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INTEGRAL DIFFUSION KERNEL FOR A CYLINDER

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ABSTRACT

Methods of integration of

$$\int_{\tau} q(P')K(|P'-P|)d\tau$$

are considered for the case in which the region τ is a cylinder, q is constant across the cylinder and the kernel is

$$K(|P',P|) = \frac{e^{-|P',P|/\lambda}}{4\pi |P',P|^2} .$$

* * * * * * * *

The writer is grateful to Rolf Landshoff and Frank Hoyt for helpful discussions and to Bengt Carlson and Max Goldstein for carrying through the numerical integrations.

[redacted]

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INTEGRAL DIFFUSION KERNEL FOR A CYLINDER

1. The kernel $K(|P',P|)$ - The average integrated flux(1), ϕ, at a point, P, of particles which originate in a region containing a continuous distribution of isotropic point sources is given by

$$\phi(P) = \int_{\tau} q(P')K(|P',P|)d\tau_{P'} , \tag{1}$$

where q is the particle source density, the kernel is defined by

$$K(|P',P|) = \frac{e^{-|P',P|/\lambda}}{4\pi|P',P|^2} , \tag{2}$$

and the integration is carried over the entire region. λ represents the total mean free path for the particles in question and $|P',P|$ denotes the magnitude of the distance between points P' and P. Note that the time of flight of the particles has been neglected.

Collision densities can be obtained from ϕ through dividing by the mean free path. For example, if λ_j is the mean free path of the particles between collisions of a certain type(2), then the collision density at P for collisions of this type is given by

$$Q_j(P) = \frac{1}{\lambda_j}\phi(P). \tag{3}$$

The integral, $\frac{1}{\lambda}\int K(|P',P|)d\tau_{P'}$, that is Q for the case $q = 1$ and $\lambda_j = \lambda$, may also be interpreted as the relative probability that a particle which originates at P will make its first collision within the region of integration.

2. Cylindrical symmetry - Consider the case where the region of integration is a circular cylinder of radius R. Let x and ρ be distances measured along the axial and radial directions, respectively, so that

$$|P',P| = \sqrt{(\rho'-\rho)^2+(x'-x)^2}$$

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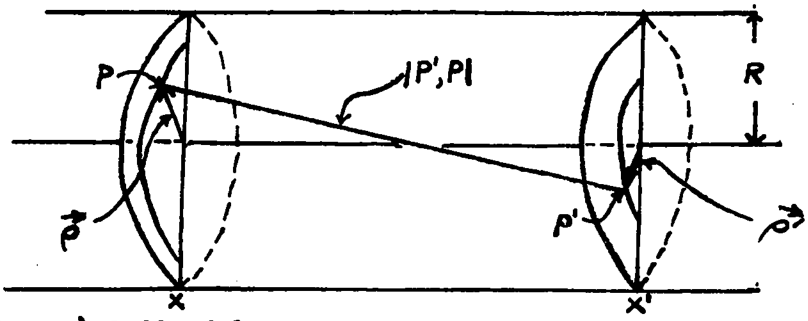
(1) By "flux", in this case, is meant the magnitude of the particle current density.

(2) This may represent, for example, elastic scattering, fission, etc.

[redacted]

If $d\sigma$ represents an element of area in a plane normal to the axis of the clyinder, then the element of volume $d\tau$ may be written $d\sigma dx$. In the case where q may be considered as a function of x only, it is convenient to use an average of ϕ (averaged across the cylinder) defined by



$$(4) \qquad \bar{\phi}(R,x) = \frac{1}{\pi R^2} \int_{\sigma} \phi(\rho,x) d\sigma_p.$$

Now introduce an average kernel

$$(5) \qquad \bar{K}(|x'-x|) \equiv \frac{1}{\pi R^2} \int_{\sigma_{p'}} \int_{\sigma_p} K(|P',P|) d\sigma_{p'} d\sigma_p$$

and (4) becomes

$$(6) \qquad \bar{\phi}(R,x) = \int q(x') \bar{K}(|x'-x|) dx'.$$

3. <u>Evaluation of the average kernel</u> - For purposes of numerical integration of (6) when the form of q is given, it is convenient to have tabulated the one dimensional kernel $\bar{K}$ as a function of $|x'-x|$. To facilitate integration make the substitutions

$$(\vec{\rho}\,'-\vec{\rho})^2 = \ell^2, \quad (x'-x)^2 = y^2.$$

The expression for the average kernel then becomes

$$\bar{K}(|x'-x|) = \bar{K}(|y|) = \frac{1}{\pi R^2} \int_{\sigma_p} \int_{\sigma_{p'}} \frac{e^{-\sqrt{\ell^2+y^2}/\lambda}}{4\pi(\ell^2+y^2)} \, d\sigma_{p'} d\sigma_p.$$

Using formula (1) derived in Appendix I and expressing ℓ in units of λ, the integral reduces to

$$(7)\quad \overline{K}(|y_o|) = \frac{1}{\pi}\int_0^{2R_o} \frac{e^{-\sqrt{\ell^2+y_o^2}}}{\ell^2+y_o^2}\left[\cos^{-1}\left(\frac{\ell}{2R_o}\right) - \left(\frac{\ell}{2R_o}\right)\sqrt{1-\left(\frac{\ell}{2R_o}\right)^2}\right]\ell\, d\ell,$$

where

$$(8)\quad R_o = R/\lambda \quad \text{and} \quad y_o = y/\lambda \ .$$

This expression has been integrated numerically with y_o ranging from .001 to 3.0 for various values of R_o between .05 and 6. The results are tabulated in Table 1. Also, the kernel $\overline{K}$ is plotted as a function of y_o for various R_o in Fig. 1, and as a function of R_o for various y_o in Fig. 2.

