[redacted]

LOS ALAMOS SCIENTIFIC LABORATORY

of the

UNIVERSITY OF CALIFORNIA

Report Written:
November 1, 1951

LA-1321

This document consists of 22 pages

[redacted]

TAYLOR INSTABILITY ON CYLINDERS AND SPHERES IN THE SMALL AMPLITUDE APPROXIMATION

Work Done By:

George I. Bell

Report Written By:

George I. Bell

Phys. + Math

[redacted]

~~Principles & Testing~~

[redacted]

Distributed: DEC 13 1951 LA-1321

Washington Document Room	1-7
J R. Oppenheimer	8
Los Alamos Report Library	9-30

[redacted]

[redacted]

ABSTRACT

We consider the growth of a small ripple on a cylindrical or spherical fluid surface which is subject to arbitrary radial motion. Differential equations for the amplitudes of the ripples as functions of time are derived under the assumption that the motion is irrotational and of amplitude small compared to the wavelength of the ripple. The liquid may be compressible. Two further assumptions prove convenient though not necessary, namely: that the wavelength of the disturbance is small compared to the thickness of the cylindrical or spherical shell, and that the densities of the fluids bordering the shell are negligibly small.

[redacted]

APPROVED FOR PUBLIC RELEASE

We derive the equations in a very simple manner. Since the motion is irrotational, there exists a velocity potential, ϕ, which satisfies Laplaces' equation, $\nabla^2\phi = 0$, if the matter is incompressible, or, for compressible material, ϕ satisfies a Poisson type equation, $\nabla^2\phi = -\dot{\rho}/\rho$. [Note: throughout we take $\nabla\phi = \vec{v}$ and not the more customary $-\nabla\phi = \vec{v}$.] We then invoke boundary conditions at the surfaces of the shell to determine ϕ. From the equations of motion

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial r}\right)^2 = \text{constant}, \tag{1}$$

the differential equations for the development of the ripple emerge directly.

A note on the large amplitude case is appended. Modifications for thin shells are also noted.

I. CYLINDRICAL SURFACE, INCOMPRESSIBLE FLUID

In cylindrical coordinates (r, θ, z) the equation $\nabla^2\phi = 0$ has the form

$$\frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2} = 0. \tag{2}$$

If ϕ is independent of z, θ:

$$\phi = A_1 \ln r + B_1. \tag{3}$$

[redacted]

[redacted]

APPROVED FOR PUBLIC RELEASE

If ϕ is independent of z:

$$\phi = \left(\frac{A_2}{r^k} + B_2 r^k\right) \cos k\,\theta. \qquad (4)$$

(We ignore the $\sin k\,\theta$ solution as adding nothing new.) If ϕ is independent of θ:

$$\phi = \left(A_3 J_0(ikr) + B_3\, H_0^{(1)}(ikr)\right) \cos k\, z. \qquad (5)$$

In general we may have

$$\phi = \cos k\,\theta \cos \ell\,\theta Z_\ell\,(ikr),$$

with Z_ℓ any Bessel function of order ℓ; but as computation with arbitrary ℓ would be tedious, we have taken our ripple independent of θ or z always. Presumably ripples independent of θ are of greatest physical interest.

a) Ripple on Inside Surface Independent of z

Let the position of the unrippled surface be given by $R_o(t)$. This corresponds to what comes out of a SEAC calculation. Let the actual rippled surface be given at any time by $R_o(t) + b(t) \cos\ k\ \theta$. Our problem is to find $b(t)$ in terms of R_o, $\dot{R}_o$, $\ddot{R}_o$, and k.

We demand that $\nabla^2\phi = 0$ and that

$$\left(\frac{\partial \phi}{\partial r}\right)_{R_o + b(t)\cos k\,\theta} = \dot{R}_o + \dot{b}(t) \cos k\,\theta. \qquad (6)$$

For the other boundary condition, we should presumably consider the outer surface, $R_1(t)$, and demand that $\partial\phi/\partial r)_{R_1} = \dot{R}_1$. However, if $(R_o/R_1)^{2k}$ is small compared to one, this amounts to setting $B_2 = 0$ in equation (4). Thus we merely demand that the disturbance due to the ripple decrease at large r. These three conditions are satisfied by

$$\phi = R_o \dot{R}_o \ln r - \left(\dot{b}(t) + b(t) \frac{\dot{R}_o}{R_o}\right) \frac{R_o^{k+1}}{k\, r^k} \cos k\theta. \tag{7}$$

