LA--251

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ABSTRACT

Following a suggestion by Kistiakowsky and Peierls (LAMS-188) an analysis has been made of the effect of a finite reaction zone on the velocity of a detonation wave, and in particular of the difference in this phenomenon between plane and expanding waves; this analysis is limited to the asymptotic behavior of the velocity as it increases toward the Chapman-Jouguet limit. In the case of plane waves the velocity defect decreases as $(d/\ell)^2$ where ℓ is the age of the wave and d is the distance behind the front at which the Chapman-Jouguet condition is approximately attained, while for expanding waves the decrease is of the order (d/ℓ). The steady velocity of a wave in a slab of explosive is also found in the limiting case where the thickness is large compared with the reaction zone (the behavior in a finite stick should be similar). In this case the law is $(d/\ell)^4$ where 2ℓ is now the thickness of the slab. Numerical values have been calculated for a particular model of the detonation process, showing that deviations of about 5% are to be expected between plane and spherical waves that have travelled distances of the order of 10 cm; calculations for slightly varied models suggest that the absolute values are not very sensitive to the choice of model.

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EFFECT OF REACTION ZONE ON DETONATION VELOCITY

1. Introduction

The propagation of a symmetric detonation wave in 1, 2, or 3 dimensions is well known in the ideal conditions that the size of the reaction zone is neglected, the Chapman-Jouguet condition is satisfied, and the equation of state of the exploded gases follows a γ-law; solutions have been given by Taylor[1] (these solutions are referred to below as the "ideal" solutions). The phenomena involved in the reaction zone are too complicated and ill understood to permit any exact mathematical treatment; we have attempted here only to find the asymptotic form of the deviations from the ideal solutions in some special problems, namely the variation of detonation velocity with age in a symmetric wave, and the steady velocity of a wave travelling along an unconfined semi-infinite slab (this being the simplest approximation to a stick of explosive).

Our results are based on a particular model of the detonation process, which, we believe, takes account of its significant features, at least in conditions not far from the steady state. This is based on the hypotheses of von Neumann as to the general structure of the wave[2]; according to this theory the detonation front consists of an initial shock in which the pressure is higher (by perhaps 50%) than the normal Chapman-Jouguet pressure, followed by a narrow region in which the reaction takes place and the pressure falls steeply. It is believed that this zone is generally rather less than 1 cm thick.

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1) G. I. Taylor, BM-49, AC-639

2) J. von Neumann, OSRD No. 549; reported in LA-165, "Shock Hydrodynamics".