In making use of the computed values of the average kernel to numerically evaluate an integral of the form (6), the integral would be replaced by a finite sum:

$$\int q(x')\overline{K}(|x'-x|)dx' \longrightarrow \Delta x\left[\sum_{\substack{j=o\\ j\neq i}}^{n} q_j\, K_{|i-j|} + q_i\,\overline{K}_o\right]$$

where

$$q_j = \tfrac{1}{2}\left[q(x_{j-1}) + q(x_{j+1})\right] \quad \text{and} \quad K_{|i-j|} = \overline{K}(|(i-j)\Delta x|) \ ; \ j \neq i.$$

It is seen that $K_{|i-j|}$, with $i=j$, is infinite so that it is necessary to use some kind of average over an interval of width Δx. Thus $\overline{K}_o$ is taken to be

$$\overline{K}_o = \frac{1}{(\Delta x)^2}\int_0^{\Delta x} dx \int_0^{\Delta x} dx'\ \overline{K}\ (|x'-x|).$$

$\overline{K}_o$ is evaluated in Appendix II, and its numerical values for various radii and interval lengths are shown in Table 3.

4. <u>Approximation through Taylor expansion of q</u> - For the case of a cylinder of infinite length an approximate expression of simple form can be obtained for the integral in (6). Expand $q(x')$ in a Taylor series about the point $x' = x$,

$$q(x') = q(x) + q'(x)(x'-x) + \frac{1}{2!}q''(x)(x'-x)^2 + \cdots \ .$$

Let $y = x'-x$ and substitute into (6). Assuming q and its derivatives to be continuous at x, there results

$$\overline{\phi}(R,x) = \overline{M}_o\left[2q(x) + \frac{2}{2!}q''(x)\overline{\lambda^2} + \frac{2}{4!}q^{IV}(x)\overline{\lambda^4} + \cdots\right],$$

[redacted]

where

$$(9)\quad \overline{M}_n(R) = \int_0^{\infty} y^n\overline{K}(|y|)dy, \quad \overline{\lambda^n} = \frac{\overline{M}_n}{\overline{M}_o}\ , \quad n = 0,1,2, \cdots \ .$$

The above expression may be rearranged as follows

$$\overline{\Phi}(R,x) = \overline{M}_0\Big\{q(x) + q'(x)\Lambda + \frac{1}{2!}q''(x)\Lambda^2 + \frac{1}{3!}q'''(x)\Lambda^3 + \frac{1}{4!}q^{IV}(x)\Lambda^4 + \cdots$$

$$+ q(x) + q'(x)(-\Lambda) + \frac{1}{2!}q''(x)(-\Lambda)^2 + \frac{1}{3!}q'''(x)(-\Lambda)^3 + \frac{1}{4!}q^{IV}(x)(-\Lambda)^4 + \cdots$$

$$+ \frac{2}{4!}\left[\overline{\lambda^4} - \Lambda^4\right]q^{IV}(x) + \cdots\Big\}.$$

where $\Lambda = \sqrt{\overline{\lambda^2}}$. If the fourth and higher derivatives of q are neglected there results

$$(10) \quad \overline{\Phi}(R,x) = \int_{-\infty}^{\infty} q(x')\overline{K}(|x'-x|)dx' \cong \overline{M}_0\left[q(x+\Lambda) + q(x-\Lambda)\right].$$

When q is assumed to be linear in $(x'-x)$, but a discontinuity in the slope of q at $x'=x$ is allowed for, a similar deduction leads to the expression

$$(11) \quad \overline{\Phi}(R,x) \cong \overline{M}_0\left[q(x+\overline{\lambda}) + q(x-\overline{\lambda})\right].$$

Which of the two expressions (10) and (11) is the more appropriate in a given case would seem to depend upon whether the function q(x) is approximated more accurately by two intersecting straight lines or by a higher degree curve.

The method of computation of the moments $\overline{M}_n$ $(n=0,1,2)$ is described in Appendix [illegible], and numerical values of these are tabulated in Table 4 for R_0 = .05, .25, .5, 1, 1.5, 2, 2.5, 3, 4, 5, 6. Graphs of $\overline{M}_0$, $\overline{M}_1$, $\overline{M}_2$ as functions of R are plotted in Fig. [illegible]. It is seen from the definition (9) and the comment at the end of Section 1, that the quantity $2\overline{M}_0/\lambda$ is the average probability (averaged across the cylinder) that a particle which originates within the cylinder will undergo at least one collision before escaping from the cylinder.

[redacted]

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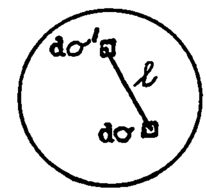
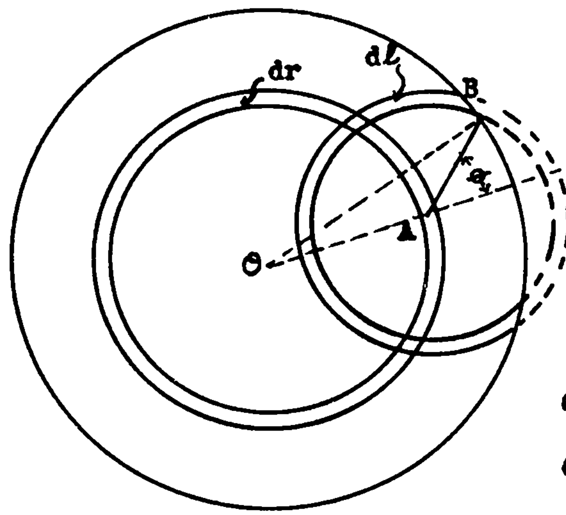
APPENDIX I. EVALUATION OF THE INTEGRAL, $I = \int_{\sigma}\int_{\sigma'} f(\ell)\,d\sigma' d\sigma$.

In this integral $d\sigma$ and $d\sigma'$ are elements of area and ℓ is the distance between them. The integral is to be evaluated over a circle of radius R. $f(\ell)$ is any integrable function of ℓ.



