LA-4822

C. 1

The Notion of Complexity

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87544

UNITED STATES
ATOMIC ENERGY COMMISSION
CONTRACT W-7405-ENG. 36

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

This report expresses the opinions of the author or authors and does not necessarily reflect the opinions or views of the Los Alamos Scientific Laboratory.

Printed in the United States of America. Available from
National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151
Price: Printed Copy $3.00; Microfiche $0.95

LA-4822
UC-32
ISSUED: December 1971

los alamos
scientific laboratory
of the University of California
LOS ALAMOS, NEW MEXICO 87544

The Notion of Complexity

by

W. A. Beyer
M. L. Stein
S. M. Ulam

THE NOTION OF COMPLEXITY

by

W. A. Beyer, M. L. Stein, and S. M. Ulam

ABSTRACT

The notion of the arithmetic complexity $|n|$ of an integer n is defined in terms of the minimum number of additions, multiplications, and exponentiations required to combine 1's to form n. The value of $|n|$ is calculated for $n < 2^{10}$. n is called complicated if $|n| > |n_1|$ for every $n_1 < n$. Of the first 19 complicated numbers, 14 are prime. A conjecture about a relation between complexity and entropy is proposed. Some computations are presented to support this conjecture.

I. INTRODUCTION

In this report we discuss notions of complexity in some algebraic structures. These notions are also applicable to more general combinatorial situations that perhaps lack any algebraic pattern in the classical sense. We concentrate on a few special cases for which we studied and calculated a special notion of complexity. Essentially, we examined a special notion of complexity for ordinary integers with a little excursion on such a notion for integers modulo a prime.

The notion of complexity, in our view, is separate, though associated with the idea of the amount of information or entropy of a system. We mention briefly a possible axiomatic approach to defining a real number called complexity for elements of a set or of a class on which certain operations are performed. These could be binary operations; our set could be a set of integers, and the operations could be addition, multiplication, and exponentiation, for example. It is this case that was examined on a computing machine and to which most of this report is devoted.

Another case would be a class of subsets of a given set, with allowed operations being the Boolean operations of union and intersection or union and complementation. One could add other operations, for example, the direct product of sets and also projection. This would correspond to allowing quantifiers in our theory. One can study a notion of complexity for vectors in a countable space or even in the continuum. An important study would be that of a relative complexity; that is to say, complexity of elements or "expressions" when the complexity of certain symbols is normalized to 1. In what has been sometimes called "speculation" on constants in physical theories, for example, the whole art seems to depend on the success of attempts to define some known important numbers, e.g., the dimensionless ratios

$$M_{proton}/M_{electron} = 1836.11...$$

and

$$e^2/hc = 137.1...$$

by use of only a few artificially introduced constants which should be as "simple" as possible. (cf. the attempts by Eddington[1] and some very recent ones by Good[2] and Wyler.[3])

Considered "genetically," a mathematical theory resembles a tree in that one obtains from a given number of symbols corresponding to "variables" and from a number of allowed operations, expressions that elongate by branching. The simplifications and abbreviations may then reduce the length of the expressions.

One could try to define complexity in a mathematical structure by postulating certain of its properties, somewhat like postulating properties of a measure.

Let the structure, S, consist of elements x, y, ... It may be finite or infinite. We have in the set S a number of, say, binary operations R_1, R_2, ... R_n. We want to assign a number $c(x) \geq 0$ to each element x of S and to each R_i $(i = 1 \ldots n)$ so that the following properties should hold.

a. If $z = R_i(x,y)$, then $c(z) = c(R_i(x,y))$ $\leq c(x) + c(y) + c(R_i)$ $\quad i = 1 \ldots n$.

b. For each element z, if $z = R_j(x,y)$, we should have for one case at least, $c(z) = c(x) + c(y) + c(R_j)$.

c. $z(x_0) = z(x_1) = \ldots z(x_n)$ for some preassigned elements $x_0 \ldots x_n$.

Needless to say, one can define analogous desiderata for the case in which the operations are more general than binary ones.

