d}{dt}\int_{x}^{x+\Delta x}\int_{y}^{y+\Delta y}\rho\, dx\, dy = -\oint\left[(\rho u)\frac{dy}{dS} - (\rho v)\frac{dx}{dS}\right]dS \tag{2.8}$$

where the line integral is taken around the boundary of the rectangle in the x, y plane bounded by the points indicated in the limits of integration in such a way that the area involved is on the left as the boundary is traversed. Again quadrature formulas may be used to obtain differential difference equations which will hold (but with different accuracy) across regions with discontinuities as well as those without discontinuities.

As an example, note that if we use the trapezoidal rule in equation (2.7) and write $x = l\,\Delta x$ and $x + \Delta x = (l + 1)\,\Delta x$, it becomes

$$\frac{\Delta x}{2}\frac{d}{dt}(\rho_{l+1} + \rho_l) = -\left[(\rho u)_{l+1} - (\rho u)_l\right] \tag{2.9}$$

This may be further approximated as complete difference equations by integrating both sides with respect to t and using a similar integration formula

$$\begin{aligned}\rho_{l+1}^{n+1} + \rho_l^{n+1} - \rho_{l+1}^{n} - \rho_l^{n} \\ = -\frac{\Delta t}{\Delta x}\left[(\rho u)_{l+1}^{n+1} - (\rho u)_l^{n+1} + (\rho u)_{l+1}^{n} - (\rho u)_l^{n}\right]\end{aligned} \tag{2.10}$$

A variety of other difference equations may be obtained. Thus, Lax has used*

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*P. D. Lax, "On Discontinuous Initial Value Problems for Nonlinear Equations and Finite Difference Schemes," Los Alamos Scientific Laboratory Report LAMS-1332, December 1952.

$$\rho_l^{n+1} = \rho_l^n + \frac{\Delta t}{2}\left[\frac{d}{dt}(\rho_{l+1} + \rho_l)\right]^n = \rho_l^n - \frac{\Delta t}{\Delta x}\left\{(\rho u)_{l+1}^n - (\rho u)_l^n\right\} \tag{2.11}$$

When we deal with Lagrangian coordinates, the equation of conservation of mass which holds irrespective of whether or not a discontinuity is present in the fixed volume is

$$\frac{d}{dt}\int_{V_1(o)+V_2(o)} \rho_o \, dV_o = 0 \tag{2.12}$$

In the one dimensional case when the "volume" is the interval between ξ and $\xi + \Delta\xi$ this may be written as

$$\frac{d}{dt}\int_{\xi}^{\xi+\Delta\xi} \rho_o \, d\xi = 0 \tag{2.13}$$

As has been mentioned before

$$\rho_o V(\xi, t) = \frac{\rho_o}{\rho} = \frac{\partial x}{\partial \xi} \tag{2.14}$$

and

$$u = \frac{\partial x}{\partial t} \tag{2.15}$$

where V is the specific volume and $x(\xi, t)$ for fixed ξ is the particle path of the particle at ξ. It follows from this interpretation of x that it is a continuous function of ξ and t whose partial derivatives may become discontinuous along a curve in the ξ, t plane. Thus, the equation

$$\rho_o \frac{\partial V}{\partial t} = \frac{\partial u}{\partial \xi} \tag{2.16}$$

holds everywhere in the ξ, t plane except on the discontinuous line. We may write an integral form of equation (2.16) obtained by integrating this equation over a region of the x, t plane or equivalently by integrating equations (2.14) and (2.15). Thus

$$\int_{\xi}^{\xi+\Delta\xi} \rho_o V \, d\xi = x_{\xi+\Delta\xi}(t) - x_{\xi}(t)$$

$$\int_{t}^{t+\Delta t} u \, dt = x_{\xi}(t+\Delta t) - x_{\xi}(t)$$

These equations hold irrespective of whether a discontinuity is present. By approximating the integrals appropriately, they lead to difference equations for V and u in terms of $x_{\xi}(t)$.