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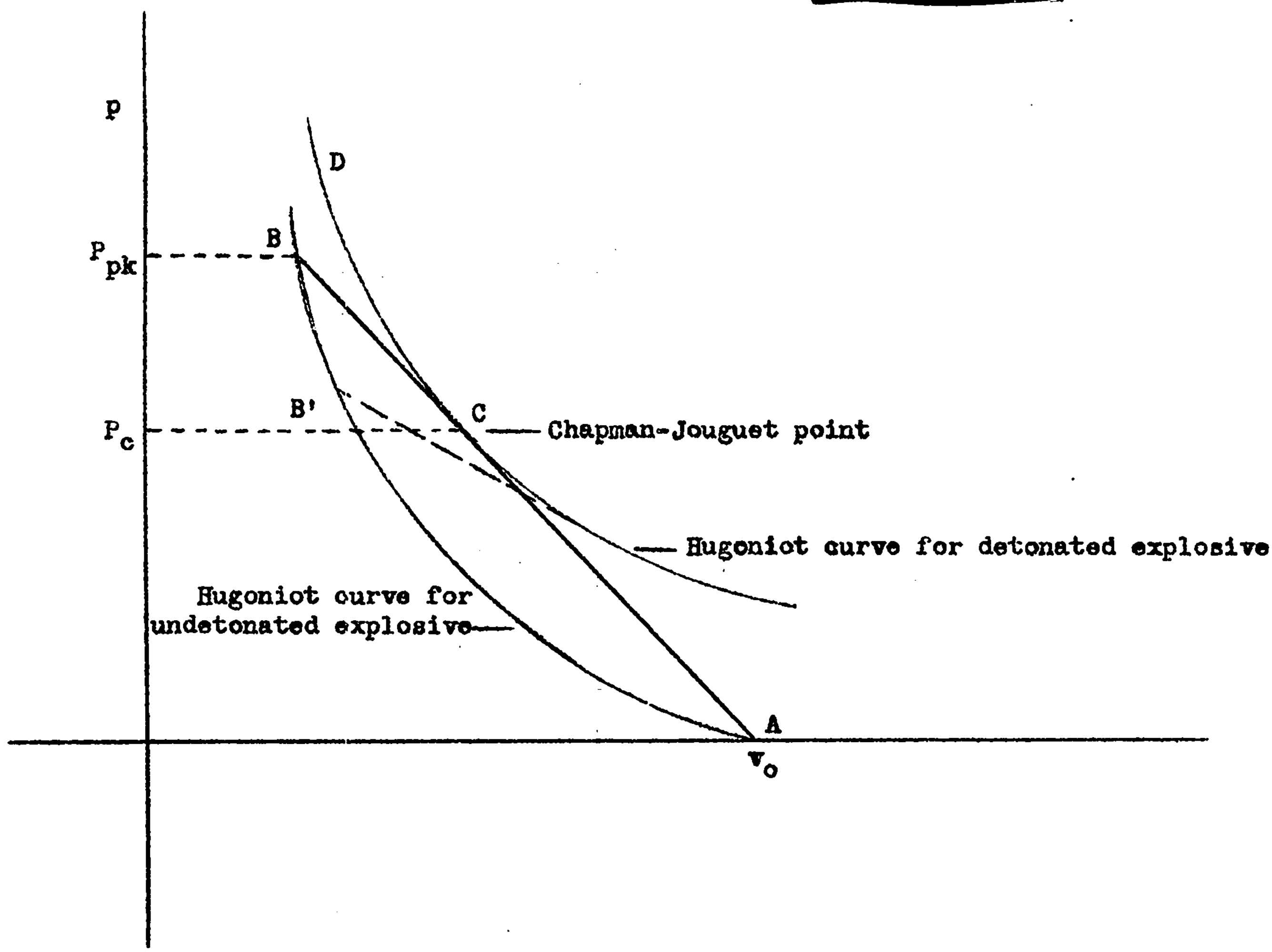

Fig. 1

This is illustrated graphically in the p-v diagram of Fig. 1. The point A represents conditions in the undetonated explosive. The shock wave raises discontinuously the pressure to a value P_{pk} on the curve AB which is the Hugoniot shock curve for the undetonated explosive. The reaction begins and the representative point in this diagram of a particle describes the line BC, C being the "Chapman-Jouguet" point on the Hugoniot curve CD for the fully detonated explosive; the slope of the line ABC is proportional to the square of the detonation velocity. This applies only to the steady state in which the pressure and density are functions only of the distance behind the front; until this is attained the initial shock will raise

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the pressure to a somewhat lower pressure (or higher pressure in the case of "persistence") corresponding to a point B' on the curve AB. From there the path of the representative point will move towards some asymptotic adiabat which will in general be different from that reached in the steady state.

We describe these phenomena mathematically as follows: the asymptotic adiabat for any particle is taken in the form

$$p = \mu(x)\, V^{-\gamma} \tag{1.1}$$

where x is a coordinate labelling the particle under consideration. Before the reaction is completed pressures will be higher than given by this equation. We therefore write

$$p = \mu(x)\left(V^{-\gamma} - \eta(x,t)\right) \tag{1.2}$$

where $\eta(x,t) \longrightarrow 0$ as $t \longrightarrow \infty$. The value of η is a measure of the amount of reaction as yet uncompleted. We now assume that the time variation of η is described by an equation

$$d\eta/dt = -\,2\eta/\tau(p,v) \tag{1.3}$$

where $\tau(p,v)$ is some given function of the pressure and volume. The factor 2 is inserted for convenience in the case when $\tau(p,v)$ is constant. Then $\eta \propto e^{-2t/\tau}$ and this τ corresponds more nearly to the usual definition of the reaction time as that time after which the reaction is essentially complete (in this case $(1-e^{-2}) = 85\%$ complete).