To carry out the integration, first hold ℓ constant and move it through all possible positions in the circle. Do this by allowing one end of ℓ, say A (see figure), to move on the ring of inner radius r and outer radius r + dr covering the area $2\pi r\,dr$, while for each point of this ring the other end of ℓ sweeps out the area

$$(2\pi - 2\theta)\,\ell\,d\ell.$$



OA = r

OB = R

AB = ℓ

Then sum as ℓ varies from 0 to 2R and r varies from 0 to R obtaining

$$I = \int_{\ell=0}^{2R}\int_{r=0}^{R} f(\ell)\cdot(2\pi r\,dr)\cdot 2(\pi-\theta)\,\ell\,d\ell$$

$$= 4\pi\int_0^{2R} d\ell\, f(\ell)\,\ell \int_0^{R} dr\,(\pi-\theta)\,r.$$

The angle θ is seen to be given by

$$\theta = \begin{cases} 0, & \text{for } r \leq R-\ell \\ \cos^{-1}\left(\dfrac{R^2-\ell^2-r^2}{2\ell r}\right), & \text{for } r > R-\ell. \end{cases}$$

Substituting into the integral there results [redacted]

$$I = 4\pi\int_0^{2R} d\ell\, f(\ell)\,\ell\left(\frac{\pi R^2}{2} - I'\right).$$

[redacted]

where

$$I' = \int_{R-\ell}^{R} r \cos^{-1}\left(\frac{R^2-\ell^2-r^2}{2\ell r}\right) dr.$$

To evaluate I', make the change of variable

$$\frac{r}{\sqrt{R^2-\ell^2}} = x$$

and obtain

$$I' = (R^2-\ell^2) \int_{\frac{a(R-\ell)}{2\ell}}^{\frac{aR}{2\ell}} x \cos^{-1}\left(\frac{1-x^2}{ax}\right) dx,$$

where

$$a = \frac{2\ell}{\sqrt{R^2-\ell^2}} .$$

Integrating by parts yields

$$I' = \frac{R^2}{2}\left[\pi - \cos^{-1}\left(\frac{\ell}{2R}\right)\right] - (R^2-\ell^2) \int_{\frac{a(R-\ell)}{2\ell}}^{\frac{aR}{2\ell}} \frac{(x^2+1)x\,dx}{2\sqrt{-x^4+(a^2+2)x^2-1}} .$$

Now make the change of variable $x^2 = y$ in the integral term and it becomes, after considerable algebraic reduction,

$$\frac{R^2-\ell^2}{4} \int_{\frac{a^2(R-\ell)^2}{(2\ell)^2}}^{\left(\frac{aR}{2\ell}\right)^2} \frac{(y+1)dy}{\sqrt{-y^2+(a^2+2)y-1}} = \frac{R^2}{2}\left[-\frac{\ell}{R}\sqrt{1-\left(\frac{\ell}{2R}\right)^2} + \cos^{-1}\left(\frac{\ell}{2R}\right)\right] .$$

Thus

$$I' = \frac{\pi R^2}{2} - R^2\left[\cos^{-1}\left(\frac{\ell}{2R}\right) - \frac{\ell}{2R}\sqrt{1-\left(\frac{\ell}{2R}\right)^2}\right]$$

[redacted]

[redacted]

and the original integral becomes

$$(1) \quad I = 4\pi R^2 \int_0^{2R} f(\ell)\left[\cos^{-1}\left(\frac{\ell}{2R}\right) - \frac{\ell}{2R}\sqrt{1-\left(\frac{\ell}{2R}\right)^2}\right]\ell\, d\ell .$$

By the substitutions

$$\ell = 2R\cos\theta \qquad \text{and} \qquad \ell = R\sqrt{2(1+\cos x)},$$

there result, respectively, the two additional forms

$$(2) \quad I = 16\pi R^4 \int_0^{\pi/2} f(2R\cos\theta)\left[\theta - \tfrac{1}{2}\sin(2\theta)\right]\sin(2\theta)\, d\theta$$

and

$$(3) \quad I = 2\pi R^4 \int_0^{\pi} f\left(R\sqrt{2(1+\cos x)}\right)\left[x\sin x - \sin^2 x\right] dx.$$

[redacted]

[redacted]

APPENDIX II. Evaluation of $\bar{K}_0$

$$\bar{K}_0 = \frac{1}{(\Delta x)^2} \int_0^{\Delta x} dx \int_0^{\Delta x} dx' \ \bar{K}(|x'-x|)$$

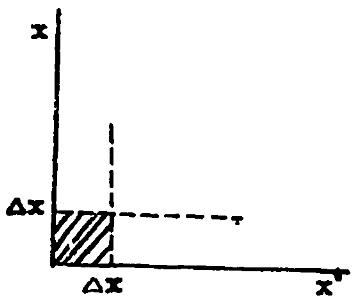
Make the change of variable

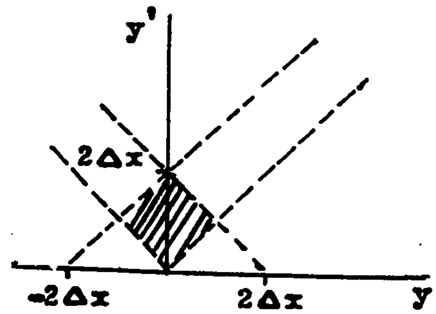
$$x'-x = y$$

$$x'+x = y' \quad ; \quad y' = y + 2x = -y + 2x'$$

so that

$$dx dx' = \left| J\left(\frac{x,x'}{y,y'}\right)\right| dy dy' = \tfrac{1}{2}\, dy dy'.$$





Then

$$\bar{K}_0 = \frac{1/2}{(\Delta x)^2}\left[\int_{-\Delta x}^{0} dy \int_{-y}^{y+2\Delta x} dy' \bar{K}(|y|) + \int_0^{\Delta x} dy \int_{y}^{-y+2\Delta x} dy' \bar{K}(|y|)\right]$$

$$= \frac{2}{\Delta x}\, I_0 - \frac{2}{(\Delta x)^2}\, I_1 .$$

where

$$I_0 = \int_0^{\Delta x} \bar{K}(|y|)dy \qquad \text{and} \qquad I_1 = \int_0^{\Delta x} y\bar{K}(|y|)dy.$$