3. THE CONSERVATION OF MOMENTUM

The equations describing the conservation of momentum are obtained by considering the equation

$$\frac{d}{dt}\int_{V(t)} \rho \, u^i \, dV = \int_{V(t)} \rho \, F^{(i)} \, dV - \int_{S(t)} p \, \lambda^i \, dS \tag{3.1}$$

where $F^{(i)}$ is the component of force per unit mass in the direction of x^i and p is the pressure acting. Equation (3.1) leads to the partial differential equations

$$\frac{\partial(\rho u^i)}{\partial x^j} + \frac{\partial(\rho u^i u^j + p\delta^{ij})}{\partial x^i} = \rho \, F^{(i)} \tag{3.2}$$

in regions where the derivatives exist and to the equations

$$(p_1 - p_2) \, \lambda^i = m(u_1^i - u_2^i) \tag{3.3}$$

across discontinuities.

It may be shown by the use of an argument analogous to that given in the derivation of equation (2.5) that for a fixed volume

$$\frac{d}{dt}\int \rho\, u^i\, dV = -\int (\rho\, u^i u^j + p\delta^{ij})\, \lambda_j\, dS + \int \rho\, F^{(i)}\, dV \tag{3.4}$$

where the surface integral is taken over the fixed surface bounding the volume. Equation (3.4) holds irrespective of whether the volume contains a surface discontinuity.

In the one dimensional case equation (3.4) becomes

$$\begin{aligned} &\frac{d}{dt}\int_x^{x+\Delta x} \rho\, u\, dx \\ &\quad = -\left[(\rho\, u^2 + p)_{x+\Delta x} - (\rho\, u^2 + p)_x\right] + \int_x^{x+\Delta x} \rho\, F\, dx \end{aligned} \tag{3.5}$$

In the two dimensional case we have

$$\begin{aligned} \frac{d}{dt}\int_x^{x+\Delta x}\int_y^{y+\Delta y} \rho\, u\, dx\, dy &= -\oint \left[(\rho\, u^2 + p)\frac{dy}{dS} - (\rho\, u\, v)\frac{dx}{dS}\right] dS \\ &+ \int_x^{x+\Delta x}\int_y^{y+\Delta y} \rho\, F^{(1)}\, dx\, dy \end{aligned} \tag{3.6}$$

$$\begin{aligned} \frac{d}{dt}\int_x^{x+\Delta x}\int_y^{y+\Delta y} \rho\, v\, dx\, dy &= -\oint \left[(\rho\, u\, v)\frac{dy}{dS} - (\rho\, v^2 + p)\frac{dx}{dS}\right] dS \\ &+ \int_x^{x+\Delta x}\int_y^{y+\Delta y} \rho\, F^{(2)}\, dx\, dy \end{aligned} \tag{3.7}$$

These equations may be used to obtain difference equations by using quadrature formulas to approximate the integrals involved.

When we deal with Lagrange coordinates, the conservation of momentum condition may be written as

$$\frac{d}{dt}\int_{V_1(o)+V_2(o)} \rho_o u^i \, dV_o$$

(3.8)

$$= -\int_{V_1(o)+V_2(o)} \frac{\rho_o}{\rho} \frac{\partial p}{\partial x^i} \, dV_o + \int_{V_1(o)+V_2(o)} \rho_o F^{(i)} \, dV_o$$

In the one dimensional case we may write this as

(3.9) $$\frac{d}{dt}\int_{\xi}^{\xi+\Delta\xi} \rho_o u \, d\xi = -p(\xi + \Delta\xi) + p(\xi) + \int_{\xi}^{\xi+\Delta\xi} \rho_o F \, d\xi$$

By approximating the integral appropriately, we obtain a differential difference equation which may be turned into a complete difference equation by integrating with respect to t and replacing the time integration by an approximation.

