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APPLICATION OF A MEAN FIELD APPROXIMATION TO TWO SYSTEMS THAT EXHIBIT SELF-ORGANIZED CRITICALITY

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In this exposition, a mean field analysis will be applied to predict and explain some of the features observed in two systems that are known to exhibit self-organized criticality: the sandpile model of Bak, Tang, and Wiesenfeld [1], and a variation using continuous values that was introduced by Zhang [2]. It will be argued that mean field of Tang and Bak [3] is problematic in that it fails to converge. The modification suggested here introduces a parameter to account for sandgrains falling off the edge of the sandpile; this balances the sandgrains which are dropped on the sandpile from above. The modified analysis is then applied to the equilibrium state of the sandpile, and to the time evolution toward equilibrium. The analysis is then extended to other systems which exhibit self-organized critical behavior.

1. Introduction

For dissipative dynamical systems with extended degrees of freedom, a variety of apparently collective phenomenon have been observed, including spatio-temporal chaos [4], robust intermittency [5], long quasistationary transients [6], attractor crowding [7] and clustering [8, 9], "amplitude death" [10], and self-organized criticality [1]. Self-organized criticality, in particular, has been invoked to explain behavior of a wide variety of physical systems, from earthquakes to avalanches.

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The mean field is one of the few analytical tools that are available for the understanding of systems with many interacting components. The approach of the mean field is to replace individual components with statistical averages. It is a relatively crude approximation which ignores the spatial aspect of the system, but it is general enough to be applied to many different systems, and it usually leads to a computationally tractible approximation.

After introducing the sandpile model, I will discuss the the mean field analysis that was developed by Tang and Bak [3], and show how a slight modification makes the analysis more sensible. The modified analysis will then be compared to simulations and the quantitative agreement (and disagreement) will be discussed. Finally, the analysis will be extended to the continuous system of Zhang [2], and it will be shown that the self-organized discretization which this system displays can be understood in terms of a mean field.

2. The sandpile model

The sandpile model was introduced by Bak, Tang, and Wiesenfeld [1], hereafter BTW, to illustrate a property they called self-organized criticality. At each site (x, y) on a square lattice, there is an integ number $z(x, y)$ of what are often, but not quite correctly, called sandgrains. (It would be only slightly more correct to call $z(x, y)$ a "local slope," but "sandgrains" provide a more concrete picture.) There is a critical value $K = 4$ above which a lattice site is unstable. If any site on the lattice is unstable, then the lattice as a whole is considered unstable, and is permitted to relax in a series of discrete steps. Four sandgrains at each unstable site are redistributed, one to each of the four neighboring sites, in a kind of diffusion that is highly nonlinear. Here,

$$\begin{aligned} z(x, y) &\rightarrow z(x, y) - 4, \\ z(x, y \pm 1) &\rightarrow z(x, y \pm 1) + 1, \\ z(x \pm 1, y) &\rightarrow z(x \pm 1, y) + 1. \end{aligned} \tag{1}$$

The relaxation process continues until every site on the lattice is stable.

The dynamics then proceeds by seeding at a single random (or a fixed [11]) site: $z(x_o, y_o) \rightarrow z(x_o, y_o) + 1$. If this leads to an unstable lattice, the lattice is allowed to relax by the above rule until it becomes stable again. Thus there are "long" time steps between each seeding, and "short" time steps, during which the lattice relaxes.

In simulations, one usually waits until the lattice is finished relaxing until the next seeding; the point of this is to ensure that the two time scales (slow seeding and fast relaxing) are fully separated. The physical process which does the seeding presumably is doing so at a slow rate that is independent of the state of the lattice.