Since this is a small amplitude approximation, we always ignore squares of $b(t)$. We see that

$$\frac{\partial \phi}{\partial r} = \frac{R_o}{r} \dot{R}_o + \left(\dot{b}(t) + b(t) \frac{\dot{R}_o}{R_o}\right) \frac{R_o^{k+1}}{r^{k+1}} \cos k\theta$$

so that

$$\left(\frac{\partial \phi}{\partial r}\right)_{r=R_o+b(t)\cos k\theta} = \frac{\dot{R}_o}{1+\frac{b(t)\cos k\theta}{R_o}} + \dot{b}(t) \cos k\theta + b(t) \frac{\dot{R}_o}{R_o} \cos k\theta$$

$$= \dot{R}_o + \dot{b}(t) \cos k\theta.$$

Thus we see that ϕ satisfies the requirements. To get the equations of motion

$$\frac{\partial \phi}{\partial t} = \ln r \left[\dot{R}_o^2 + R_o \ddot{R}_o\right] - \ddot{b}(t) \frac{R_o^{k+1}}{k\, r^k} \cos k\theta$$

$$- \dot{b}(t) \dot{R}_o \frac{R_o^k}{k\, r^k} \cos k\theta - \frac{k+1}{k} \dot{b}(t) \frac{R_o^k \dot{R}_o}{r^k} \cos k\theta$$

$$- b(t) \left[\frac{\ddot{R}_o R_o^k + k \dot{R}_o^2 R_o^{k-1}}{k\, r^k}\right] \cos k\theta.$$

The equations of motion have the form

$$-\frac{p}{\rho} = \frac{\partial \phi}{\partial t} + \frac{1}{2} \left(\frac{\partial \phi}{\partial r}\right)^2 + F(t).$$

At the interface p is taken to be a constant, independent of θ. Equating the terms independent of θ at $r = R_o + b(t) \cos k\theta$, we have merely a first integral of the equation $-\frac{dp}{dr} = \rho \ddot{r}$. However, equating the terms proportional to $\cos k\theta$ and using

$$\ln \left(R_o + b(t) \cos k\theta\right) = \ln R_o + \frac{b(t) \cos k\theta}{R_o},$$

we have:

$$0 = \frac{k\, b(t)}{R_o}\left[\dot{R}_o^2 + R_o \ddot{R}_o\right] - \ddot{b}(t)\, R_o - \dot{b}(t)\, \dot{R}_o - (k+1)\, \dot{b}\, \dot{R}_o$$

$$- b(t)\, \ddot{R}_o - k\, b(t) \frac{\dot{R}_o^2}{R_o} + k\, \dot{b}(t)\, \dot{R}_o\,,$$

whence

$$(k-1)\, b(t)\, \ddot{R}_o - \ddot{b}(t)\, R_o - 2\, \dot{b}(t)\, \dot{R}_o = 0 \qquad (8)$$

This is the differential equation for $b(t)$. It is easily interpreted in simple cases. If the wavelength of the disturbance is small compared to the radius of curvature, R_o, one expects the solution for a plane to emerge. The wavelength, $\lambda = R_o/k$, and for $R_o \gg \lambda$ we may write our equation as

$$\frac{1}{\lambda}\, b(t)\, \ddot{R}_o - \ddot{b}(t) - 2\, \dot{b}(t) \frac{\dot{R}_o}{R_o} = 0.$$

The last term will usually be unimportant for large R_o and the plane solution

$$b(t) = e^{\pm\sqrt{\ddot{R}_o/\lambda}\; t}$$

is, indeed, formally valid.

The last term in (8) is peculiar to the cylindrical motion. Its influence is easily seen for $\ddot{R}_o = 0$. With a ripple on an unaccelerated plane, one has $b(t) = b_o + b_1 t$. On the cylinder the solution of (8) is $b(t) = \frac{b_o'}{R_o} + b_1'$. We note that the mass of the material displaced in the ripple is proportional to $b\, R_o$ so that the displaced mass behaves as $b_o + b_1 t$, just as in the plane case. This suggests that the equations in Lagrangean coordinates might be simpler.

Equation (8) takes on a much simpler form when the substitution $b(t) = c(t)/R_o$ is carried through:

$$\boxed{k\, c(t)\, \ddot{R}_o - \ddot{c}(t)\, R_o = 0} \tag{8a}$$

$$b(t) = c(t)/R_o.$$

We see that the growth of the amplitude $c(t)$ is just that for a plane ripple of variable wavelength.

If one wished to derive the differential equation for a ripple amplitude when the densities of the media or either side of the interface are comparable, then one would have to set down the velocity potentials in both media. One would infer from Taylor's treatment that this would lead to something like multiplying the first term in (8a) by $\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$ where ρ_1 and ρ_2 are the densities.