One further assumption is needed to render these equations definite, namely an equation to fix the initial value $\eta_o(x)$ of a particle when it is reached by the shock front. We have here assumed for simplicity that

$$\eta_o(x) = \text{constant} \tag{1.4}$$

but it could be replaced by any relation between $\eta(x)$ and $\mu(x)$. We are now ready to set up the equations of our problem.

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2. Units, Coordinates, and Equations for a Symmetric Wave

We take as unit of density the density of the undetonated explosive; the unit of velocity will be the ultimate detonation velocity (which in the case under consideration, a velocity approached asymptotically from below, will be shown to be the Chapman-Jouguet velocity as we should expect); the unit of length is not specified.

The coordinate x will denote the position of a particle when it is crossed by the shock; at a later time t, this particle "x" has a position y; p, v denote pressure and specific volume. Then in k dimensions ($k = 1, 2, 3$) the equation of motion is

$$\partial^2 y/\partial x^2 = - (y/x)^{k-1}\, \partial p/\partial x \tag{2.1}$$

and the equation of conservation

$$(y/x)^{k-1}\, \partial y/\partial x = v \tag{2.2}$$

These two together with (1.2), (1.3) are four equations for y, p, v, η as function of x, t. To these must be added some boundary conditions; these we impose only at the shock front, for the rest we are interested in investigating only solutions asymptotic to the "ideal" Taylor solutions. We have already assumed (1.4); in addition we have the Hugoniot conditions at the shock. The path of the shock will be given by

$$x = f(t) \tag{2.3}$$

Then

$$y[f(t), t] = f(t) \tag{2.4}$$

$$p_s = (1 - v_s)[f'(t)]^2 \tag{2.5}$$

$$H(p_s, v_s) = 0 \tag{2.6}$$

Here $p_s = p(t,t)$, $v_s = v(t,t)$ the values of p, v immediately behind the shock and $H(p,v) = 0$ is the equation of the Hugoniot curve AB of Fig. 1.

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We shall find it convenient to change the independent variable x so that the path of the shock is fixed in our coordinates; accordingly we define $s(x)$ by the equation

$$x = f(s) \tag{2.7}$$

where $f(s)$ is as yet an unknown function. We now rewrite our equations taking (s,t) as independent variables; the equations are

$$f'(s)\,\frac{\partial^2 y}{\partial t^2} = -\left(\frac{y}{f(s)}\right)^{k-1}\frac{\partial p}{\partial s} \tag{2.8}$$

$$\left(\frac{y}{f(s)}\right)^{k-1}\frac{\partial y}{\partial s} = vf'(s) \tag{2.9}$$

$$\partial\eta/\partial t = -2\eta/\tau(p,v) \tag{2.10}$$

$$y(t,t) = f(t) \tag{2.11}$$

$$p_s = (1-v_s)\left[f'(s)\right]^2 \tag{2.12}$$

$$H(p_s,v_s) = 0 \tag{2.13}$$

$$\eta_o(s) = \eta_o = \text{constant} \tag{2.14}$$

$$\mu(s) = (v_s^{-\gamma} - \eta_o)^{-1} \cdot p_s \tag{2.15}$$

$$p = \mu(s)\left(v^{-\gamma} - \eta(s,t)\right) \tag{2.16}$$

3. A Transformation of the Equations

We shall rearrange and combine (2.8) to (2.16) in the form of an integro-differential equation for v; thereby we absorb most of the boundary conditions and have the equations in a form suitable for an iterative method of solution.

From (2.9) and (2.12)

$$y(s,t) = f(t) - \int_s^t v(\omega,t)\,\rho(\omega,t)\,f'(\omega)\,d\omega \tag{3.1}$$

where $\rho(s,t)$ denotes the "weight factor" $\left[f(s)/y(s,t)\right]^{k-1}$. Therefore