Obviously, in the case to which our exercise is devoted, these postulates are satisfied. Moreover, they define the complexity uniquely if, as must be the case in general, the complexity was normalized for some elements. (In our case, we assume the complexity of the integer 0 to be equal to 1. We hope to study this notion more thoroughly for the more general case and also to perform experiments to determine complexity functions for the case in which S is a class of sets.) Ultimately, one would wish to discuss the complexity of genetic codes and biological organisms quantitatively.

("Integer" always means a positive integer.)

II. ARITHMETIC COMPLEXITY OF INTEGERS

The arithmetic complexity $|n|$ of an integer n is defined as the fewest number of operators: +, x, xx (addition, multiplication, and exponentiation) which combine 1's to form n. Thus, $|1| = 0$; $|2| = 1$ since $2 = 1 + 1$; and $|5| = 4$ since $5 = (1 + 1)\text{xx}(1 + 1) + 1$ and not fewer than four operators with 1's will form five. Obviously, for a and b integers, $|a + b|$, $|ab|$, and $|a^b|$ are each not more than $|a| + |b| + 1$. For an infinity of integers n, the relation $|n + 1| = |n| + 1$ holds.

For the purpose of calculating the complexity of some integers, all correct formulas (up to some number of operators) involving +, x, xx, and the number 1 were enumerated using parenthesis-free notation on a computer. It required one hour of computer time to enumerate the integers with complexity ≤ 6. Ralph Cooper made the following observation. Each correct formula involving n (> 0) operators is the composition of two formulas, one formula with n_1 operators and one formula with n_2 operators such that $n = n_1 + n_2 + 1$. One generates the integers of complexity n by first generating tables of integers of complexity $< n$. One partitions $n - 1$ into $n_1 + n_2$ in all ways and combines the integers of complexity n_1 with the integers of complexity n_2 to produce integers of complexity not larger than n. This method is considerably more efficient than the previous method. Table I lists the complexity of all integers $< 2^{10}$.

From the above construction, one sees that an upper bound $\ell^1(k)$ to $\ell(k)$, the number of integers of complexity k, is given by the solution of

$$\ell^1(k + 1) = \sum_{j=0}^{k} \ell^1(j)\, \ell^1(k - j)\ ,$$

with $\ell^1(0) = 1$. The solution to this equation is given by

$$\ell^1(k) = \frac{1}{k+1}\binom{2k}{k} 2^{-k},$$

which implies that

$$\ell(k) \leq \frac{2^k}{k\sqrt{\pi k}} + 0\left(2^k\ k^{-5/2}\right).$$

Two additional forms of complexity have been considered and calculated.

a. Complement complexity. To make complexity symmetric in 0's and 1's, we introduce a slightly different complexity, the complement complexity $\overline{K}(y|n)$. Define the complement operation C by $C(x|n) = 2^n - 1 - x$. $\overline{K}(y|n)$ is defined as the fewest operations of addition, multiplication, exponentiation, and complementation that combine 1's to form y. In the count of operations, the

TABLE I. COMPLEXITY OF INTEGERS $< 2^{10}$.