<u>b) Ripple on Outside Surface Independent of z</u>

The method goes exactly as before. Now

$$\phi = R_o \dot{R}_o \ln r + \frac{1}{k}\left(\dot{b}(t) + b(t)\frac{\dot{R}_o}{R_o}\right)\frac{r^k}{R_o^{k-1}} \cos k\theta \tag{9}$$

[redacted]

APPROVED FOR PUBLIC RELEASE

leads to

$$b(t)\,(k+1)\,\ddot{R}_o + \ddot{b}(t)\,R_o + 2\,\dot{b}(t)\,\dot{R}_o = 0. \tag{10}$$

If we substitute $b(t) = c(t)/R_o$ we find the convenient equation

$$\boxed{k\,c(t)\,\ddot{R}_o + \ddot{c}(t)\,R_o = 0} \tag{10a}$$

and the interpretation is just as before.

c) Inside Ripple Independent of θ

$$\phi = R_o\,\dot{R}_o \ln r + \frac{i}{k}\left(\dot{b}(t) + b(t)\,\frac{\dot{R}_o}{R_o}\right)\cos kz\,\frac{H_0^{(1)}(i\,k\,r)}{H_1^{(1)}(i\,k\,R_o)}. \tag{11}$$

Because of the involvement of the Bessel functions, the resulting differential equation is more complicated:

$$\begin{aligned} & k\,\dot{b}(t)\,\dot{R}_o + k\,b(t)\,\frac{\dot{R}_o^2}{R_o} + b(t)\,k\,\ddot{R}_o \\ & + i\left\{\ddot{b}(t) + 2\,\dot{b}(t)\,\frac{\dot{R}_o}{R_o} + b(t)\,\frac{\ddot{R}_o}{R_o}\right\}\frac{H_0^{(1)}(i\,k\,R_o)}{H_1^{(1)}(i\,k\,R_o)} \\ & + \left\{k\,\dot{R}_o\,\dot{b}(t) + k\,b(t)\,\frac{\dot{R}_o^2}{R_o}\right\}\cdot\left[\frac{H_0^{(1)}(i\,k\,R_o)}{H_1^{(1)}(i\,k\,R_o)}\right]^2 = 0. \end{aligned} \tag{12}$$

In many practical cases one will have $k\,R_o \gg 1$, in which case we have

$$\frac{H_0^{(1)}(i\,k\,R_o)}{H_1^{(1)}(i\,k\,R_o)} = i - \frac{i}{2k\,R_o} + \frac{3\,i}{8(k\,R)^2} - - - -$$

$$\left(\frac{H_0^{(1)}(i\,k\,R_o)}{H_1^{(1)}(i\,k\,R_o)}\right)^2 = -1 + \frac{1}{k\,R} - \frac{1}{k^2\,R^2} + - - - .$$

Using these expansions, we find to lowest order in $1/k\,R_o$:

$$-\ddot{b}(t) + b(t)\, k\, \ddot{R}_o - \dot{b}(t)\, \frac{\dot{R}_o}{R_o} + b(t)\, \frac{\dot{R}_o^2}{R_o^2} = 0. \tag{13}$$

In the plane approximation, $\frac{\dot{R}_o}{R} = 0$, and the proper behavior, $b(t) \sim e^{\sqrt{k\, \ddot{R}_o}\; t}$, prevails. When $\ddot{R}_o = 0$, $b(t) = b_o'/R_o + b_1'\, R_o$ which allows $R_o\, b(t)$ to behave as $b_o + b_1 t + b_2 t^2$. Just why the quadratic time behavior arises is not at present understood. Here the substitution $c(t) = R_o b(t)$ leads to

$$\boxed{c(t)\, k\, \ddot{R}_o - \ddot{c}(t) + \dot{c}(t)\, \frac{\dot{R}_o}{R_o} = 0} \tag{13a}$$

The solution to this for $\ddot{R}_o = 0$ can also be written $c(t) = c_o + c_1 \int_o^t R_o(s)\, ds$.

To the next order in $1/k\, R_o$, we have

$$-\ddot{b}(t)\left(1 - \frac{1}{2\, k\, R}\right) + b(t)\, k\, \ddot{R}_o \left(1 - \frac{1}{k\, R_o}\right) - \dot{b}(t)\, \frac{\dot{R}_o}{R_o} + b(t)\, \frac{\dot{R}_o^2}{R_o^2} \left[1 - \frac{1}{k\, R_o}\right] = 0\,. \tag{14}$$

d) Outside Ripple Independent of θ

Here the velocity potential

$$\phi = R_o\, \dot{R}_o \ln r + \frac{1}{k} \left(b(t) + b(t)\, \frac{\dot{R}_o}{R_o}\right) \cos k\, z\; \frac{J_o(i\, k\, r)}{J_1(i\, k\, R_o)} \tag{15}$$