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$$\partial y(s,t)/\partial t = f'(t)\left[1-v_s\right] - \int_s^t (\partial \rho v/\partial t) \quad f'(\omega)\, d\omega \tag{3.2}$$

and

$$\frac{\partial^2 y(s,t)}{\partial t^2} = f''(t)\left[1-v_s\right] - f'(t)\,\frac{dv_s}{dt} - \left\{\left(\frac{\partial v}{\partial t}\right)_s f'(t) - v_s\, f'(t) \cdot \frac{(k-1)(\partial y/\partial t)_s}{f(t)}\right\}$$
$$- \int_s^t \frac{\partial^2(\rho v)}{\partial t^2}\, f'(\omega)\, d\omega \tag{3.3}$$

Now $\left(\frac{\partial y}{\partial t}\right)_s = f'(t)\left[1-v_s\right]$ $\left(\text{from } (3.2)\right)$ (3.4)

and

$$\left(\frac{\partial v}{\partial t}\right)_s = \frac{dv_s}{dt} - \left(\frac{\partial v}{\partial s}\right)_s \tag{3.5}$$

So

$$\frac{\partial^2 y(s,t)}{\partial t^2} = \left[f''(t)(1-v_s) - 2f'(t)\,\frac{dv_s}{dt} + \frac{(k-1)\left(f'(t)\right)^2}{f(t)}\, v_s(1-v_s)\right]$$
$$+ \left(\frac{\partial v}{\partial s}\right)_s f'(t) - \int_s^t \frac{\partial^2(\rho v)}{\partial t^2}\, f'(\omega)\, d\omega \tag{3.6}$$

Also from (2.8),

$$p(s,t) = p_s + \int_s^t \rho(\omega,t)\,\frac{\partial^2 y}{\partial t^2}\, f'(\omega)\, d\omega \tag{3.7}$$

Combining (3.6), (3.7), we get

$$p(s,t) = p_s + \left\{\left[f''(t)(1-v_s) - 2f'(t)\frac{dv_s}{dt}\right] + \left[\frac{(k-1)[f'(t)]^2\, v_s(1-v_s)}{f(t)} + \left(\frac{\partial v}{\partial s}\right)_s f'(t)\right]\right\}\int_s^t \rho f'(\omega) d\omega$$
$$- \int_s^t \frac{\partial^2(\rho v)}{\partial t^2}\, f'(\omega)\left(\int_s^\omega \rho f'(\varpi)\, d\varpi\right) d\omega \tag{3.8}$$

Consider now

$$\int_s^t \frac{\partial^2(\rho v)}{\partial \omega^2}\, f'(\omega)\left(\int_s^\omega \rho f'(\varpi)\, d\varpi\right) d\omega =$$

$$= \left[\frac{\partial(\rho v)}{\partial \omega}\right]_s f'(t) \int_s^t (\rho f')\, d\omega + \int_s^t \frac{\partial(\rho v)}{\partial \omega}\left[1-\left(f'(\omega)\right)^2 - f''(\omega) \int_s^\omega \rho f'(\varpi)\, d\varpi\right] d\omega \tag{3.9}$$
$$+ \int_s^t \frac{\partial(\rho v)}{\partial \omega}\,(1-\rho)\left[f'(\omega)\right]^2 d\omega - v_s + \rho(s,t)\, v(s,t)$$

If we now add (3.8), (3.9) and substitute for p from (2.10) and (2.11) we get the desired equation,

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$$
\begin{aligned}
\mu(s)\, v^{-\gamma} &+ v(s,t) - 1 \\
&= \mu(s)\, \eta_0 \left\{ \exp\left(- \int_s^t \frac{2dt}{\eta(p,v)} \right) \right\} + \left\{ p_s + v_s - 1 \right\} + v(s,t) \left[1 - \rho(s,t) \right] \\
&+ \left[f''(t)(1-v_s) - 2f'(t) \frac{dv_s}{dt} \right] \int_s^t \rho f'(\omega)\, d\omega \\
&- \int_s^t \frac{\partial(\rho v)}{\partial \omega} \left[1 - \left(f'(\omega) \right)^2 - f''(\omega) \int_s^\omega \rho f'(\bar{\omega})\, d\bar{\omega} \right] d\omega \qquad (3.10) \\
&- \int_s^t \frac{\partial(\partial v)}{\partial \omega} (1-\rho) \left(f'(\omega) \right)^2 d\omega \\
&- \int_s^t \left[\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \omega^2} \right) \rho v \right] f'(\omega) \left(\int_s^\omega \rho f'(\bar{\omega})\, d\bar{\omega} \right) d\omega
\end{aligned}
$$

This formula simplifies considerably in the plane case when $\rho \equiv 1$. The boundary conditions remaining so that a solution of (3.10) may be a solution of our problem are (2.13), (2.14), (2.16) which determine p_s, v_s and $\mu(s)$ in terms of the unknown function $f(s)$.