Make the substitution (8) so that

$$I_0 = \lambda \int_0^{\Delta x_0} \bar{K}(|y_0|)dy_0 \quad \text{and} \quad I_1 = \lambda^2 \int_0^{\Delta x_0} y_0 \bar{K}(|y_0|)dy_0.$$

[redacted]

where

$$\Delta x_0 = \frac{\Delta x}{\lambda} \ .$$

[redacted]

Thus

$$\frac{2}{\Delta x} I_o = \frac{2}{\Delta x_o} \int_0^{\Delta x_o} \overline{K}(|y_o|)dy_o \quad \text{and} \quad \frac{2}{(\Delta x)^2} I_1 = \frac{2}{(\Delta x_o)^2} \int_0^{\Delta x_o} y_o \overline{K}(|y_o|)dy_o$$

The I_o and I_1 have been integrated numerically making use of the previous evaluation of $\overline{K}(|y_o|)$, and computed values of $\frac{2}{\Delta x} I_o$ and $\frac{2}{(\Delta x)^2} I_1$ for various Δx_o and R_o are tabulated in Table 2. Computed values of $\overline{K}_o$ are also tabulated, in Table 3.

[redacted]

APPENDIX XII. COMPUTATION OF THE $\overline{M}_n$ FOR n=0,1,2.

1. n = 0

$$\overline{M}_o(R) = \int_0^\infty \overline{K}(|y|)dy = \frac{1}{\pi R^2}\int_0^\infty dy \int_\sigma \int_{\sigma'} \frac{e^{-\sqrt{\ell^2+y^2}/\lambda}}{4\pi(\ell^2+y^2)}\, d\sigma' d\sigma,$$

where

$$\ell^2 = (\vec{\rho}\,' - \vec{\rho})^2.$$

To carry out the integration with respect to y, let

$$y^2 = \ell^2 \sinh^2 u$$

so

$$\ell^2 + y^2 = \ell^2 \cosh^2 u, \qquad dy = \ell \cosh u \, du$$

and the integral becomes

$$\overline{M}_o(R) = \frac{1}{4\pi^2 R^2}\int_\sigma \int_{\sigma'} d\sigma' d\sigma \int_0^\infty \frac{e^{-\frac{\ell}{\lambda}\cosh u}}{\ell \cosh u}\, du.$$

The integral

$$Ki_n(x) = \int_0^\infty \frac{e^{-x\cosh u}}{\cosh^n u}\, du$$

has been tabulated(3) so that it is possible to write

$$\overline{M}_o(R) = \frac{1}{4\pi^2 R^2}\int_\sigma \int_{\sigma'} \frac{Ki_1\left(\frac{\ell}{\lambda}\right)}{\ell}\, d\sigma' d\sigma.$$

Applying the formula derived in Appendix I and making the change of variable

$$x = \frac{\ell}{2R}$$

there results

$$\overline{M}_o(R) = \frac{2R}{\pi}\int_0^1 Ki_1\left(\frac{2R}{\lambda}x\right)\left[\cos^{-1} x - x\sqrt{1-x^2}\right] dx.$$ [redacted]

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(3) W. G. Bickley and J. Nayler, Phil. Mag., S.7, 20 (1935). This function is equal to the Bessel function of imaginary argument for n=0, and $Ki_n(x) = \int_x^\infty Ki_{n-1}(t)dt$. See also reference (4), below.

[redacted]

This expression has been numerically integrated for various values of $\frac{R}{\lambda} = R_o$, that is, various values of the radius expressed in units of a mean free path, and the results are shown in Table 4 and Fig. 3.

2. <u>n = 1</u>

$$\overline{M}_1(R) = \int_0^\infty y\overline{K}(|y|)dy = \frac{1}{\pi R^2}\int_0^\infty y\,dy \int_\sigma\int_{\sigma'} \frac{e^{-\sqrt{\ell^2+y^2}/\lambda}}{4\pi(\ell^2+y^2)}\,d\sigma' d\sigma$$

Make the substitution

$$y^2 = \ell^2(u^2-1), \qquad -ydy = \ell^2\,udu$$

and obtain

$$\overline{M}_1(R) = \frac{1}{4\pi^2R^2}\int_\sigma\int_{\sigma'} d\sigma' d\sigma \int_1^\infty \frac{e^{-\frac{\ell}{\lambda}u}}{u}\,du.$$

The exponential integral

$$E_n(x) = \int_1^\infty \frac{e^{-xu}}{u^n}\,du$$

has been tabulated so that this may be written

$$\overline{M}_1(R) = \frac{1}{4\pi^2R^2}\int_\sigma\int_{\sigma'} E_1\left(\frac{\ell}{\lambda}\right)d\sigma' d\sigma\ .$$

Again making use of the formula derived in Appendix I, this becomes

$$\overline{M}_1(R) = \frac{4R^2}{\pi}\int_0^1 E_1\left(\frac{2R}{\lambda}x\right)\left(\cos^{-1}x - x\sqrt{1-x^2}\right)x\,dx.$$

The numerical integration of this expression leads to the results shown in Table 4 and Fig. 3.