yields the differential equation:

$$k \dot{R}_o \dot{b}(t) + k\, b(t) \frac{\dot{R}_o^2}{R_o} + b(t)\, k \ddot{R}_o$$
$$+ i \left(\ddot{b}(t) + 2 \dot{b} \frac{\dot{R}_o}{R_o} + b(t) \frac{\ddot{R}_o}{R_o}\right) \frac{J_o(i\, k\, R_o)}{J_1(i\, k\, R_o)} \tag{16}$$
$$+ \left(k \dot{R}_o \dot{b}(t) + k\, b(t) \frac{\dot{R}_o^2}{R_o}\right)\left(\frac{J_o(i\, k\, R_o)}{J_1(i\, k\, R_o)}\right)^2 = 0\,.$$

The appropriate expansions for large k R are:

$$i\, \frac{J_o\,(i\, k\, R_o)}{J_1\,(i\, k\, R_o)} = 1 + \frac{1}{2k\, R_o} + \frac{3}{8(k\, R_o)^2} + \ldots$$

$$\left(\frac{J_o(i\, k\, R_o)}{J_1(i\, k\, R_o)}\right)^2 = -\left(1 + \frac{1}{k\, R} + \frac{1}{k^2\, R^2} + \ldots\right),$$

and we have the lowest order in $1/kR$

$$b(t)\, k \ddot{R}_o + \ddot{b}(t) + \dot{b}(t) \frac{\dot{R}_o}{R_o} - b(t) \frac{\dot{R}_o^2}{R_o^2} = 0. \tag{17}$$

$$\boxed{c(t)\, k \ddot{R}_o + \ddot{c}(t) - \dot{c}(t) \frac{\dot{R}_o}{R_o} = 0} \tag{17a}$$

To next order we have

$$b(t)\, k \ddot{R}_o \left[1 + \frac{1}{k\, R_o}\right] + \ddot{b}(t) \left[1 + \frac{1}{2k\, R_o}\right] + \dot{b}(t) \frac{\dot{R}_o}{R_o}$$
$$- b(t) \frac{\dot{R}_o^2}{R_o^2} \left[1 + \frac{1}{k\, R_o}\right] = 0. \tag{18}$$

II. CYLINDRICAL SURFACE, COMPRESSIBLE FLUID

Now the velocity potential no longer satisfies $\nabla^2 \phi = 0$, but rather $\nabla^2 \phi = - \dot{\rho}/\rho$. Let us first consider the simple case $-\dot{\rho}/\rho = F(t)$, i. e., independent of position within the fluid. A velocity potential of the form

$$\phi = \ln r \left[R_o \dot{R}_o - \frac{R_o^2}{2} F(t) \right] + \frac{r^2}{4} F(t)$$

satisfies $\nabla^2 \phi = F(t)$. The radial velocity is then

$$V_r = \partial \phi / \partial r = \frac{R \dot{R}_o}{r} - \left(\frac{R_o^2 - r^2}{2r} \right) F(t).$$

To see that this is the correct form let $\dot{R}_o = 0$, in which case

$$V_r = \dot{\rho}/\rho \cdot \frac{R_o^2 - r^2}{2r} .$$

But $2\pi r \rho V_r$ is the flux per unit length through a cylindrical surface at r. Also $\pi \dot{\rho} (R_o^2 - r^2)$ is the rate of change of mass per unit length inside a cylindrical volume of radii R_o and r. Hence V_r is indeed the correct value to insure continuity.

One can now devise a velocity potential ϕ which satisfies 1) $\nabla^2 \phi = F(t)$, 2) $\left.\frac{\partial \phi}{\partial r}\right)_{R_o b(t) \cos k_z^\theta} = \dot{R}_o + \dot{b}(t) \cos k_z^\theta$, and 3) has a disturbance which for an inside ripple decreases for increasing r or for an outside ripple decreases for decreasing r.

One cannot, under the present theory, take into account variations of density with r inside the liquid. There is no difficulty in finding solutions of $\nabla^2 \phi = \sum_{n=0}^{\infty} r_n F_n (t) = F(r,t)$ and incorporating these solutions in the equations of motion to find the differential equation for b. The difficulty arises from the fact that in a small amplitude theory, only the value of F and $\dot{F}$ near the interface $(r = R_o)$ can affect the solution. Physically one would expect that an initial ripple of the form $\cos k\,\theta$ in material of uniform density could not maintain this form if one portion of the material were compressed. Since our formalism allows only ripples of form $\cos k\,\theta$, it must be unaffected by variations of ρ with r at R_o. All this emerges if one carries through the theory with $F(r, t)$. Perhaps by taking a more general form of F, involving θ, one could improve this.