4. Stationary Solution in Plane Case

A stationary solution is one for which $f'(s) = 1$ and v is a function only of $(t-s)$; this can only exist for $k = 1$ and then since $\partial^2 y/\partial t^2 = \partial^2 y/\partial s^2$, (3.10) shows that

$$p + v = 1 \qquad (4.1)$$

and therefore

$$\mu_0\, v^{-\gamma} + v - 1 = \eta_0 \exp\left(- \int_s^t \frac{dt}{\tau(1-v,v)} \right) \qquad (4.2)$$

This is an ordinary differential equation for v which may be solved numerically for any given $\tau(p,v)$.

5. Similarity Solutions in Ideal Case

a) Plane case: When $k = 1$ the ideal solution has a simple analytic representation. In our notation,

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$$\left.\begin{aligned} y &= \frac{\gamma}{\gamma+1}\, s^{1-2/(\gamma+1)}\, t^{2/(\gamma+1)} - \frac{t}{\gamma-1} \\ p &= \frac{1}{\gamma+1}\left(\frac{s}{t}\right)^{\gamma-2\gamma/(\gamma+1)} \\ v &= \frac{\gamma}{\gamma+1}\left(\frac{s}{t}\right)^{2/(\gamma+1)-1} \end{aligned}\right\} \qquad (5.1)$$

with

$$\mu_0 = p_s\, v_s^{\,3} = \gamma^{\gamma}/(\gamma+1)^{\gamma+1} \qquad (5.2)$$

b) Cylindrical and Spherical Cases: For $k=2,\ 3$ no simple analytic representation is possible; the solutions have been determined numerically by Taylor[3]. These differ essentially from the plane solution in that there is an infinite pressure gradient immediately behind the shock front; the following expansions have been determined in the case $\gamma=3$,

k = 2(cyl)

$$\left.\begin{aligned} p &= 1/4 - 3/8\sqrt{2}\ \theta - 5/64\ \theta^2 - \dots \\ v &= 3/4 + 3/8\sqrt{2}\ \theta + 17/64\ \theta^2 + \dots \end{aligned}\right\} \qquad (5.3)$$

k = 3(spl)

$$\left.\begin{aligned} p &= 1/4 - 3/8\ \theta - 3/16\ \theta^2 - \dots \\ v &= 3/4 + 3/8\ \theta + 9/16\ \theta^2 + \dots \end{aligned}\right\} \qquad (5.4)$$

Here θ denotes $+\sqrt{1-s/t}$.

6. Perturbation of Ideal Solution (Plane Case)

If we suppose there is a small disturbance y_1, p_1, v_1 in the values of y, p, v given by (5.1) and substitute these values into our equations we find

$$p_1 = -(\gamma p/v)\, v_1 = -(s/t)^{\gamma-1}\, v_1 \qquad (6.1)$$

Consequently at the front $p_1 + v_1 = 0$ and there is no first order change in velocity. Therefore we may put $f' = 1$ in (2.8), (2.9) and eliminating p_1 v_1 we arrive at the differential equation for y_1,

$$\frac{\partial^2 y_1}{\partial t^2} - \frac{\partial}{\partial s}\left[\left(\frac{s}{t}\right)^{\gamma-1}\frac{\partial y_1}{\partial s}\right] = 0 \qquad (6.2)$$

3) G. I. Taylor, ibid, and [illegible]

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For the case $\gamma = 3$ this may be solved immediately, having the general solution

$$y_1 = \phi(s/t) + t\psi(st) \qquad (6.3)$$

where ϕ, ψ are arbitrary functions. The condition (2.12) eliminates the second term. Consequently

$$\left.\begin{aligned} v_1 &= \frac{1}{t}\,\phi'(s/t) \\ p_1 &= -\frac{s^2}{t^3}\,\phi'(s/t) \end{aligned}\right\} \qquad (6.4)$$