4. THE CONSERVATION OF ENERGY

We shall assume that the forces per unit mass are derivable from a potential function, that is, there exists an Ω such that

(4.1) $$F^{(i)} = -\frac{\partial \Omega}{\partial x^i}$$

and that Ω is independent of t

(4.2) $$\frac{\partial \Omega}{\partial t} = 0$$

The specific internal energy of the gas will be denoted by E and for a perfect gas we shall write

(4.3) $$E = \frac{1}{\gamma - 1} \frac{p}{\rho}$$

We shall also write

$$q^2 = u^i u^j \delta_{ij}$$

That is, we shall denote the square of the speed of the gas by q^2.

The conservation of energy equation then takes the form

(4.4) $$\frac{d}{dt} \int_{V(t)} \rho \left(\frac{q^2}{2} + E + \Omega \right) dV = - \int p\, u^i \lambda_i \, dS$$

This leads to the differential equation

(4.5) $$\frac{\partial}{\partial t} \left[\rho \left(\frac{q^2}{2} + E + \Omega \right) \right] = - \frac{\partial}{\partial x^i} \left[\rho\, u^i \left(\frac{q^2}{2} + E + \Omega + \frac{p}{\rho} \right) \right]$$

and to the condition

(4.6) $$m \left(\frac{q_1^2}{2} + E_1 \right) - m \left(\frac{q_2^2}{2} + E_2 \right) = p_1 u_1^i \lambda_i - p_2 u_2^i \lambda_i$$

across discontinuities.

In view of the equations of conservation of mass and momentum an equivalent form for equation (4.5) is

(4.7) $$\rho \left(\frac{\partial E}{\partial t} + u^i \frac{\partial E}{\partial x^i} \right) + p \frac{\partial u^i}{\partial x^i} = 0$$

In case the gas is incompressible, that is, in case

$$E \equiv 0$$

equation (4.7) becomes

(4.8) $$\frac{\partial u^i}{\partial x^i} = 0$$

By using the same argument as that given in the two preceeding sections, it follows from equations (4.5) and (4.6) that

(4.9) $$\frac{d}{dt}\int \rho\left(\frac{q^2}{2} + E + \Omega\right) dV = -\int \rho\, u^i \left(\frac{q^2}{2} + E + \Omega + \frac{p}{\rho}\right)\lambda_i\, dS$$

where the volume integral is over a fixed volume which may contain a surface of discontinuity, and the surface integral is over the fixed surface of the volume.

In the case of the incompressible fluid we integrate equation (4.8) over the subvolumes $W_1(t)$ and $W_2(t)$ (cf. Section 2), and we obtain

$$\int u^i\, \lambda_i\, dS + \int [u^i]\, \lambda_{\Sigma i}\, d\Sigma = 0$$

Where the first integral is taken over the surface bounding the fixed volume and the second integral is over the surface of discontinuity. However, in the incompressible case we have

$$[u^i]\, \lambda_{\Sigma i} = 0$$

Hence equation (4.8) may be written as

(4.10) $$\int u^i\, \lambda_i\, dS = 0$$

where the integral is taken over the surface bounding a fixed volume which may contain a discontinuity.

In the one dimensional case equation (4.9) may be written as

$$\frac{d}{dt}\int_{x}^{x+\Delta x} \rho\left(\frac{u^2}{2} + E + \Omega\right)dx$$

(4.11)

$$= -\left[\rho\, u\left(\frac{u^2}{2} + E + \Omega + \frac{p}{\rho}\right)\right]_{x+\Delta x} + \left[\rho\, u\left(\frac{u^2}{2} + E + \Omega + \frac{p}{\rho}\right)\right]_{x}$$

In the two dimensional case we have

$$\frac{d}{dt}\iint \rho\left(\frac{q^2}{2} + E + \Omega\right)dx\,dy$$

(4.12)

$$= -\oint \rho\left(\frac{q^2}{2} + E + \Omega + \frac{p}{\rho}\right)\left(u\,\frac{dy}{dS} - v\,\frac{dx}{dS}\right)dS$$

The two dimensional form of equation (4.10) is

(4.13) $$\oint \left(u\,\frac{dy}{dS} - v\,\frac{dx}{dS}\right)dS = 0$$

In both equations (4.12) and (4.13) the line integrals are taken in directions which keep the area being bounded on the left.