a) Inside Ripple Independent of z

Our three conditions are satisfied by

$$\phi = \ln r \left[R_o \dot{R}_o - \frac{R_o^2}{2} F(t) \right] + \frac{r^2}{4} F(t) + \tag{19}$$
$$+ \frac{1}{k} \left[\dot{b}(t) + b(t) \frac{\dot{R}_o}{R_o} - b(t)\, F(t) \right] \cos k\,\theta \; \frac{R_o^{k+1}}{r^k} .$$

The differential equation for $b(t)$ becomes

$$(k-1)\, b(t)\, \ddot{R}_o - \ddot{b}(t)\, R_o - 2\, \dot{b}(t)\, \dot{R}_o + \frac{\partial}{\partial t} \big(b\, R_o\, F(t) \big) = 0. \tag{20}$$

The substitution $b(t) = c(t)/\rho R_o$ leads to the simpler equation

$$\boxed{c(t)\, \frac{k}{R_o}\, \ddot{R}_o - \ddot{c}(t) + \dot{c}(t)\, \frac{\dot{\rho}}{\rho} = 0.} \tag{20a}$$

Here, of course, we have used $F = -\dot{\rho}/\rho$. If $\ddot{R}_o = 0$, the solution is $c(t) = c_o + c_1 \int_o^t \rho(s)\, ds$.

[redacted]

b) Outside Ripple Independent of z

$$\phi = \ln r \left[R_o \dot{R}_o - \frac{R_o^2}{2} F(t) \right] + \frac{r^2}{4} F(t)$$

$$+ \frac{1}{k} \left[\dot{b}(t) + b(t) \frac{\dot{R}_o}{R_o} - b(t) \; F(t) \right] \frac{r^k}{R_o^{k-1}} \cos k\theta \tag{21}$$

leads to the equation

$$b(t)\,(k+1)\,\ddot{R}_o + \ddot{b}(t)\,R_o + 2\,\dot{b}(t)\,\dot{R}_o - \frac{\partial}{\partial t}(F\,b\,R_o) = 0. \tag{22}$$

In terms of $c(t) = \rho(t)\,R_o(t)\,b(t)$, this becomes

$$\boxed{c(t)\,\frac{k}{R_o}\,\ddot{R}_o + \ddot{c}(t) - \dot{c}(t)\,\frac{\dot{\rho}}{\rho} = 0.} \tag{22a}$$

c) Inside Ripple Independent of θ

$$\phi = \ln r \left[R_o \dot{R}_o - \frac{R_o^2}{2} F(t) \right] + \frac{r^2}{4} F(t)$$

$$+ \frac{1}{k} \left[\dot{b}(t) + b(t) \frac{\dot{R}_o}{R_o} - b(t) \; F(t) \right] \cos kz \; \frac{H_o^{(1)}(i\,k\,r)}{H_1^{(1)}(i\,k\,R_o)} \tag{23}$$

leads to the equation:

$$k\,\dot{b}(t)\,\dot{R}_o + k\,b(t)\,\frac{\dot{R}_o^2}{R_o} + b(t)\,k\,\ddot{R}_o - k\,b(t)\,\dot{R}_o\,F$$

$$+ i \left\{ \ddot{b}(t) + 2\dot{b}(t)\,\frac{\dot{R}_o}{R_o} + b(t)\,\frac{\ddot{R}_o}{R_o} - \dot{b}\,F - \dot{F}\,b - b\,F\,\frac{\dot{R}_o}{R_o} \right\} \frac{H_o^{(1)}(i\,k\,R_o)}{H_1^{(1)}(i\,k\,R_o)} \tag{24}$$

$$+ \left[\frac{H_o^{(1)}(i\,k\,R_o)}{H_1^{(1)}(i\,k\,R_o)} \right]^2 \left[k\,\dot{R}_o\,\dot{b} + k\,b\,\frac{\dot{R}_o^2}{R_o} - k\,b(t)\,\dot{R}_o\,F \right] = 0.$$

[redacted]

For $k R >> 1$, we use the asymptotic expansions on page 9 and find:

$$-\ddot{b}(t) + b(t)\, k\, \ddot{R}_o - \dot{b}(t)\, \frac{\dot{R}_o}{R_o} + b(t)\, \frac{\dot{R}_o^2}{R_o^2} + \frac{\partial}{\partial t}\, b(t)\, F(t) = 0. \tag{25}$$

The substituting $b(t) = c(t)/\rho(t)\, R_o(t)$ gives

$$\boxed{c(t) k\, \ddot{R}_o - \ddot{c}(t) + \dot{c}\left(\frac{\dot{\rho}}{\rho} + \frac{\dot{R}_o}{R_o}\right) = 0,} \tag{25a}$$

so that in this case also the compressibility leads to a term $\dot{c}\, \dot{\rho}/\rho$. The solution for $\ddot{R}_o = 0$ is now $c(t) = c_o + c_1 \int_o^t \rho(s)\, R_o(s)\, ds$.