Complexity	Integer																			
0	1																			
1	2																			
2	3																			
3	4																			
4	5	6	8	9																
5	7	10	16	27																
6	11	12	17	18	25	28	32	36	64	81	256	512								
7	13	14	15	19	20	24	26	29	33	37	49	54	65	82	100	125	128	216	243	257
	513	729	1024																	
8	21	22	30	34	38	48	50	55	56	66	72	83	101	121	126	129	144	162	217	244
	258	289	324	343	514	625	730	784	1000											
9	23	31	35	39	40	45	51	52	57	58	67	73	74	75	84	96	98	102	108	122
	127	130	145	163	164	169	192	196	200	218	225	245	250	259	290	325	344	361	400	432
	486	515	576	626	676	731	768	785	841	1001										
10	41	42	44	46	53	59	60	63	68	76	78	80	85	87	90	97	99	103	109	110
	111	112	123	131	132	135	146	147	165	166	170	193	195	197	201	202	219	226	242	246
	251	252	260	288	291	300	326	345	362	375	384	401	433	434	441	484	487	488	516	577
	578	627	648	677	686	732	769	771	786	842	900	1002								
11	43	47	61	62	69	70	77	79	86	88	89	91	104	113	114	116	124	133	134	136
	140	148	150	153	160	167	168	171	180	189	194	198	203	204	220	224	227	247	249	253
	254	261	262	264	265	270	292	301	303	320	327	328	338	346	363	376	378	385	387	392
	402	405	435	436	442	450	485	489	490	500	517	518	520	521	529	579	580	628	649	650
	651	678	687	688	722	733	770	772	774	787	800	843	864	867	901	961	972	1003		
12	71	92	93	95	105	106	115	117	118	119	120	137	141	149	151	152	154	156	161	172
	174	175	176	181	185	190	199	205	206	208	221	222	228	232	234	248	255	263	266	271
	272	280	283	293	294	296	297	302	304	306	321	329	330	332	333	339	340	347	360	364
	366	377	379	381	386	388	390	393	394	403	404	406	410	437	438	443	448	451	452	459
	491	492	501	502	504	507	519	522	528	530	539	567	581	582	585	588	600	629	640	652
	654	656	675	679	689	690	723	724	734	735	737	738	750	756	773	775	777	788	801	802
	810	844	865	866	868	870	882	902	962	968	973	974	975	976	1004					
13	94	107	138	142	155	157	158	159	173	177	178	182	186	187	191	207	209	223	229	231
	233	235	240	267	268	273	274	275	281	284	295	298	305	307	308	309	322	331	334	336
	337	341	342	348	349	351	352	365	367	369	370	380	382	389	391	395	396	407	408	411
	415	416	425	439	440	444	449	453	454	455	460	464	476	493	494	495	498	503	505	506
	508	510	523	524	531	537	540	544	548	568	574	583	584	586	589	591	592	593	594	601
	602	603	605	606	612	630	631	633	634	641	645	653	655	657	664	680	691	692	700	702
	704	720	725	726	736	739	745	747	751	752	753	757	776	778	780	783	789	790	792	793
	803	804	808	811	820	845	869	871	872	873	875	883	884	891	896	903	909	963	969	970
	977	978	980	999	1005	1006	1008	1009												
14	139	143	179	183	184	188	210	212	230	236	237	238	241	269	276	279	282	285	286	299
	310	312	315	316	319	323	335	350	353	356	359	368	371	372	383	397	398	399	409	412
	417	420	426	445	456	461	462	465	468	472	475	477	480	496	499	509	511	525	526	527
	532	536	538	541	542	545	549	550	560	561	566	569	575	587	590	595	604	607	608	609
	610	613	632	635	637	642	646	658	660	665	666	672	681	682	684	685	693	694	701	703
	705	707	715	721	727	728	740	741	746	748	754	755	758	759	761	762	765	779	781	791
	794	795	805	806	809	812	815	816	821	825	830	832	833	846	847	849	850	874	876	880
	885	886	892	897	904	910	918	924	925	928	936	960	964	971	979	981	982	984	985	1007
	1010	1014	1016																	
15	211	213	214	239	277	278	287	311	313	314	317	318	354	355	357	373	374	413	414	418
	421	423	424	427	429	446	447	457	458	463	466	469	470	473	478	481	483	497	533	534
	543	546	551	555	562	570	596	597	599	611	614	615	616	618	621	624	636	638	643	644
	647	659	661	662	663	667	668	670	673	674	683	695	696	698	706	708	714	716	742	743
	744	749	760	763	764	766	782	796	798	807	813	814	817	822	824	826	829	831	834	836
	837	840	848	851	854	855	857	858	877	878	879	881	887	888	889	893	898	905	906	908
	911	912	913	919	920	926	927	929	931	935	937	945	950	952	957	965	983	986	987	988
	990	996	1011	1012	1015	1017	1018	1020												
16	215	358	419	422	428	430	467	471	474	479	482	535	547	552	556	557	558	559	563	564
	565	571	572	573	598	617	619	620	622	639	669	671	697	699	709	711	712	713	717	718
	767	797	799	818	819	823	827	828	835	838	852	853	856	859	861	890	894	899	907	914
	915	916	917	921	922	930	932	938	944	946	951	953	954	958	966	967	989	991	992	993
	997	998	1013	1019	1021	1022	1023													
17	431	553	554	623	710	719	839	860	862	895	923	933	939	940	941	942	947	948	949	955
	956	959	994	995																