The consequent change in detonation velocity is

$$\frac{c[\phi'(1)]^2}{t^2} \qquad (6.5)$$

where

$$c = \left(2\,\frac{\partial^2 H}{\partial p \partial v} - \frac{\partial^2 H}{\partial p^2} - \frac{\partial^2 H}{\partial v^2}\right) \Bigg/ \frac{\partial H}{\partial p} \qquad (6.6)$$

$H(p,v)$ is the Hugoniot curve. The solution in higher orders is completely determined by the one arbitrary function $\phi(s/t)$, as we might expect.

The important point is that p_1, v_1 die away as $1/t$ and this may be expected to be true for all γ.

7. Perturbation Caused by Reaction Zone in Plane Case

Our objective is to find the asymptotic form of the perturbation in velocity for a wave of finite age. We wish to find a solution p,v of our equations which

a) approximates to the solution (5.1) at a distance from the front at late times.

b) Approaches the stationary solution (4.2) in the immediate neighborhood of the shock.

Consider now Eq. (3.10). In view of (b) and the ideal solution, we can be sure that $(\partial^2/\partial t^2 - \partial^2/\partial s^2)v$ will be of small order near the front. If we insert the ideal solution in this term we get $(p + v - 1)_{id}$. For small deviations we have

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$$(p_s + v_s - 1) = - 2(1-v_s)\ g(t) \tag{7.1}$$

where $g(t) = 1 - f'(t)$ is the change in velocity. The third and sixth terms in (3.10) vanish identically, and the fourth is of order $(t-s)\ g'(t)$ and may therefore be neglected in comparison with $g(t)$ when we are concerned only with values of $(t-s)$ that are $o(t)$.

In the fifth term the bracket is approximately equal $2g(\omega)$, and so long as $(t-s) = o(t)$ we may write this term as

$$- 2g(t) \int_s^t \frac{\partial(\rho v)}{\partial \omega}\, d\omega = - 2g(t)\left[v_s - v(s,t)\right] \tag{7.2}$$

If now we make these simplifications (3.10) becomes

$$\mu(s)\ v^{-\gamma} + v(s,t) - 1 = \mu(s)\ \eta(s,t) - 2g(t)\left[1 - v(s,t)\right] + \left[p + v - 1\right]_{id} \tag{7.3}$$

Forget about the second term; then for $(t-s) \ll t$, (7.3) is essentially the equation for the stationary case. Again if $t = s \gg t_0$ the last term dominates and a solution is obviously $p = p_{id}$, $v = v_{id}$.

<u>However</u>, this is not sufficient since the left of (7.3) has a minimum at $v = v_c$, say, for which its value is $\left[(\gamma+1)/\gamma\right] v_c - 1$. The right-hand side must therefore pass through this value <u>and</u> also have a zero derivative with respect to t at constant s, at this same point; else we shall be on the wrong "branch", and the solution will not approach the ideal solution. If $g(t)$ be so chosen that these conditions are satisfied then the solution of (7.3) as an algebraic equation in v is our first approximation to the solution.

Now $\mu(s)$ is defined by (2.16). From (2.13), (2.14) we have

$$\delta p + \delta v = - 2(1-v_s)\ g(t)$$
$$\delta p + (- \partial p/\partial v)_H\ \delta v = 0 \tag{7.4}$$

So

$$\frac{\delta p}{- (- \partial p/\partial v)_H} = \frac{\delta v}{1} = \frac{2(1-v_s)\ g(t)}{(- \partial p/\partial v)_H - 1} \tag{7.5}$$

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Therefore from (2.16)

$$\frac{\delta\mu}{\mu_o} = \frac{\delta p}{p_s} + \frac{\gamma v_s^{-\gamma+1}\,\delta v}{v_s^{-\gamma} - \eta_o} = -2rg(s) \qquad (7.6)$$

where

$$r = \frac{|\partial p/\partial v|_H - |\partial p/\partial v|_s}{|\partial p/\partial v|_H - 1} \qquad (7.7)$$