d) Outside Ripple Independent of θ

$$\phi = \ln r \left[R_o \dot{R}_o - \frac{R_o^2}{2} F(t)\right] + \frac{r^2}{4} F(t) + \frac{i}{k}\left[\dot{b}(t) + b(t)\, \frac{\dot{R}_o}{R_o} - b(t)\, F(t)\right] \cos k z\, \frac{J_o(i k r)}{J_1(i k R_o)} \tag{26}$$

leads to

$$k\, \dot{b}(t)\, \dot{R}_o + k\, b(t)\, \frac{\dot{R}_o^2}{R_o} + b(t)\, k\, \ddot{R}_o - k\, b(t)\, \dot{R}_o\, F + i \left\{ \ddot{b}(t) + 2\, \dot{b}(t)\, \frac{\dot{R}_o}{R_o} + b(t)\, \frac{\ddot{R}_o}{R_o} - \dot{b}(t)\, F - b(t)\, \dot{F} - b(t)\, F\, \frac{\dot{R}_o}{R_o} \right\}$$

$$\frac{J_o(i k R_o)}{J_1(i k R_o)} + \left[\frac{J_o(i k R_o)}{J_1(i k R_o)}\right]^2 \left[k\, \dot{b}(t)\, \dot{R}_o + k\, b(t)\, \frac{\dot{R}_o^2}{R_o} - k\, b(t)\, \dot{R}_o\, F\right] = 0. \tag{27}$$

For $k R_o >> 1$ we use the asymptotic expansion on page 11 to find:

$$\ddot{b}(t) + b(t)\, k\, \ddot{R}_o + \dot{b}(t)\, \frac{\dot{R}_o}{R_o} - b(t)\, \frac{\dot{R}_o^2}{R_o^2} - \dot{b}(t)\, F - b(t)\, \dot{F}(t) = 0. \tag{28}$$

In this case the substitution $b(t) = c(t)/\rho(t)\, R_o(t)$, leads to

$$\boxed{c(t)\, k\, \ddot{R}_o + \ddot{c}(t) - \dot{c}\left(\frac{\dot{\rho}}{\rho} + \frac{\dot{R}_o}{R_o}\right) = 0.}$$

Again for $\ddot{R}_o = 0$, $c(t) = c_o + c_1 \int_o^t \rho(s)\, R_o(s)\, ds$.

III. SPHERICAL SURFACES

We wish solutions of $\nabla^2 \phi = 0$ in spherical coordinates. To avoid confusion we call the azimuthal angle ψ, instead of the customary ϕ which we reserve for the velocity potential. If ϕ is independent of θ, ψ:

$$\phi(r) = A_1 \frac{1}{r} + B \,. \tag{29}$$

If ϕ is independent of ψ:

$$\phi = (Ar^{\ell} + B\, r^{-\ell-1})\, P_{\ell}(\cos\theta). \tag{30}$$

For general ϕ,

$$\phi = (A\, r^{\ell} + B\, r^{-\ell-1})\, P_{\ell}^{m}(\cos\theta) \cos m\,\psi.$$

In developing our formulae we shall take $m = 0$ for it seems likely that any sphere will have an axis of symmetry. Also the radial dependence of ϕ is independent of m, so there is no real loss of generality as those results which hold for a ripple proportional to $P_{\ell}(\cos\theta)$ will hold as well for a ripple proportional to $P_{\ell}^{m}(\cos\theta) \cos m\,\psi$.

The equation $\nabla^2 \phi = F(t)$ has a particular solution

$$\phi = \frac{r^2}{6}\, F(t). \tag{31}$$

We are now in a position to proceed exactly as before.

a) Inside Ripple Incompressible

$$\phi = -\frac{R_o^2\,\dot R_o}{r} - \frac{1}{\ell+1}\left(\dot b(t) + \frac{2\dot R_o b}{R_o}\right)\frac{R_o^{\ell+2}}{r^{\ell+1}}\;P_\ell(\cos\theta). \qquad (32)$$

This function satisfies $\nabla^2\phi = 0$ and the boundary conditions, for

$$\left.\frac{\partial\phi}{\partial r}\right)_{r=R_o+b(t)P_\ell(\cos\theta)} = \dot R_o + \dot b(t)\,P_\ell(\cos\theta).$$

We obtain:

$$b(t)\,\ddot R_o(\ell-1) - \ddot b(t)\,R_o - 3\,\dot b(t)\,\dot R_o = 0. \qquad (33)$$