2.1. Notation

The lattice in the BTW system needn't be two dimensional, nor need it even be a lattice. Let $\mathbf{x}$ represent a site on the lattice, and $z(\mathbf{x})$ be the value at that site. Each site is connected to its neighbors $\mathbf{x}' \in \mathcal{N}_{\mathbf{x}}$. (Note that there is a bi-directionality in the case of a lattice: $\mathbf{x}' \in \mathcal{N}_{\mathbf{x}}$ implies $\mathbf{x} \in \mathcal{N}_{\mathbf{x}'}$.) It is possible to represent the external seeding by a field term $h_t(\mathbf{x})$, which is zero almost all of the time, but equal to one when the $\mathbf{x}$ is seeded. (The t here corresponds to short time steps.) Further, write $\xi_t(\mathbf{x})$ as the "backflow" from neighboring sites. Let $b = |\mathcal{N}_{\mathbf{x}}|$ be the number of bonds per site (also called the "coordination number"), and let K be the threshold value (note $b = K = 4$ for the original two-dimensional BTW system).

$$
\begin{aligned}
z_{t+1}(\mathbf{x}) &= z_t(\mathbf{x}) + h_t(\mathbf{x}) - bF(z_t(\mathbf{x})) + \xi_t(\mathbf{x}) && (2)\\
\xi_t(\mathbf{x}) &= \sum_{\mathbf{x}' \in \mathcal{N}_{\mathbf{x}}} F(z_t(\mathbf{x}')) && (3)
\end{aligned}
$$

where $F(z)$ is the threshold function: its value is 0 for $z < K$ (inactive), and 1 for $z \geq K$ (active).

3. The mean field of Tang and Bak

In the mean field introduced by Tang and Bak [3], hereafter TB, the approximation is made that the sites are statistically independent and identically distributed. The site value z is treated as a random variable whose evolution is given by the equation

$$z_{t+1} = z_t - bF(z_t) + h_t + \xi_t \tag{4}$$

where h_t is a random variable that takes a value of 1 with probability h and a value of zero the rest of the time. The backflow, ξ_t, is a random variable given by

$$\xi_t := \sum_{i=1}^{b} F(z_t^{(i)}) \tag{5}$$

where $z_t^{(i)}$ are random variables that correspond to the neighbors of z_t. By the assumption of the mean field, these are taken to be independent but with the same distribution as z_t. Now ξ_t takes on a value of r when exactly r of the site's neighbors are active. Thus,

$$\mathcal{P}\{\xi = r\} = \binom{b}{r} A^r I^{b-r} \tag{6}$$

where, following TB, I have written $A := \mathcal{P}\{F(z) = 1\} = \mathcal{P}\{z \geq K\}$ as the probability that a site is active, and $I := 1 - A = \mathcal{P}\{z < K\}$ as the probability that a site is inactive.

Now, I can write a transition matrix for the Markov process. Let $T_t(z \leftarrow z')$ $T_t[z][z'] := \mathcal{P}\{z_{t+1} = z \mid z_t = z'\}$ be the probability of a transition at the fiducial site from a value of z' to a value of z. In general, I have

$$\mathcal{T}_t[z][z'] = \begin{cases} \mathcal{P}\{h_t + \xi_t = z - z'\} & \text{for } z' < K, \\ \mathcal{P}\{h_t + \xi_t = z - z' + b\} & \text{for } z' \geq K, \end{cases} \tag{7}$$

which we can write $\mathcal{T}_t[z][z'] = \mathbf{T}_t[z - z' + bF(z')]$ where the transition vector $\mathbf{T}$ is defined by

$$\begin{aligned} \mathbf{T}[x] &:= \mathcal{P}\{h + \xi = x\} && (8) \\ &= (1-h)\mathcal{P}\{\xi = x\} + h\mathcal{P}\{\xi = x - 1\} && (9) \\ &= (1-h)\binom{b}{x} A^x I^{b-x} + h\binom{b}{x-1} A^{x-1} I^{b-x+1}. && (10) \end{aligned}$$