($|\partial p/\partial v|_s$ is the slope of the adiabat through p_s, v_s). Now v_c is defined by $v_c^{\gamma+1} = \gamma\,\mu(s)$. So

$$\frac{\delta v_c}{v_c} = \frac{1}{\gamma+1}\,\frac{\delta\mu}{\mu} = -\frac{2rg(s)}{\gamma+1}$$

Therefore the minimal value of the left side of (7.3) is

$$\frac{\gamma+1}{\gamma}\,v_c - 1 = \frac{\gamma+1}{\gamma}\,\delta v = -\frac{2rg(s)}{\gamma+1} \qquad (7.8)$$

We are interested here in the case of velocities approached asymptotically from below, i.e., $g(t) > 0$. Therefore the minimum value of (7.3) is <u>negative</u>; the first and last terms are positive and $g(t)$ must supply the difference. Clearly as $t \longrightarrow \infty$ this minimum must tend to zero i.e., $v_c \longrightarrow \gamma/(\gamma+1)$, the Chapman-Jouguet value. (If $g(t) < 0$ then since $r < 1$, Eq. (7.3) obviously cannot be satisfied at the minimum; that is, the velocity can only exceed the C-J value if it is maintained by an artificial high-pressure backing.)

Denote the right of (7.3) by $Q(s,t)$. Then for a certain $s = s_0(t)$ we demand that

$$\left.\begin{aligned} -\frac{2rg(s)}{\gamma+1} &= Q(s,t) \\ 0 &= \frac{\partial Q(s,t)}{\partial t} \end{aligned}\right\} \qquad (7.9)$$

These equations suffice to determine the asymptotic form of $g(t)$. Clearly we may put $\mu(s) = \mu_o$ in Q. If we differentiate the first of Eqs. (7.9) totally with respect to t and subtract the second it follows that

$$-\frac{2rg'(s)}{\gamma+1}\,\frac{ds}{dt} = \frac{\partial Q}{\partial s}\cdot\frac{ds}{dt} \qquad (7.10)$$

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or

$$\mu_0 \frac{\partial \eta}{\partial s} + 2g(t)\left(\frac{\partial v}{\partial s}\right) + \frac{\partial}{\partial s}\left[p + v - 1\right]_{id} + \frac{2rg'(s)}{\gamma + 1} = 0 \qquad (7.11)$$

Now obviously if $t - s = o(t)$ the 2nd and 4th terms are of lower order than the first and third. Thus the function $s(t)$ is defined by

$$\mu_0 \frac{\partial \eta(s,t)}{\partial s} + \frac{\partial}{\partial s}(p + v - 1)_{id} = 0 \qquad (7.12)$$

Then we find

$$\frac{\partial(1-r)}{\gamma+1} g(t) = \mu_0 \, \eta(s,t) + \left[p \quad v - 1\right]_{id} \qquad (7.13)$$

$\left(\text{since } g(s) \approx g(t)\right)$.

If we introduce a perturbation (of the type considered in 6) into our "ideal" solution we shall introduce terms of higher order into the term $(\partial^2 v/\partial t^2 - \partial^2 v/\partial s^2)$; therefore the velocity etc. will only be affected in a higher order. The defect in velocity (7.13) is <u>independent</u> (asymptotically) of initial conditions (unless we are concerned with "persistence" effects, $g < 0$).

The procedure outlined above clearly supplies an iterative method for determining higher-order effects; in the next approximation, we should have to include terms arising from the departure of $\zeta(p,v)$ from its stationary value, and neglected terms in (3.10).

8. Perturbation of Expanding Waves Due to Reaction Zone

The treatment in the cases $k = 2, 3$ is essentially the same but there are some differences due to the extra terms in (3.10) and their different relative orders of magnitude. Of the terms on the right of (3.10) the second and fourth are treated as before; the fifth will be

$$-2g(t)\int_s^t \frac{\partial(\rho v)}{\partial \omega}\, d\omega = -2g(t)\left[v_s - \partial(s,t)\, v(s,t)\right] \qquad (8.1)$$

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Consider the third term; $y(s,t) = f(t) - \bar{v}(t-s)$ where $\bar{v}$ is some average value of v over the region between (s,t) and the shock.