3. <u>n = 2</u> [redacted]

$$\overline{M}_2(R) = \int_0^\infty y^2\overline{K}(|y|)dy = \frac{1}{\pi R^2}\int_0^\infty y^2dy \int_\sigma\int_{\sigma'} \frac{e^{-\sqrt{\ell^2+y^2}/\lambda}}{4\pi(\ell^2+y^2)}\,d\sigma' d\sigma\ .$$

Make the same change of variable as in Section 1 and obtain

[redacted]

$$\overline{M}_2(R) = \frac{1}{4\pi^2 R^2} \int_{\sigma}\int_{\sigma'} d\sigma' d\sigma \int_0^{\infty} \ell \left(\cosh u \; e^{-\frac{\ell}{\lambda}\cosh u} - \frac{e^{-\frac{\ell}{\lambda}\cosh u}}{\cosh u} \right) du,$$

or

$$\overline{M}_2(R) = \frac{1}{4\pi^2 R^2} \int_{\sigma}\int_{\sigma'} d\sigma' d\sigma \, \ell \left[K_1\left(\frac{\ell}{\lambda}\right) - Ki_1\left(\frac{\ell}{\lambda}\right) \right],$$

where the second term inside the square brackets is defined in section 1 and the first term is the Bessel function of imaginary argument[4] defined by

$$K_n(x) = \int_0^{\infty} e^{-x \cosh u} \cosh(n u) du.$$

Proceeding then as before, the expression becomes

$$\overline{M}_2(R) = \frac{8R^3}{\pi} \int_0^1 \left[K_1\left(\frac{2R}{\lambda} x\right) - Ki_1\left(\frac{2R}{\lambda} x\right) \right] \left(\cos^{-1}x - x\sqrt{1-x^2}\right) x^2 dx.$$

Results of the final numerical integration are to be seen in Table 4 and Fig. 4.

(4) See G. N. Watson, "Theory of Bessel Functions", Eq. (5), page 181, and Table II.

[redacted]

TABLE 1

Table of $\bar{K}(|y_o|)$

y_o \ R_o	.05	.25	.5	1.0	2.0	3.0	4.0	6.0
0			∞	∞	∞	∞	∞	∞
.001	1.694	2.4143	2.6737	2.8760	3.0106	3.0610	3.0870	3.1132
.002	1.357	2.0701	2.3286	2.5304	2.6648	2.7150	2.7410	2.7671
.003	1.163	1.8698	2.1273	2.3287	2.4628	2.5130	2.5388	2.5650
.004	1.028	1.7283	1.9849	2.1859	2.3196	2.3697	2.3956	2.4217
.005	.927	1.6192	1.8749	2.0752	2.2089	2.2589	2.2846	2.3107
.006	.845	1.5304	1.7851	1.9850	2.1184	2.1683	2.1942	2.2201
.007	.777	1.4556	1.7095	1.9090	2.0420	2.0919	2.1177	2.1436
.008	.720	1.3913	1.6442	1.8432	1.9760	2.0259	2.0515	2.0774
.009	.670	1.3348	1.5868	1.7853	1.9179	1.9676	1.9932	2.0190
.01	.627	1.2845	1.5356	1.7336	1.8659	1.9156	1.9411	1.9670
.02	.371	.9613	1.2034	1.3967	1.5266	1.5755	1.6006	1.6261
.03	.251	.7813	1.0149	1.2036	1.3311	1.3792	1.4039	1.4290
.04	.182	.6598	.8849	1.0692	1.1943	1.2417	1.2660	1.2907
.05	.136	.5700	.7869	.9669	1.0899	1.1366	1.1604	1.1848
.06	.105	.5000	.7091	.8849	1.0058	1.0517	1.0752	1.0992
.07	.081	.4437	.6451	.8169	.9356	.9809	1.0040	1.0275
.08	.066	.3971	.5912	.7591	.8758	.9204	.9430	.9663
.09	.054	.3579	.5448	.7089	.8238	.8676	.8901	.9129
.1	.046	.3244	.5045	.6649	.7779	.8211	.8431	.8656
.2	.012	.1457	.2717	.3988	.4933	.5333	.5517	.5715
.3	.0051	.0779	.1681	.2698	.3459	.3843	.4010	.4181
.4	.0024	.0465	.1111	.1939	.2612	.2896	.3059	.3207
.5	.0015	.0296	.0767	.1432	.2015	.2251	.2400	.2532
.6	.0010	.0186	.0547	.1084	.1583	.1792	.1919	.2037
.7	.0006	.0136	.0398	.0836	.1268	.1447	.1559	.1651
.8	.0004	.0096	.0280	.0652	.1028	.1179	.1279	.1368
.9	.0003	.0066	.0223	.0515	.0836	.0975	.1058	.1138
1.0	.0002	.0054	.0171	.0411	.0687	.0810	.0882	.0953
1.2	.0001	.0034	.0104	.0266	.0472	.0564	.0623	.0678
1.5	.00006	.0016	.0053	.0146	.0277	.0345	.0382	.0421
2.0	.00002	.0006	.0019	.0057	.0122	.0157	.0177	.0198
3.0	.000003	.0001	.0003	.0011	.0027	.0037	.0044	.0051

Table 2

Table of $\frac{2}{\Delta x} I_o$

Δx_o \ R_o	.05	.25	.5	1.0	2.0	3.0	4.0	6.0
0								
.001	4.38	5.83	6.34	6.75	7.02	7.12	7.18	7.23
.002	3.69	5.14	5.66	6.06	6.33	6.43	6.48	6.53
.003	3.30	4.73	5.25	5.65	5.92	6.02	6.08	6.13
.004	3.02	4.45	4.96	5.37	5.64	5.74	5.79	5.84
.005	2.81	4.23	4.74	5.15	5.41	5.51	5.57	5.62
.006	2.63	4.05	4.56	4.96	5.23	5.33	5.38	5.44
.007	2.49	3.89	4.41	4.81	5.08	5.18	5.23	5.28
.008	2.36	3.76	4.28	4.68	4.95	5.05	5.10	5.15
.009	2.26	3.65	4.16	4.56	4.83	4.93	4.98	5.03
.01	2.16	3.54	4.06	4.46	4.72	4.82	4.88	4.93
.02	1.56	2.89	3.38	3.77	4.04	4.14	4.18	4.24
.03	1.24	2.49	2.99	3.38	3.64	3.74	3.79	3.84
.04	1.04	2.24	2.71	3.10	3.36	3.46	3.50	3.56
.05	.894	2.03	2.50	2.89	3.14	3.24	3.29	3.34
.06	.785	1.87	2.34	2.71	2.97	3.07	3.11	3.16
.07	.700	1.74	2.20	2.57	2.82	2.92	2.97	3.02
.08	.630	1.63	2.08	2.44	2.70	2.79	2.84	2.89
.09	.573	1.53	1.97	2.34	2.58	2.68	2.73	2.78
.1	.527	1.45	1.88	2.24	2.49	2.58	2.63	2.68
.2	.273	.938	1.31	1.63	1.86	1.95	2.03	2.04
.3								
.4	.142	.553	.830	1.09	1.29	1.37	1.43	1.45
.5								
.6	.0957	.389	.606	.827	.813	1.06	1.11	1.13
.7								
.8	.0722	.298	.474	.662	.809	.870	.913	.936
.9								
1.0	.0579	.252	.388	.551	.381	.736	.773	.794