first three are given the value 1 and the last is given the value zero. Thus $\overline{K}(y|n) = \overline{K}(2^n - 1 - y|n)$. Table II gives the values of $\overline{K}(y|n)$ for $y < 2^{10}$ and $n = 10$.

b. Modulo a prime p complexity. In addition to the operations of +, x, and xx, the operation of mod_p is allowed and is defined by $\mathrm{mod}_p(x) = x - p[x/p]$ where p is a fixed prime and [] denotes the greatest integer. Table III gives the modulo prime $p = 137$ complexity for integers < 137. Table IV gives the modulo prime $p = 1009$ complexity for integers < 1009.

III. COMPLICATED NUMBERS

One defines n to be a complicated number if $|n| > |n_1|$ for every $n_1 < n$. The complicated numbers $< 2^{10}$ are 1, 2, 3, 4, 5, 7, 11, 13, 21, 23, 41, 43, 71, 94, 139, 211, 215, 431, and 863. (Those underlined are also prime.) Obviously, there are an infinity of complicated numbers. We propose the following conjectures.

a. There exists K such that all complicated numbers $K_1 > K$ are prime.

b. Every sufficiently large integer n is the sum of $k < \log n$ complicated integers.

c. There exists c such that every sufficiently large n satisfies $|n| < c + \sqrt{\log n}$.

IV. COMPLEXITY AND ENTROPY

Kolmogorov[4,5] has introduced the notion of complexity of a finite string over a given alphabet. For simplicity, suppose the alphabet to be {0,1}. Let A be an algorithm that transforms finite binary sequences into binary sequences. By an algorithm is meant any of the various equivalent concepts used in logic. For a binary string x, one defines the complexity by

$$K_A(x) = \begin{cases} \min\limits_{A(p)=x} \ell(p) \\ \\ \infty \\ \text{if no p exists such that } A(p) = x, \end{cases}$$

where $\ell(p)$ denotes the length of the binary string p. Analogously, one defines conditional complexity. Let A(p,x) be an algorithm defined from pairs of binary strings to binary strings. Put

$$K_A(y|x) = \begin{cases} \min\limits_{A(p,x)=y} \ell(p) \\ \\ \infty \\ \text{if no p exists such that } A(p,x) = y. \end{cases}$$

$K_A(y|x)$ is called the conditional complexity of y with respect to x. Kolmogorov regards complexity as analogous to entropy. We make the following conjecture.

Conjecture. Let a discrete binary information source S in the sense of Shannon[6] be given with entropy $H = -p \log p - (1-p) \log (1-p)$ where probability (0) = p and probability (1) = 1 -p; $0 < p < 1$. Let $\{x_1, x_2, \ldots, x_{2^n}\}$ be the set of all binary strings of length n arranged in order of decreasing probability. Let k(n) be the least integer so that $\sum_{i=1}^{k(n)} \text{prob}\ (x_i) > r$ where $1/2 < r < 1$. Then asymptotically for large n,

$$H \approx \frac{1}{k(n)} \sum_{i=1}^{k(n)} K_A(x_i|n). \qquad (1)$$

(In Eq. (1), K_A should be normalized so that when $p = 1/2$,

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} K_A(x_i|n) = 1.)$$

In other words, the most likely sequences from A have complexity approximately equal to the entropy of S.