5. TWO DIMENSIONAL INCOMPRESSIBLE FLOWS

As an example of the method described above for obtaining difference and differential difference equations, we consider the case of the two dimensional flow of an incompressible fluid subject to a uniform gravitational field. The equations we have to consider are the conservation of mass

(5.1) $$\frac{d}{dt}\iint \rho\,dx\,dy = -\oint \rho\left(u\,\frac{dy}{dS} - v\,\frac{dx}{dS}\right)dS$$

The conservation of momentum

(5.2) $$\frac{d}{dt}\iint \rho\,u\,dx\,dy = -\oint \left[(\rho\,u^2 + p)\,\frac{dy}{dS} - \rho\,u\,v\,\frac{dx}{dS}\right]dS$$

(5.3) $$\frac{d}{dt}\iint \rho\, v\, dx\, dy = -\oint \left[\rho\, u\, v \frac{dx}{dS} - (\rho\, v^2 + p)\frac{dy}{dS}\right] dS - \iint g\, \rho\, dx\, dy$$

and the conservation of energy

(5.4) $$\oint \left(u \frac{dy}{dS} - v \frac{dx}{dS}\right) dS = 0$$

where g is the acceleration of gravity.

Our program is to approximate the integrals by quadrature formulas and to study the ensuing equations. We shall use the trapezoidal rule for the approximation of the integrals and shall ignore the existence of discontinuities in the region. This implies that our approximation is not equally good for regions where a discontinuity (or abrupt transition) occurs as for other regions. The form of the trapezoidal rule we shall use is the following one: We shall replace any linear integral by the value of the function at the mid-point of the interval times the length of the interval. We evaluate equations (5.1) to (5.4) by this method, using as our region of integration the rectangle of sides $2\ \Delta x$ and $2\ \Delta y$, centered at the point $x = i\ \Delta x$, $y = j\ \Delta y$.

Equation (5.1) becomes

$$\frac{d}{dt}\int_{i-1}^{i+1}\int_{j-1}^{j+1} \rho\, dx\, dy = 2\,\Delta y \frac{d}{dt}\int_{i-1}^{i+1} \rho(x, j\,\Delta y)\, dx = 4\,\Delta x\,\Delta y \frac{d}{dt}\rho_{ij}$$

$$= \int_{i-1}^{i+1} (\rho\, v)\,[x, (j-1)\,\Delta y]\, dx - \int_{j-1}^{j+1} (\rho\, u)\,[(i+1)\,\Delta x, y]\, dy$$

$$- \int_{i-1}^{i+1} (\rho\, v)\,[x, (j+1)\,\Delta y]\, dx + \int_{j-1}^{j+1} (\rho\, u)\,[(i-1)\,\Delta x, y]\, dy$$

$$= 2\,\Delta x\left[(\rho\, v)_{i,j-1} - (\rho\, v)_{i,j+1}\right] + 2\,\Delta y\left[(\rho\, u)_{i-1,j} - (\rho\, u)_{i+1,j}\right]$$

That is

$$(5.5)\qquad \frac{d}{dt}\rho_{ij} = \frac{1}{2\,\Delta y}\left[(\rho\, v)_{i,j-1} - (\rho\, v)_{i,j+1}\right] + \frac{1}{2\,\Delta x}\left[(\rho\, u)_{i-1,j} - (\rho\, u)_{i+1,j}\right]$$