The transition matrix $\mathcal{T}_t$ allows us to evolve the vector of current probabilities $\mathbf{P}_t[z] := \mathcal{P}\{z_t = z\}$. We have, in matrix notation, $\mathbf{P}_{t+1} = \mathcal{T}_t\mathbf{P}_t$, or more explicitly

$$\begin{aligned} \mathbf{P}_{t+1}[z] &= \sum_{z'} \mathcal{T}_t[z][z']\mathbf{P}_t[z'] && (11) \\ &= \sum_{z'} \mathbf{T}_t[z - z' + bF(z')]\mathbf{P}_t[z'] && (12) \end{aligned}$$

Thus, we can derive the time evolution of the probability distribution of the random variable z. Because the the matrix $\mathcal{T}_t$ depends on the vector $\mathbf{T}_t$, which depends on A_t and I_t, both of which in turn depend on $\mathbf{P}_t$, one can think of the evolution of probabilities in Eq. 12 as a nonlinear map:

$$\mathbf{P}_{t+1} = f(\mathbf{P}_t). \tag{13}$$

3.1. The problem of nonconvergence

The problem with this version of the mean field is that it does not converge. For any $h > 0$, there is no solution $\mathbf{P}_\infty[z]$ which satisfies $\mathbf{P} = f(\mathbf{P})$. In particular, it is straightforward to show that for this model, the average value $\langle z_t \rangle := \sum_z z\mathbf{P}_t[z]$ does not converge. Instead, we have, from Eqs. (4) and (5), that

$$\langle z_{t+1} \rangle = \langle z_t \rangle + h. \tag{14}$$

This is in retrospect not really surprising, since the dynamics preserves the total number of "sandgrains" (each sand grain lost at site $\mathbf{x}$ is picked up again at a neighboring site $\mathbf{x}'$), and a steady field h is being added at each step. In an actual BTW simulation, this is not a problem, since avalanches occasionally carry those added grains of sand off the edge.

4. Modified mean field: introducing edges

One solution to the problem of nonconvergence is to mimic what happens in an actual BTW lattice, and to let sandgrains fall off the edge. It isn't instantly obvious how to do this, since a mean field approximation by definition eliminates the lattice structure from the problem, and fails to distinguish edges from interior sites. The suggestion here is to state that a fraction $e > 0$ of the sites are on the edge; we can later take e very small if that is desired, but we need some mechanism to "lose" those grains of sand that are being added by the field. Having a finite fraction of edge sites alters the equation for the backflow ξ. It is still the case that ξ is equal to the number of active neighbors, but the number of available neighbors depends now on whether the site is on the boundary or not. With probability e it is on the boundary, and has $b-1$ available neighbors; with probability $1-e$ it is in the interior, and has b available neighbors. Thus,

$$\mathcal{P}\{\xi = x\} = (1-e)\binom{b}{x} A^x I^{b-x} + e\binom{b-1}{x} A^x I^{b-1-x}. \tag{15}$$

and the modified transition vector is given by

$$\begin{aligned}\mathbf{T}_t[x] &= (1-h)\left[(1-e)\binom{b}{x} A_t^x I_t^{b-x} + e\binom{b-1}{x} A_t^x I_t^{b-1-x}\right] \\ &+ h\left[(1-e)\binom{b}{x-1} A_t^{x-1} I_t^{b-x} + e\binom{b-1}{x-1} A_t^{x-1} I_t^{b-1-x}\right].\end{aligned} \tag{16}$$

The first thing to note is that $e = 0$ leads to the original TB mean field. However, in this case, we have that the average evolves according to

$$\langle z_{t+1}\rangle = \langle z_t\rangle + h - eA_t. \tag{17}$$

Thus, a steady state can be achieved, with the balance $h = eA$ corresponding to external input on the left hand side, and loss over the edge on the right hand side. Note that this balance equation does not make sense in the original model of TB, where $e = 0$.