In order to test the conjecture expressed in Eq. (1), we replaced $K_A(x_i|n)$ by $\lambda\overline{K}(y|n)$, where λ is selected so that when $p = 1/2$,

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} \lambda\overline{K}(x_i|n) = 1.$$

Graphs of $H_1 = -p \log p - (1 - p) \log (1 - p)$ and

$$H_2 = \frac{1}{k(n)} \sum_{i=1}^{n} \lambda\overline{K}(x_i|n)$$

when $n = 10$ and $r = .75$ are shown in Fig. 1

TABLE II. COMPLEMENT COMPLEXITY OF INTEGERS $< 2^{10}$.

Complement Complexity	Integer																			
0	0	1	1022	1023																
1	2	1021																		
2	3	1020																		
3	4	1019																		
4	5	6	8	9	1014	1015	1017	1018												
5	7	10	16	27	996	1007	1013	1016												
6	11	12	15	17	18	25	26	28	32	36	64	81	256	511	512	767	942	959	987	991
	995	997	998	1005	1006	1008	1011	1012												
7	13	14	19	20	24	29	31	33	35	37	49	54	63	65	80	82	100	125	128	216
	243	255	257	294	510	513	729	766	768	780	807	895	898	923	941	943	958	960	969	974
	986	988	990	992	994	999	1003	1004	1009	1010										
8	21	22	23	30	34	38	48	50	52	53	55	56	62	66	72	79	83	99	101	121
	124	126	127	129	144	162	215	217	225	239	242	244	254	258	289	293	295	324	343	347
	398	509	514	625	676	680	699	728	730	734	765	769	779	781	784	798	806	808	861	879
	894	896	897	899	902	922	924	940	944	951	957	961	967	968	970	971	973	975	985	989
	993	1000	1001	1002																
9	39	40	45	47	51	57	58	61	67	70	71	73	74	75	78	84	96	98	102	108
	120	122	123	130	143	145	160	161	163	164	169	182	192	196	200	214	218	224	226	238
	240	241	245	250	253	259	288	290	292	296	323	325	342	344	346	348	361	397	399	400
	432	435	447	486	508	515	537	576	588	591	623	624	626	662	675	677	679	681	698	700
	727	731	733	735	764	770	773	778	782	783	785	797	799	805	809	823	827	831	841	854
	859	860	862	863	878	880	893	900	901	903	915	921	925	927	939	945	948	949	950	952
	953	956	962	965	966	972	976	978	983	984										
10	41	42	44	46	59	60	68	69	76	77	85	87	90	93	95	97	103	104	105	106
	107	109	110	111	112	119	131	132	135	141	142	146	147	158	159	165	166	168	170	181
	183	189	191	193	195	197	198	199	201	202	213	219	223	227	237	246	248	249	251	252
	260	287	291	297	300	322	326	329	337	341	345	349	360	362	375	384	396	401	430	431
	433	434	436	437	441	445	446	448	450	478	484	485	487	488	494	507	516	529	535	536
	538	539	545	573	575	577	578	582	586	587	589	590	592	593	622	627	639	648	661	663
	674	678	682	686	694	697	701	723	726	732	736	763	771	772	774	775	777	786	796	800
	804	810	821	822	824	825	826	828	830	832	834	840	842	853	855	857	858	864	865	876
	877	881	882	888	891	892	904	911	912	913	914	916	917	918	919	920	926	928	930	933
	936	938	946	947	954	955	963	964	977	979	981	982								
11	43	86	88	89	91	92	94	113	114	116	118	133	134	136	138	140	148	150	153	156
	157	167	171	180	184	186	188	190	194	203	204	208	212	220	222	228	229	234	236	247
	261	262	264	265	270	286	298	299	301	303	306	320	321	327	328	330	331	335	336	338
	339	340	350	359	363	364	372	373	374	376	377	378	381	383	385	387	392	395	402	405
	428	429	438	439	440	442	443	444	449	451	452	476	477	479	480	482	483	489	490	493
	495	500	502	503	504	505	506	517	518	519	520	521	523	528	530	533	534	540	541	543
	544	546	547	571	572	574	579	580	581	583	584	585	594	595	618	621	628	631	636	638
	640	642	645	646	647	649	650	651	659	660	664	673	683	684	685	687	688	692	693	695
	696	702	703	717	720	722	724	725	737	753	758	759	761	762	776	787	789	794	795	801
	803	811	815	819	820	829	833	835	837	839	843	852	856	866	867	870	873	875	883	885
	887	889	890	905	907	909	910	929	931	932	934	935	937	980						
12	115	117	137	139	149	151	152	154	155	172	174	175	176	179	185	187	205	206	207	209
	210	211	221	230	231	232	233	235	263	266	267	269	271	272	273	279	280	282	283	284
	285	302	304	305	307	309	315	316	318	319	332	333	334	351	358	365	366	367	369	371
	379	380	382	386	388	390	391	393	394	403	404	406	410	416	423	426	427	453	454	456
	459	474	475	481	491	492	496	498	499	501	522	524	525	527	531	532	542	548	549	564
	567	569	570	596	597	600	607	613	617	619	620	629	630	632	633	635	637	641	643	644
	652	654	656	657	658	665	672	689	690	691	704	705	707	708	714	716	718	719	721	738
	739	740	741	743	744	750	751	752	754	756	757	760	788	790	791	792	793	802	812	813
	814	816	817	818	836	838	844	847	848	849	851	868	869	871	872	874	884	886	906	908
13	173	177	178	268	274	275	276	277	278	281	308	310	312	314	317	352	353	354	355	356
	357	368	370	389	407	408	409	411	415	417	418	420	421	422	424	425	455	457	458	460
	463	464	465	468	472	473	497	526	550	551	555	558	559	560	563	565	566	568	598	599
	601	602	603	605	606	608	612	614	615	616	634	653	655	666	667	668	669	670	671	706
	709	711	713	715	742	745	746	747	748	749	755	845	846	850						
14	311	313	412	413	414	419	461	462	466	467	469	470	471	552	553	554	556	557	561	562
	604	609	610	611	710	712														