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Table 2 (Continued)

Table of $\frac{2}{(\Delta x)^2} I_1$

R_o / Δx_o	.05	.25	.5	1.0	2.0	3.0	4.0	6.0
0								
.001	1.69	2.40	2.68	2.88	3.00	3.06	3.08	3.12
.002	1.58	2.30	2.56	2.76	2.90	2.94	2.98	3.00
.003	1.37	2.09	2.35	2.55	2.68	2.73	2.76	2.79
.004	1.26	1.97	2.23	2.43	2.56	2.61	2.64	2.67
.005	1.15	1.85	2.11	2.31	2.45	2.50	2.52	2.55
.006	1.07	1.77	2.03	2.23	2.36	2.41	2.44	2.46
.007	1.00	1.69	1.95	2.15	2.28	2.34	2.36	2.39
.008	.944	1.63	1.89	2.09	2.22	2.27	2.30	2.32
.009	.889	1.57	1.83	2.03	2.16	2.21	2.24	2.27
.01	.884	1.53	1.78	1.98	2.11	2.16	2.19	2.23
.02	.555	1.18	1.43	1.63	1.76	1.81	1.83	1.86
.03	.416	1.01	1.25	1.44	1.57	1.62	1.65	1.67
.04	.325	.876	1.11	1.30	1.43	1.48	1.50	1.53
.05	.266	.783	1.01	1.20	1.33	1.38	1.40	1.43
.06	.220	.705	.931	1.11	1.24	1.29	1.31	1.34
.07	.187	.644	.864	1.05	1.17	1.22	1.24	1.26
.08	.159	.590	.805	.984	1.11	1.15	1.18	1.20
.09	.139	.546	.756	.932	1.05	1.10	1.12	1.15
.1	.122	.506	.711	.885	1.00	1.05	1.07	1.10
.2	.0479	.281	.440	.587	.692	.734	.755	.776
.3								
.4	.0159	.129	.235	.347	.431	.468	.485	.504
.5								
.6	.00789	.0738	.147	.234	.303	.333	.349	.364
.7								
.8	.00472	.0475	.100	.168	.226	.250	.264	.277
.9								
1.0	.00312	.0328	.0721	.126	.175	.195	.207	.219

Table 3 - $\bar{K}_o$

Table of $\frac{2}{\Delta x} I_o - \frac{2}{(\Delta x)^2} I_1 = \bar{K}_o$

R_o / ΔX_o	.05	.25	.5	1.0	2.0	3.0	4.0	6.0
0								
.001	2.69	3.43	3.66	3.87	4.02	4.06	4.10	4.11
.002	2.11	2.84	3.10	3.30	3.43	3.49	3.50	3.53
.003	1.93	2.64	2.90	3.10	3.24	3.29	3.32	3.34
.004	1.76	2.48	2.73	2.94	3.08	3.13	3.15	3.17
.005	1.66	2.38	2.63	2.84	2.96	3.01	3.05	3.07
.006	1.56	2.28	2.53	2.73	2.87	2.92	2.94	2.98
.007	1.49	2.20	2.46	2.66	2.80	2.84	2.87	2.89
.008	1.42	2.13	2.39	2.59	2.73	2.78	2.80	2.83
.009	1.37	2.08	2.33	2.53	2.67	2.72	2.74	2.76
.01	1.28	2.01	2.28	2.48	2.61	2.66	2.69	2.70
.02	1.00	1.71	1.95	2.14	2.28	2.33	2.35	2.38
.03	.824	1.48	1.74	1.94	2.07	2.12	2.14	2.17
.04	.715	1.36	1.60	1.80	1.93	1.98	2.00	2.03
.05	.628	1.25	1.49	1.69	1.81	1.86	1.89	1.91
.06	.525	1.16	1.41	1.60	1.73	1.78	1.80	1.82
.07	.513	1.10	1.34	1.52	1.65	1.70	1.73	1.76
.08	.471	1.04	1.28	1.46	1.59	1.64	1.66	1.69
.09	.434	.984	1.21	1.41	1.53	1.58	1.61	1.63
.1	.405	.944	1.17	1.36	1.49	1.53	1.56	1.58
.2	.225	.657	.870	1.04	1.17	1.22	1.28	1.26
.3								
.4	.126	.424	.595	.743	.859	.902	.945	.946
.5								
.6	.0878	.315	.459	.593	.510	.727	.761	.766
.7								
.8	.0675	.250	.374	.494	.583	.620	.649	.658
.9								
1.0	.0548	.219	.316	.425	.506	.541	.566	.575