Similarly, we may show that equations (5.2), (5.3) and (5.4) become

$$(5.6)\qquad \frac{d}{dt}(\rho\, u)_{ij} = \frac{1}{2\,\Delta y}\left[(\rho\, u\, v)_{i,j-1} - (\rho\, u\, v)_{i,j+1}\right] + \frac{1}{2\,\Delta x}\left[(\rho\, u^2 + p)_{i-1,j} - (\rho\, u^2 + p)_{i+1,j}\right]$$

$$(5.7)\qquad \frac{d}{dt}(\rho\, v)_{ij} = \frac{1}{2\,\Delta y}\left[(\rho\, v^2 + p)_{i,j-1} - (\rho\, v^2 + p)_{i,j+1}\right] + \frac{1}{2\,\Delta x}\left[(\rho\, u\, v)_{i-1,j} - (\rho\, u\, v)_{i+1,j}\right] - g\,\rho_{i,j}$$

and

$$(5.8)\qquad \frac{1}{2\,\Delta y}\left[v_{i,j-1} - v_{i,j+1}\right] + \frac{1}{2\,\Delta x}\left[u_{i-1,j} - u_{i+1,j}\right] = 0$$

respectively.

Note that if we had applied the trapezoidal rule by replacing a linear integral by the average of the integrand at the two end points of the interval times the length of the interval, the differential difference equations (5.5) to (5.8) would be quite different. In fact, equation (5.5) would be replaced by

$$\frac{d}{dt}\left[\rho_{i+1,j+1} + \rho_{i-1,j+1} + \rho_{i+1,j-1} + \rho_{i-1,j-1}\right]$$

$$(5.9)\qquad = \frac{1}{\Delta y}\left[(\rho\, v)_{i+1,j-1} + (\rho\, v)_{i-1,j-1} - (\rho\, v)_{i+1,j+1} - (\rho\, v)_{i-1,j+1}\right]$$

$$+ \frac{1}{\Delta x}\left[(\rho\, u)_{i-1,j-1} + (\rho\, u)_{i-1,j+1} - (\rho\, u)_{i+1,j+1} - (\rho\, u)_{i+1,j-1}\right]$$

Similar modifications occur in equations (5.6) to (5.8).

The choice between the differential difference equations of the type of equation (5.5) and the type of equation (5.9) must be made on the basis of a stability analysis. We shall not attempt this here. Instead, we shall outline a computation procedure for dealing with the system (5.5) to (5.8). Similar procedures would exist for the system involving equations of the type of (5.9).

6. THE EQUATION (5.8)

If we define u_{ij} and v_{ij} in terms of a quantity ψ_{ij} by means of the equations

$$u_{ij} = \frac{1}{2\,\Delta y}\left[\psi_{i,j+1} - \psi_{i,j-1}\right] \tag{6.1}$$

$$v_{ij} = \frac{-1}{2\,\Delta x}\left[\psi_{i+1,j} - \psi_{i-1,j}\right] \tag{6.2}$$

then equation (5.8) is satisfied for every choice of the quantities ψ_{ij}, as may readily be verified.

If the other application of the trapezoidal rule were made, there would be analogous expressions for u_{ij} and v_{ij} in terms of ψ which would automatically satisfy the analogue of equation (5.8).

On fixed boundaries we must have the component of the velocity of the fluid normal to the boundary vanish. If the fixed boundaries make up a rectangular region, this may be accomplished by defining

$$\psi_{i,-1} = \psi_{i,1}$$

$$\psi_{-1,j} = \psi_{1,j}$$

7. THE EQUATIONS (5.6) AND (5.7)

These equations may be written as

$$\frac{1}{2\,\Delta x}\left(p_{i+1,j} - p_{i-1,j}\right) = -\frac{d}{dt}(\rho\,u)_{ij} + T_{ij} \tag{7.1}$$

and

$$\frac{1}{2\,\Delta y}\left(p_{i,j+1} - p_{i,j-1}\right) = -\frac{d}{dt}(\rho\,v)_{ij} + S_{ij} \tag{7.2}$$