We try the substitution $b(t) = c(t)/R_o^2$ to obtain

$$\boxed{c(t)\,\ddot R_o\,\frac{\ell+1}{R_o} - \ddot c(t) + \dot c(t)\,\frac{\dot R_o}{R_o} = 0.} \qquad (33a)$$

The simplification is not marked, but the equation strongly resembles our previous ones suggesting that the factor R_o^{-2} in the substitution is correct for spheres. For $\ddot R_o = 0$, $c(t) = c_o + c_1 \int_o^t R_o(s)\,ds$.

b) Outside Ripple Incompressible

$$\phi = -\frac{R_o^2\dot R_o}{r} + \frac{1}{\ell}\left(\dot b(t) + \frac{2\dot R_o b(t)}{R_o}\right)\frac{r^\ell}{R_o^{\ell-1}}\,P_\ell(\cos\theta). \qquad (34)$$

This gives

$$b(t)\,\ddot R_o\,\frac{\ell+2}{R_o} + \ddot b(t) + 3\dot b\,\frac{\dot R_o}{R_o} = 0. \qquad (35)$$

In terms of $c(t) = R_o^2\,b(t)$, this is:

$$\boxed{c(t)\,\ddot R_o\,\frac{\ell}{R_o} + \ddot c(t) - \dot c(t)\,\frac{\dot R_o}{R_o} = 0.} \qquad (35a)$$

[redacted]

c) Inside Ripple Compressible

$$\phi = \frac{1}{r}\left[-R_o^2\,\dot{R}_o + \frac{R_o^3}{3}F(t)\right] + \frac{r^2}{6}F(t)$$
$$- \frac{1}{(\ell+1)}\left[\dot{b}(t) + \frac{2\dot{R}_o\,b(t)}{R_o} - b(t)\,F(t)\right]\frac{R_o^{\ell+2}}{r^{\ell+1}}\;)P_\ell(\cos\theta) \tag{36}$$

leads to

$$b(t)\,\ddot{R}_o(\ell-1) - \ddot{b}(t)\,R_o - 3\,\dot{b}(t)\,\dot{R}_o + \frac{\partial}{\partial t}\left(R_o(t)\,b(t)\,F(t)\right) = 0. \tag{37}$$

The substitution $b(t) = c(t)/R_o^2\rho$ produces

$$\boxed{c(t)\,\ddot{R}_o\,\frac{\ell+1}{R_o} - \ddot{c}(t) + \dot{c}(t)\left(\frac{\dot{R}_o}{R_o} + \frac{\dot{\rho}}{\rho}\right) = 0}$$

which is the form we by now expect.

d) Outside Ripple Compressible

$$\phi = \frac{1}{r}\left[-R_o^2\,\dot{R}_o + \frac{R_o^3}{3}F(t)\right] + \frac{r^2}{6}F(t)$$
$$+ \frac{1}{\ell}\left[\dot{b}(t) + \frac{2\dot{R}_o\,b(t)}{R_o} - b\,F(t)\right]\frac{r^\ell}{R_o^{\ell-1}}\,P_\ell(\cos\theta) \tag{38}$$

leads to

$$b(t)\,\dot{R}_o\,\frac{\ell+2}{R_o} + \ddot{b}(t) + 3\dot{b}(t)\,\frac{\dot{R}_o}{R_o} - \frac{1}{R_o}\,\frac{\partial}{\partial t}\left(R_o(t)\,b(t)\,F(t)\right) = 0. \tag{39}$$

When we substitute $b(t) = c(t)/R_o^2\rho$ we find

$$\boxed{c(t)\,\ddot{R}_o\,\frac{\ell}{R_o} + \ddot{c}(t) - \dot{c}(t)\left(\frac{\dot{R}_o}{R_o} + \frac{\dot{\rho}}{\rho}\right) = 0.} \tag{39a}$$

When the final formulae are collected as expressed in terms of the proper $c(t)$ s, great regularity is noted. In the following tables "In" denotes a ripple on an inside surface, "Out" denotes a ripple on an outside surface, z(or θ) means the ripple is independent of z(or θ), I denotes incompressible and C denotes compressible.