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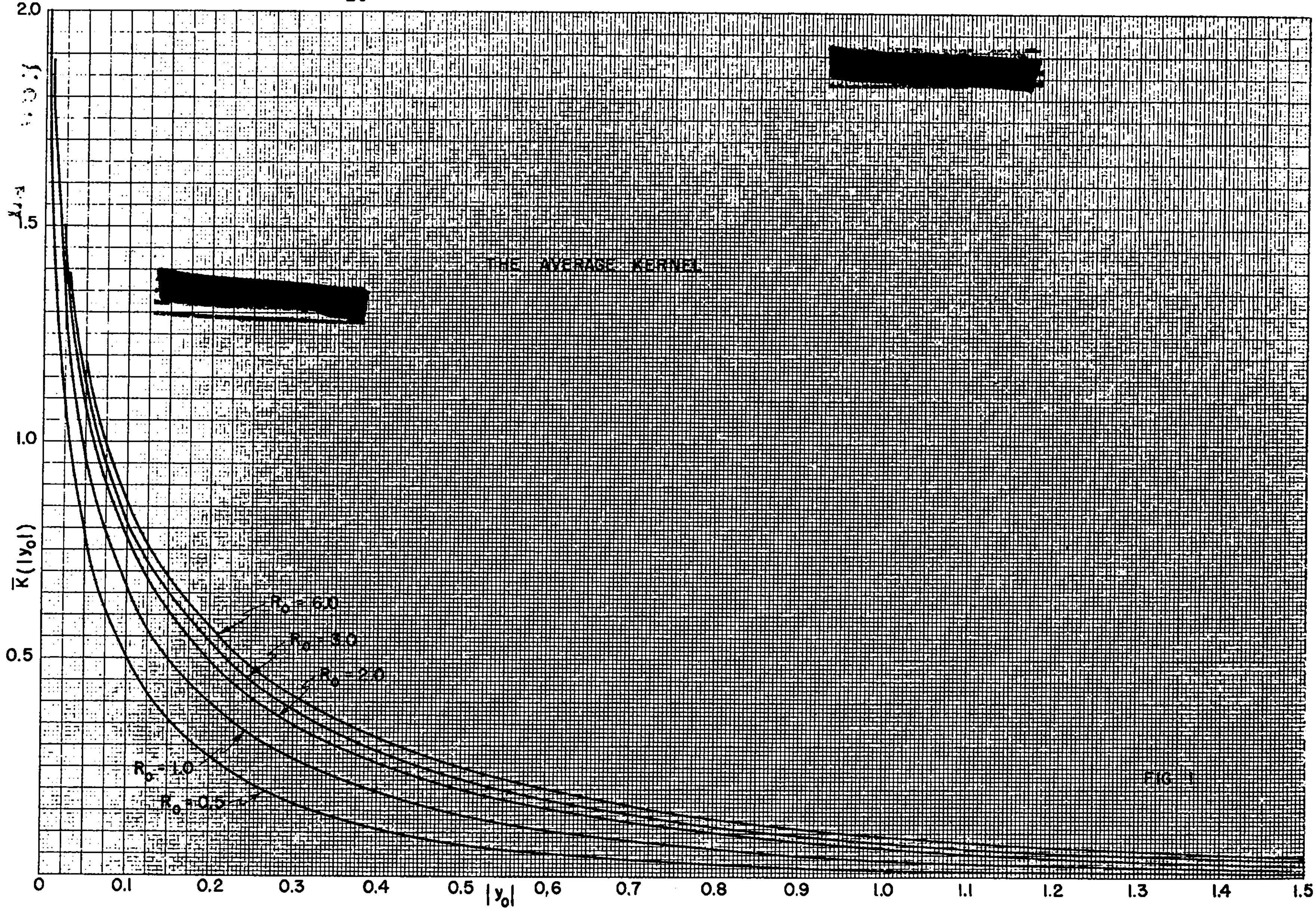
[redacted]

Table 4

Table of Moments of $\overline{K}$

R/λ	$\overline{M}_o/\lambda$	$\overline{M}_1/\lambda^2$	$\overline{M}_2/\lambda^3$
.05	.0506	.0017	.0005
.25	.1235	.0199	.0115
.5	.2021	.0477	.0351
.75	.2563	.0736	.0620
1.0	.2965	.0954	.0883
1.5	.3491	.130	.134
2.0	.3815	.153	.168
2.5	.4089	.170	.193
3.0	.4190	.182	.212
4.0	.4383	.198	.242
5.0	.4502	.208	.261
6.0	.4585	.213	.272

[redacted]