where

$$\begin{aligned} T_{ij} &= \frac{1}{2\,\Delta y}\left[(\rho\,u\,v)_{i,j-1} - (\rho\,u\,v)_{i,j+1}\right] \\ &\quad + \frac{1}{2\,\Delta x}\left[(\rho\,u^2)_{i-1,j} - (\rho\,v^2)_{i+1,j}\right] \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} S_{ij} &= \frac{1}{2\,\Delta y}\left[(\rho\,v^2)_{i,j-1} - (\rho\,v^2)_{i,j+1}\right] \\ &\quad + \frac{1}{2\,\Delta y}\left[(\rho\,u\,v^2)_{i-1,j} - (\rho\,u\,v)_{i+1,j}\right] - g\,\rho_{ij} \end{aligned} \tag{7.4}$$

We may eliminate p_{ij} from equations (7.1) and (7.2) in a manner suggested by the differential equations analogous to these equations and obtain

$$\frac{1}{2\,\Delta x}\left(T_{i,j+1} - T_{i,j-1}\right) - \frac{1}{2\,\Delta x}\left(S_{i+1,j} - S_{i-1,j}\right)$$
$$= \frac{d}{dt}\left[\frac{(\rho\,u)_{i,j+1} - (\rho\,u)_{i,j-1}}{2\,\Delta y}\right] - \frac{d}{dt}\left[\frac{(\rho\,v)_{i+1,j} - (\rho\,v)_{i-1,j}}{2\,\Delta x}\right]$$

This equation may be written as

$$\frac{1}{2\,\Delta y}\left(\rho_{i,j+1}\frac{du_{i,j+1}}{dt}-\rho_{i,j-1}\frac{du_{i,j-1}}{dt}\right)$$

(7.5)

$$+\frac{1}{2\,\Delta x}\left(\rho_{i+1,j}\frac{dv_{i+1,j}}{dt}-\rho_{i-1,j}\frac{dv_{i-1,j}}{dt}\right)=B_{ij}$$

where

$$B_{ij}=\frac{1}{2\,\Delta y}\left(T_{i,j+1}-T_{i,j-1}\right)-\frac{1}{2\,\Delta x}\left(S_{i+1,j}-S_{i-1,j}\right)$$

(7.6)

$$-\frac{1}{2\,\Delta y}\left(\frac{d\rho_{i,j+1}}{dt}u_{i,j+1}-\frac{d\rho_{i,j-1}}{dt}u_{i,j-1}\right)$$

$$+\frac{1}{2\,\Delta x}\left(\frac{d\rho_{i+1,j}}{dt}v_{i+1,j}-\frac{d\rho_{i-1,j}}{dt}v_{i-1,j}\right)$$

Equation (5.5) may be used to replace the terms involving $d\rho_{ij}/dt$ in equation (7.6). Hence, B_{ij} is a function of the ρ_{ij} and the ψ_{ij} alone.

Equation (7.5) may be written as

$$\frac{1}{4\,\Delta x\,\Delta y}\left[\rho_{i,j+1}\left(\dot{\psi}_{i,j+2}-\dot{\psi}_{ij}\right)-\rho_{i,j-1}\left(\dot{\psi}_{i,j}-\dot{\psi}_{i,j-2}\right)\right.$$

(7.7)

$$\left.-\rho_{i+1,j}\left(\dot{\psi}_{i+2,j}-\dot{\psi}_{i,j}\right)+\rho_{i-1,j}\left(\dot{\psi}_{i,j}-\dot{\psi}_{i-2,j}\right)\right]=B_{ij}$$

where

(7.8) $$\dot{\psi}_{ij}=\frac{d\psi_{ij}}{dt}$$

Equation (7.7) is a linear equation for $\dot{\psi}$ of the form

(7.9) $$A\,\dot{\psi}=b$$

where A is a matrix and $\dot{\psi}$ and b are vectors, and if it is solved subject to the proper boundary conditions ($\dot{\psi}=0$ on the boundary of the domain under consideration), we may then determine $\dot{\psi}$. Equation (5.8) enables us to determine

$$\dot{\rho}_{ij} = \frac{d\rho_{ij}}{dt}$$

Time integrations will then enable us to follow the behavior of the fluid in time.