TABLE OF FORMULAE

CYLINDERS

In	z	I	$b = c/R_o$	$kc\frac{\ddot{R}_o}{R_o} - \ddot{c} = 0$		(8a)
Out	z	I	$b = c/R_o$	$kc\frac{\ddot{R}_o}{R_o} + \ddot{c} = 0$		(10a)
In	θ	I	$b = c/R_o$	$kc\ddot{R}_o - \ddot{c} + \dot{c}\frac{\dot{R}_o}{R_o} = 0$	$kR \gg 1$	(13a)
Out	θ	I	$b = c/R_o$	$kc\ddot{R}_o + \ddot{c} - \dot{c}\frac{\dot{R}_o}{R_o} = 0$		(17a)
In	z	C	$b = c/\rho R_o$	$kc\frac{\ddot{R}_o}{R_o} - \ddot{c} + \dot{c}\frac{\dot{\rho}}{\rho} = 0$		(20a)
Out	z	C	$b = c/\rho R_o$	$kc\frac{\ddot{R}_o}{R_o} + \ddot{c} - \dot{c}\frac{\dot{\rho}}{\rho} = 0$		(22a)
In	θ	C	$b = c/\rho R_o$	$kc\ddot{R}_o - \ddot{c} + \dot{c}\left(\frac{\dot{R}_o}{R_o} + \frac{\dot{\rho}}{\rho}\right) = 0$	$kR \gg 1$	(25a)
Out	θ	C	$b = c/\rho R_o$	$kc\ddot{R}_o + \ddot{c} - \dot{c}\left(\frac{\dot{R}_o}{R_o} + \frac{\dot{\rho}}{\rho}\right) = 0$		(28a)

[redacted]

TABLE OF FORMULAE

SPHERES

In	I	$b = c/R_o^2$	$(\ell+1)c \dfrac{\ddot{R}_o}{R_o} - \ddot{c} + \dot{c} \dfrac{\dot{R}_o}{R_o} = 0$	(33a)
Out	I	$b = c/R_o^2$	$\ell c \dfrac{\ddot{R}_o}{R_o} + \ddot{c} - \dot{c} \dfrac{\dot{R}_o}{R_o} = 0$	(35a)
In	C	$b = c/\rho R_o^2$	$(\ell+1)c \dfrac{\ddot{R}_o}{R_o} - \ddot{c} + \dot{c} \left(\dfrac{\dot{R}_o}{R_o} + \dfrac{\dot{\rho}}{\rho}\right) = 0$	(37a)
Out	C	$b = c/\rho R_o^2$	$\ell c \dfrac{\ddot{R}_o}{R_o} + \ddot{c} - \dot{c} \left(\dfrac{\dot{R}_o}{R_o} + \dfrac{\dot{\rho}}{\rho}\right) = 0$	(39a)

[redacted]

NOTE ON LARGE AMPLITUDE APPROXIMATION

Taylor's theory and experiments indicate that for ripples on a plane fluid surface subject to a constant acceleration of ~ 509, the amplitude grows exponentially with time until $b(t) \sim .4\lambda$ after which it approaches a constant velocity approximately given by $C_1\sqrt{a\lambda/2}$ where a is the acceleration and C_1 is a number slightly larger than unity. If one were to extrapolate this to cylindrical and spherical surfaces subject to much greater accelerations, one would probably assume that $c(t)$ increased according to the small amplitude theory up to $c(t) \sim .4\lambda$ after which time $c(t) \longrightarrow C_1\sqrt{a\lambda/2}$. It follows that at any given time that ripple is developing most rapidly which is just reaching the end of its exponential phase. As the acceleration, ripples of longer wavelength, will catch up with and devour those of smaller wavelength and roughly the most dangerous waves would be those which at that instant were reaching the end of their exponential or small amplitude growth.

NOTE ON THIN SHELLS

When the wavelength of the disturbance is of the same magnitude as the thickness of the shell, we expect some modifications in amplitude growth. G. I. Taylor* has analyzed this problem in detail for a plane slab of liquid under constant acceleration. He finds that the ripple will appear on both surfaces even if initially one surface is completely flat. If we confine our attention to ripples which have grown to many

* Proc. Roy. Soc. A 201, 1950, page 192 (especially equations 27 and 28).

times their initial amplitude, we find that the ripples on both surfaces have the same time behavior as one would calculate for a ripple on the unstable surface of a thick shell. The amplitude of the ripple on the unstable surface is $\frac{1}{1-e^{-2h/\lambda}}$ times greater than before, but because of the ripple on the stable surface, the thickness of the liquid is greater than before, i. e., the decrease in its thickness is less by the factor

$$\frac{1-e^{-h/\lambda}}{1-e^{-2h/\lambda}}.$$

Here h is the initial liquid thickness, λ the disturbance wavelength. This tells us that a disturbance long compared to the liquid thickness may grow to a substantial amplitude without greatly thinning the material shell. If one is interested in the time at which rupture of the shell is likely, it appears that specific thin shell effects are not very important.

For cylindrical and spherical shells, one might expect the same conclusions to hold, although different factors will be involved.

UNCLASSIFIED

UNCLASSIFIED

APPROVED FOR PUBLIC RELEASE