TABLE III. MODULO PRIME p = 137 COMPLEXITY OF INTEGERS < 137.

Complexity	Integer																			
0	1																			
1	2																			
2	3																			
3	4																			
4	5	6	8	9																
5	7	10	16	27																
6	11	12	17	18	25	28	32	36	64	81	101	119								
7	13	14	15	19	20	24	26	29	33	37	44	49	50	54	61	65	79	82	92	100
	102	106	120	122	125	128														
8	21	22	30	34	38	41	45	48	51	55	56	60	62	63	66	68	69	72	77	80
	83	88	93	99	103	107	109	117	118	121	123	126	129	130	132	133				
9	23	31	35	39	40	42	46	47	52	53	57	58	59	67	70	73	74	75	76	78
	84	87	89	94	96	98	104	108	110	111	112	113	115	124	127	131	134	136		
10	0	43	71	85	86	90	95	97	105	114	116	135								
11	91																			

V. COMPLEXITY OF N-TUPLES OF INTEGERS

Matijasevič[7] has proved the following theorem. There exists a fifth-degree polynomial $Q(y_1, \ldots, y_k; z)$ with integer coefficients such that any enumerable set m of natural numbers (for example, the set of prime numbers) coincides with the set of natural values of the polynomial $Q(y_1, \ldots, y_k; a_m)$ where a_m is a certain number effectively constructed for the set m. From the result, it follows that if one could discuss complexity of n-tuples of integers, then one could discuss the complexity of enumerable sets of natural numbers by equating such complexity to the complexity of the associated polynomial Q.

REFERENCES

1. A. S. Eddington, Fundamental Theory (Cambridge University Press, 1946).

2. I. J. Good, "The Proton and Neutron Masses and a Conjecture for the Gravitational Constant," Phys. Let. 33A, 383-384 (1970).

3. A. Wyler,"Les Groupes des Potentiels de Coulomb et de Yukawa," Compt. Rend. Acad. Sci. Paris, 271, 186-188 (1971).

4. A. Kolmogorov, "Three Approaches for Defining the Concept of Information Quantity," Problems of Information Transmission 1, 3-11 (1965).

5. A. Kolmogorov, "Logical Basis for Information Theory and Probability Theory," IEEE Trans. Information Theory IT-14, 662-664 (1968).