4.1. Finding the self-consistent solution

Having defined the modified mean field equations (12) and (16), it is straightforward to solve for the probabilities $\mathbf{P}$. Numerically, one need only evolve the probabilities forward in time until a steady state is reached. The solution exists for nonzero h and e as long as $h < e$.

In particular, it is possible to show [12] that in the simultaneous limit $h \to 0$ and $e \to 0$, with $h/e \to A$

$$\text{for } 0 \le z < b, \qquad \mathbf{P}[z] = (1/b)\sum_{k=0}^{z}\binom{b}{k} A^k(1-A)^{b-k} \tag{18}$$

$$\text{for } b \le z < 2b, \qquad \mathbf{P}[z] = (1/b) \sum_{k=z-b}^{b} \binom{b}{k} A^k (1-A)^{b-k} \tag{19}$$

$$\text{for } z > 2b, \qquad \mathbf{P}[z] = 0. \tag{20}$$

This leads to the result

$$\langle z \rangle = (b-1)/2 + bA, \tag{21}$$

which although linear in A, is in fact is exact for $0 \le A < 1$. The sandpile model does not re-seed until the lattice is finished relaxing; this is equivalent to saying $h \to 0$ for fixed, but arbitrarily small, e. Thus, the $A \to 0$ limit is appropriate as an approximation of the sandpile model. This in fact is the same equilibrium probabilities that TB claim to get in the $h \to 0$ limit for their analysis.

4.2. Some comments on computing exponents

In TB, four equations are presented, Eqs. (4a-d) in their paper, with four unknowns (P_0, P_1, P_2, P_3) and one parameter (h). A second parameter (θ, which is the same as $\langle z \rangle$) is introduced as a function of the four variables. All of these are combined into a single equation, Eq. (5), which appears to exhibit two degrees of freedom; that is, both h and θ are treated as though they were free to vary independently of each other. From this, a variety of exponents are computed. But h and θ are not independent. As Obukhov [13] has pointed out, simply adding a field h brings the system away from from the critical point, so that "the parameters which describe both the proximity to the critical point and the magnetic field are coupled together." However, I would argue that not only are there not two independent degrees of freedom; there is not even one. For while there is complete degeneracy in the solution when $h = 0$ (constrained only by $\mathbf{P}[0] + \cdots + \mathbf{P}[b-1] = 1$ and $\mathbf{P}[b] = \mathbf{P}[b+1] = \cdots = 0$), taking h to any nonzero value leads to inconsistent equations with no solution at all. The edge parameter e provides one mathematically valid way to break the degeneracy.

For the mean field analysis presented here, the edge parameter is considered fixed, so there is really only one degree of freedom, in the control parameter h. So I cannot compute exponents as they are defined in TB. Because of this, I have concentrated not on predicting exponents (even though these are arguably the most interesting features), but rather the relative frequencies of the values on the lattice, $\mathbf{P}[z]$, and in particular, the average $\langle z \rangle$. I should comment that there are other approaches which also go by the name "mean field" but which are based on treating an avalanche as a branching process: these are discussed by Alstrøm [14] and Obukhov [13]. Unlike the mean field analysis presented here, these approaches *are* able to predict nontrivial exponents.

4.3. Comparison with Simulation

The mean field analysis predicts $\mathbf{P}[x] = 1/b$ for $x < b$ and $\mathbf{P}[x] = 0$ for $x \geq b$, giving $\langle z \rangle = (b-1)/2$. It is of obvious interest to compare these predictions with values obtained from numerical simulations of the BTW sandpile. The assumption of the mean field is that is that sites are statistically independent. Disagreements between mean field theory and simulations therefore point to the importance of site-to-site correlations.

For the one dimensional sandpile (with $b = K = 2$) the sandpile self-organizes into a state in which almost all of the sites are minimally stable. In the limit of large lattice size, $\mathbf{P}[1] \to 1$, and $\langle z \rangle \to 1$. By contrast, the mean field predicts $\mathbf{P}[0] = \mathbf{P}[1] = 1/2$, and $\langle z \rangle = 1/2$.