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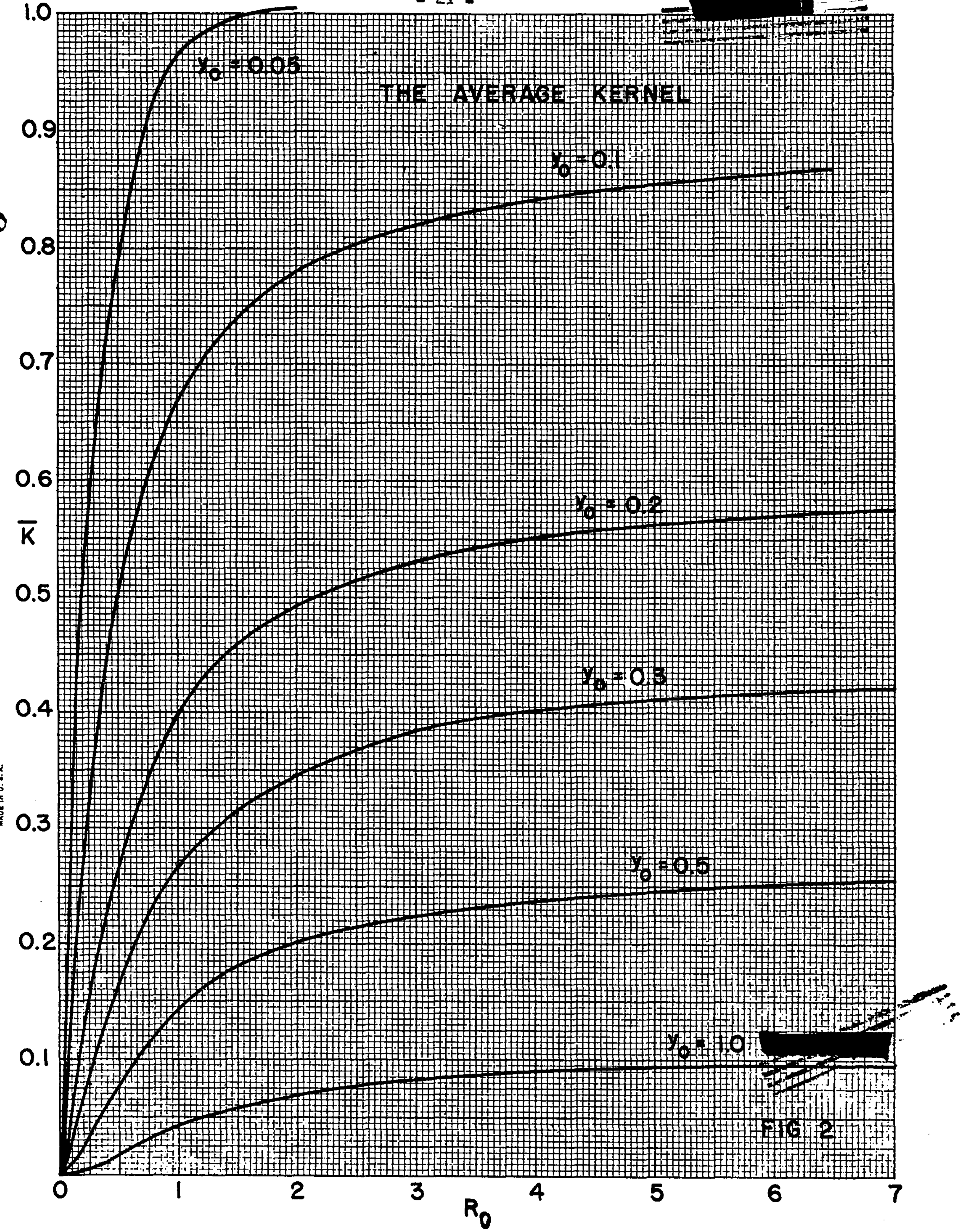
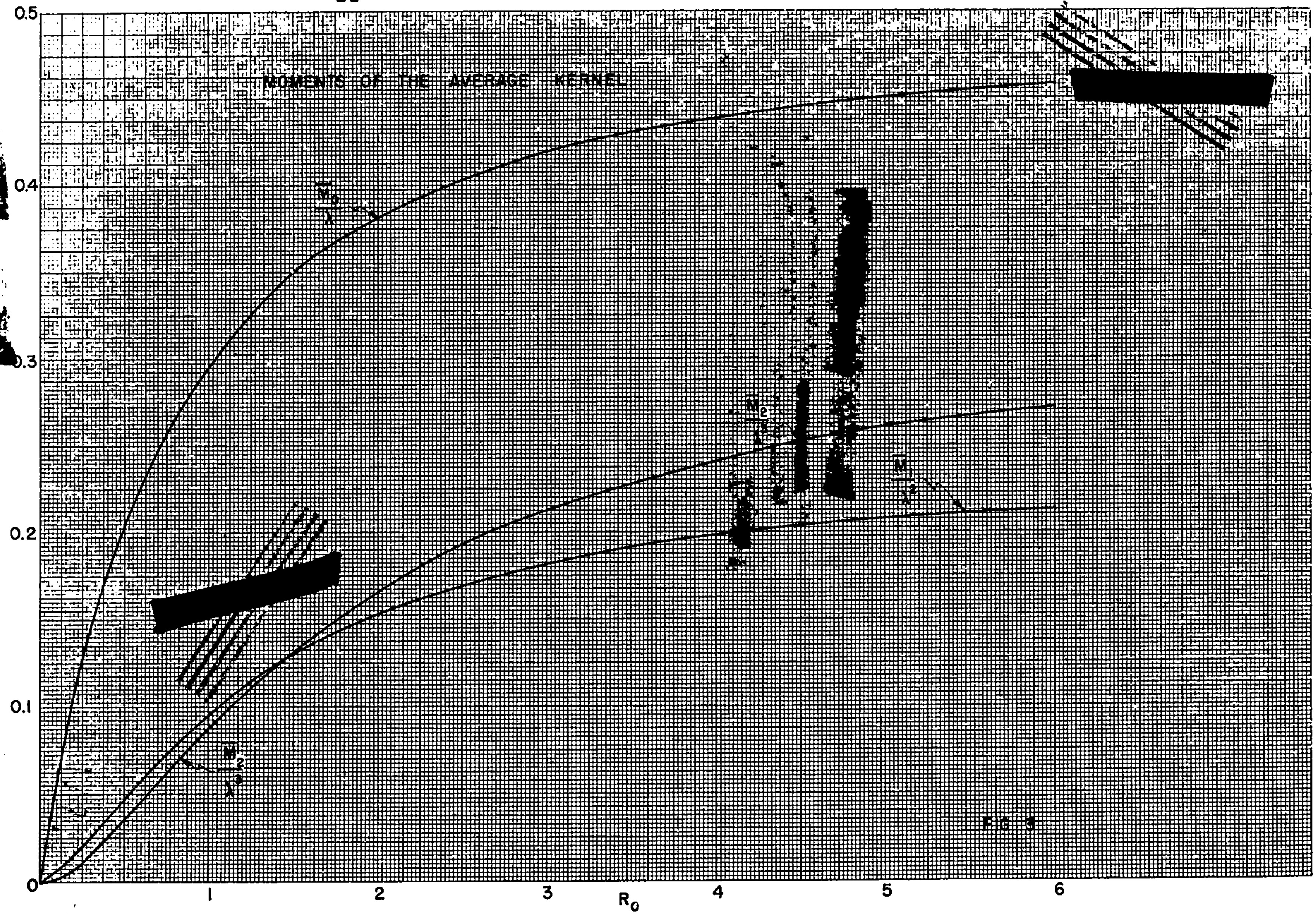

[redacted]
THE AVERAGE KERNEL
$y_0 = 0.05$
$y_0 = 0.1$
$y_0 = 0.2$
$y_0 = 0.3$
$y_0 = 0.5$
$y_0 = 1.0$
[redacted]
$\bar{K}$
R_0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7

FIG 2



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