6. C. E. Shannon and Warren Weaver, The Mathematical Theory of Communication (The University of Illinois Press, Urbana, 1949).

7. Ju V. Matijasevič, "Enumerable Sets are Diophantine," Soviet Math. Dokl. 11, 354-358 (1970).

ADDITIONAL REFERENCES TO COMPLEXITY NOT USED IN TEXT

1. E. L. Lawler, "The Complexity of Combinatorial Computations: A Survey," Proceedings of 1971 Polytechnic Institute of Brooklyn Symposium on Computers and Automata.

2. D. W. Loveland, "On Minimal Program Complexity Measures," ACM Symp. Theory of Computing, Marina del Rey, California, May 5-7, 1969.

3. P. Young, "A Note on Dense and Nondense Families of Complexity Classes," Math. Systems Theory 5, 66-70 (1971).

TABLE IV. MODULO PRIME p = 1009 COMPLEXITY OF INTEGERS < 1009.

Complexity	Integer																			
0	1																			
1	2																			
2	3																			
3	4																			
4	5	6	8	9																
5	7	10	16	27																
6	11	12	17	18	25	28	32	36	64	81	256	512								
7	13	14	15	19	20	24	26	29	33	37	49	54	65	82	100	125	128	216	243	257
	507	513	548	729	960															
8	21	22	30	34	38	48	50	55	56	60	66	72	74	83	87	101	121	126	129	137
	144	162	169	217	244	258	287	289	324	343	383	384	508	514	527	549	625	710	730	763
	784	813	911	961	993	1000														
9	23	31	35	39	40	45	51	52	57	58	61	67	73	75	80	84	88	96	98	102
	106	120	122	127	130	138	142	145	148	163	164	170	173	174	189	192	196	200	218	225
	240	242	245	250	259	270	271	274	288	290	322	325	344	360	361	385	400	411	432	449
	464	480	486	490	509	515	528	538	550	572	573	576	617	626	631	635	640	670	676	707
	711	713	719	731	764	766	768	782	785	787	808	814	829	841	859	877	893	898	912	919
	928	962	977	985	994	1001														
10	41	42	44	46	53	59	62	63	68	76	78	85	89	90	97	99	103	105	109	110
	111	112	123	131	132	135	139	143	146	147	149	165	166	171	175	177	179	180	185	186
	190	193	195	197	201	202	203	219	222	226	241	246	251	252	253	254	260	272	275	280
	284	286	291	296	300	309	320	323	326	328	338	345	348	362	375	386	394	401	404	412
	421	423	431	433	434	435	441	443	450	451	454	465	481	482	484	487	488	491	497	506
	510	516	517	519	523	529	530	539	540	551	555	556	559	574	577	578	605	607	609	616
	622	627	632	636	641	648	663	671	675	677	686	708	709	712	714	715	720	726	732	741
	755	765	767	769	771	777	783	786	788	791	805	809	815	822	824	830	835	842	847	860
	861	862	878	881	882	886	894	896	899	900	906	913	920	922	927	929	935	937	942	945
	955	963	972	978	979	986	991	995	999	1002	1006	1007								
11	43	47	69	70	71	77	79	86	91	104	106	113	114	116	124	133	134	136	140	150
	153	160	167	168	172	176	178	181	187	191	194	198	199	204	205	206	209	210	211	212
	213	220	223	224	227	247	249	255	261	262	264	265	268	269	273	276	281	283	285	292
	297	301	302	303	310	313	314	321	327	329	331	332	334	335	336	337	339	340	346	349
	353	355	363	374	376	378	379	382	387	392	395	398	402	405	406	409	410	413	417	418
	422	424	429	436	442	444	448	452	453	455	456	466	479	483	485	489	492	494	498	500
	501	511	518	520	521	524	531	533	541	542	545	546	552	557	558	560	561	565	568	575
	579	580	583	591	592	597	599	600	606	608	610	611	615	619	620	623	628	633	634	637
	638	642	644	649	650	651	654	656	661	662	664	666	668	672	673	674	678	681	685	687
	688	689	692	693	694	696	706	716	721	722	727	733	735	738	742	745	748	753	756	758
	759	762	770	772	774	778	789	792	793	800	803	804	806	810	816	823	825	831	836	840
	843	845	848	852	863	864	865	866	867	870	875	879	883	884	887	895	897	901	904	907
	908	914	915	921	923	930	934	936	938	940	943	946	949	951	953	956	959	964	967	968
	973	980	981	982	987	989	992	996	1003	1008										
12	0	92	93	95	107	115	117	118	119	141	151	152	154	156	157	161	182	183	188	207
	208	214	221	228	232	234	235	248	263	266	277	278	282	293	294	295	298	299	304	306
	311	315	317	319	330	333	341	342	347	350	354	356	357	358	364	366	369	370	372	377
	380	381	388	390	393	396	397	399	403	407	414	415	419	420	425	426	430	437	438	445
	457	459	460	461	463	467	469	471	472	473	493	495	499	502	503	504	505	522	525	532
	534	536	543	544	547	553	554	562	563	566	567	569	571	581	582	584	585	588	593	598
	601	603	604	612	613	618	621	624	629	630	639	643	645	646	652	655	657	665	667	669
	679	682	690	695	697	700	717	718	723	724	725	728	734	736	737	739	743	746	747	749
	750	754	757	760	773	775	779	790	794	797	799	801	802	807	811	817	818	820	826	832
	837	844	846	849	853	858	868	869	871	872	876	880	885	888	889	902	905	909	916	917
	924	925	931	939	941	944	947	950	952	954	957	965	966	969	974	975	976	983	988	990
	997	1004	1005																	
13	94	155	158	159	184	215	229	231	233	236	237	267	279	305	307	308	312	316	318	351
	352	359	365	367	368	371	373	389	391	408	416	427	428	439	440	446	447	458	462	468
	470	474	475	476	496	526	535	537	564	570	586	587	589	590	594	602	614	647	653	658
	659	680	683	691	698	701	702	704	740	744	751	752	761	776	780	781	795	796	798	812
	819	821	827	833	834	838	850	851	854	857	873	874	890	891	903	910	918	926	932	948
	958	970	984	998																
14	230	238	239	477	478	595	596	660	684	699	703	705	828	839	855	892	933	971		
15	856																			