A two dimensional square lattice has $b = K = 4$, and the mean field predicts $\mathbf{P}[0] = \mathbf{P}[1] = \mathbf{P}[2] = \mathbf{P}[3] = 1/4$, and $\langle z \rangle = 1.5$. However, precise numerical simulations by Manna [15] find $\mathbf{P} = (0.073, 0.174, 0.307, 0.446) \pm 0.003$ and $\langle z \rangle = 2.124$.

Because the mean field is better at predicting the equilibrium state of a two dimensional system than a one dimensional system, one is led to presume that it will be even better for higher dimensions. It is often the case for critical phenomena that there is an upper critical dimension, beyond which all behavior is independent fo dimension and depends only on coordination number. Indeed, a number of authors have suggested that there is an upper critical dimension for the sandpile [2, 3, 13, 14, 16, 17]. However, these authors are concerned with the critical exponents; I do not know of work which suggests that the individual probabilities $\mathbf{P}$ will be exact for lattices above a critical dimension.

4.4. Time evolution

The mean field analysis not only predicts a self-consistent equilibrium state, but also the time evolution of the system as it evolves toward equilibrium. In comparing the mean field to simulations on the BTW sandpile, however we must first make sure that equivalent notions of "time" are used.

The t in the mean field approximation refers to the short time steps during which the lattice is relaxing. Since h is the probability of seeding a single site at a given short time step, $1/h$ is the average number of short time steps between seedings of a particular site. If there are a total of S sites, then $1/(hS)$ is the average number of short time steps between each seeding of the whole lattice; that is, the length of the long time step. Then, $\tau = hSt$ is time measured in units of long time steps.

In Fig. 1(a), the time evolution of a 10×10 sandpile lattice is simulated, starting with the initial condition of $z(x, y) = 0$ for every site. Plotted are the frequencies of occurrence of site values 0, 1, 2, and 3. The time plotted is number long time steps divided by the lattice size: $\tau/S = \tau/100$.

In Fig. 1(b), the mean field is evolved using the evolution equation Eq. 13 with very small h. Here, the time axis is ht where t is the short time step in Eq. 13. The mean field

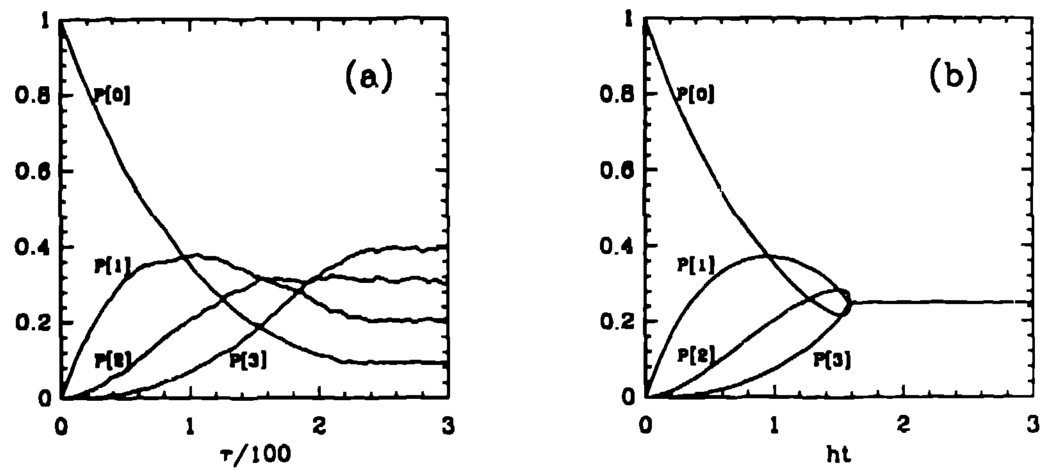

Figure 1: **(a)** An average of 50 simulations of a 10x10 lattice, showing the probabilities $P_t[z]$ for the stable values $z = 0, 1, 2, 3$ as a function of time τ scaled by the area of the lattice $S = 100$. The initial condition was for all sites to be zero. **(b)** Evolution of the mean field equations, with time t scaled by the field parameter h.

model is in good agreement with the simulation for short times, and does reasonably well predict the time required for equilibrium to be reached.