8. TIME INTEGRATIONS

In the above discussion we have shown how to calculate $\dot{\psi}$ and $\dot{\rho}$ as functions of ψ and ρ. We may consider these as ordinary differential equations and apply an integration procedure to them. Since we have already approximated spatial integrations by quadrature rules, we apply the same considerations to time integrals. The equation

$$\frac{d\psi}{dt} = \dot{\psi}$$

is equivalent to the equation

$$\psi(t + \Delta t) - \psi(t) = \int_t^{t+\Delta t} \dot{\psi}\, dt \tag{8.1}$$

Even if we use the trapezoidal rule we may obtain two approximations for this integral

$$\psi(t + \Delta t) = \psi(t) + \dot{\psi}\left(t + \frac{\Delta t}{2}\right)\Delta t \tag{8.2}$$

or

$$\psi(t + \Delta t) = \psi(t) + \frac{\Delta t}{2}\left[\dot{\psi}(t + \Delta t) + \dot{\psi}(t)\right] \tag{8.3}$$

In the method outlined above for dealing with our problem neither $\dot{\psi}(t + \Delta t)$ nor $\dot{\psi}[t + (\Delta t/2)]$ are available at time t. Therefore, neither equation (8.2) or (8.3) may be applied. There are at least three possibilities open to us:

1. Replace equation (8.1) by

$$\psi(t + \Delta t) = \psi(t) + \Delta t \, \dot{\psi}(t) + \frac{(\Delta t)^2}{2} \ddot{\psi}(t)$$

where $\ddot{\psi}(t)$ is computed by differencing $\dot{\psi}(t)$ or by differentiating equation (7.7) with respect to t and substituting appropriate expressions for $\dot{\psi}$ and $\dot{\rho}$.

2. In equation (8.3) replace the term $[\dot{\psi}(t+\Delta t) + \dot{\psi}(t)]/2$, which is a time average of ψ over the interval Δt, by an appropriate space average of ψ. That is write as an integration formula

$$\psi_{ij}(t + \Delta t) = \psi_{ij}(t) + \frac{\Delta t}{4}\left[\psi_{i-1,j} + \psi_{i+1,j} + \psi_{i,j-1} + \psi_{i,j+1}\right]$$

A procedure analogous to this has been suggested by Lax* in dealing with the hydrodynamic equations of compressible flow.

3. Recast the previous discussion completely in terms of difference equations instead of differential difference equations. In essence, this means that it is not strictly true that we do not have $\dot{\psi}(t+\Delta t)$ available to us. We have from equation (7.9)

$$\dot{\psi}(t + \Delta t) = A^{-1}(t + \Delta t) \, b(t + \Delta t)$$

If this were substituted into equation (8.3), we would have an equation for $\psi(t + \Delta t)$. However, it is easier to achieve this equation by transforming equations (5.1) and (5.4) wholly into difference equations instead of into differential difference equations.

*Ibid.

In subsequent reports we shall discuss the various difference equations that may be obtained in Eulerian coordinates. These equations will be based on approximations to the integral formulation discussed earlier and hence will hold irrespective of whether discontinuities are present. Of course, in regions where discontinuities are present the accuracy with which they represent the equations derived will be less than in the continuous regions.

The existence of a unified method for dealing with continuous and discontinuous regions in Eulerian coordinates indicates that it is feasible to deal with Eulerian coordinates in obtaining numerical solutions to multidimensional hydrodynamics problems. If such coordinates are used, the difficulties encountered when Lagrangian coordinates are used in problems where slippage occurs are obviated. A detailed discussion of numerical schemes for doing hydrodynamical problems by use of Eulerian coordinates will be given subsequently.