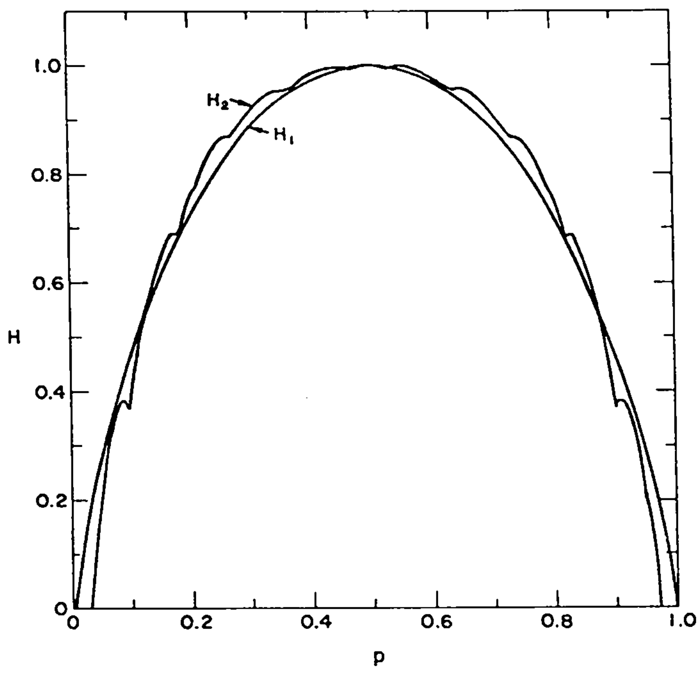

Fig. 1. Comparison of entropy $H_1 = - \sum p_i \log p_i$ and complement complexity H_2 as defined and discussed in text.

KT/bk: 304 (50)