5. Continuous Lattice system of Zhang

A variation of the sandpile model of BTW was proposed by Zhang [2]. This model uses continuous instead of discrete values z and whenever the value at a site exceeds a critical value K, the site relaxes according to the rule

$$\begin{aligned} z(x,y) &\to 0 \\ z(x \pm 1, y) &\to z(x \pm 1, y) + z(x,y)/4 \\ z(x, y \pm 1) &\to z(x, y \pm 1) + z(x,y)/4 \end{aligned} \tag{22}$$

Seeding is done by addition of a random input uniformly chosen in the range $[0, 2K/b]$ (so that the average input seed is equal to K/b which corresponds to the BTW system) at a random site. Zhang notes that, like the BTW sandpile, this system also exhibits self-organized criticality.

Another kind of self-organization is also observed. After a period of evolution, one finds that a histogram of z values exhibits distinct peaks. For a two dimensional square lattice, there are four such peaks, which is just how many distinct states are available to a site in the original BTW sandpile. (See Fig. 2(a)) One of the peaks is a delta function at $z = 0$, and the others are sharp but of finite width.

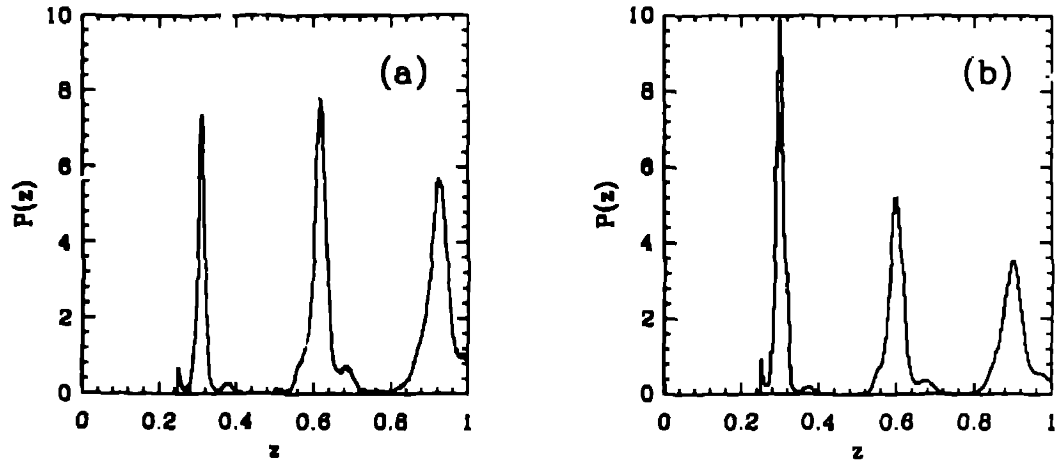

Figure 2: **(a)** Simulation of the Zhang model on a two-dimensional 50x50 square lattice. Not shown is the delta function peak at $z = 0$. **(b)** Mean field approximation for the Zhang model with $e = 0.05$.

While leaving the details to a forthcoming paper [12], I comment that the modified mean field analysis which was used in §4. for the sandpile model is readily adapted to Zhang's continuous model. As Fig. 2(b) shows, the mean field is able to predict the discretization observed in the simulations. As in the case of the BTW sandpile, the details of the distribution of $P(z)$ are only approximately captured. Further experiments [12] indicate that the widths of the peaks can be attributed to the finite size of the lattice; they are made sharper in the mean field approximation by decreasing the edge parameter e.

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