Therefore

$$\left(1 - \rho(s,t)\right) = \left\{1 - \left[\frac{f(s)}{f(t) - \bar{v}(t-s)}\right]^{k-1}\right\} \approx \frac{(k-1)(1-\bar{v})(t-s)}{t}$$

because of the exponential character of the similarity solution we may replace $\bar{v}$ by v_c with small error. So the second term is

$$\left[(k-1)v(1-v)(t-s)\right]/t \qquad (8.2)$$

The last two terms remain; if in the last term we put the similarity solution it becomes of order $t^{-3/2}$; in the stationary solution it is zero, so in this case we may neglect the last term altogether in first approximation. The sixth term is also of order $t^{-3/2}$ in the ideal solution; but it does not vanish for the stationary solution, for which however most of its value arises from immediately behind the shock, and since conditions there will be almost stationary at late times we may insert the stationary solution into this term.

We have in place of (7.3),

$$\begin{aligned} \mu(s)\, v^{-\gamma} + v - 1 &= Q(s,t) \\ &= \mu(s)\, \eta(s,t) - 2g(t)\left[1 - \rho v(s,t)\right] + v(s,t)\left[1 - \rho(s,t)\right] \qquad (8.3) \\ &\quad - \int_s^t \left[\frac{\partial(\rho v)}{\partial \omega}(1-\rho)\right]_{sty} d\omega \end{aligned}$$

The same argument applies about the Chapman-Jouguet velocity; and Eq. (7.10) yields in the same approximations as (7.12)

$$\mu_o \frac{\partial\eta(s,t)}{\partial s} - v(s,t)\frac{\partial\rho(s,t)}{\partial s} = 0 \qquad (8.4)$$

Now

$$\frac{\partial\rho(s,t)}{\partial s} = (k-1)\,\rho\left[\frac{f'(s)}{f(s)} - \frac{\rho v(s,t)}{y(s,t)}\right]$$

Therefore we have

$$\mu_o\, \partial\eta/\partial s = \left[(k-1)\, v_c(1-v_c)\right]/t \qquad (8.5)$$

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and also

$$\frac{2(1-r)}{\gamma+1}\, g(t) = \mu_0\, \eta(s,t) + v_c(1-\rho_c) - \int_{s_c}^{t} \left[\frac{\partial(\rho v)}{\partial \omega}\,(1-\rho)\right]_{sty} d\omega \tag{8.6}$$

9. Numerical Evaluation

a) Consider first the case when $\gamma = 3$, τ is constant and $\eta_o = 1$, so that the peak pressure is about 64% greater than the Chapman-Jouguet pressure. Then in the plane case, (5.1) gives, for $(t-s) \ll t$,

$$\left[p + v - 1\right]_{id} = \frac{3}{8}\left(\frac{t-s}{t}\right)^2 \tag{9.1}$$

while

$$\eta = e^{2(s-t)/\tau} \tag{9.2}$$

The Eq. (7.12) is then

$$\frac{27}{128}\, e^{-z} = \frac{3}{8}\, \frac{z}{(t/\tau)^2} \tag{9.3}$$

defining $z = 2(t-s)/\tau$ as a function of t/τ and

$$g(t) = \frac{g_o'(t)}{1-r} = \frac{\left|\partial p/\partial v\right|_H - 1}{\left|\partial p/\partial v\right|_s - 1}\, g_o'(t) \tag{9.4}$$

where

$$g_o'(t) = \frac{3/8\ z + 3/16\ z^2}{(t/\tau)^2} \tag{9.5}$$

b) Now consider the cases $k = 2, 3$; the Eq. (8.5) gives

$$\frac{27}{128}\, e^{-z} = \frac{3(k-1)}{16} \cdot \frac{\tau}{t} \tag{9.6}$$

in place of (9.3). To find g(t) we have to estimate the integral in (8.6). Since all variables are functions of (t-s) we have

$$-\int_{s}^{t} \frac{\partial(\rho v)}{\partial \omega}\,(1-\rho)\, d\omega \approx \int_0^{\infty} \frac{\partial v}{\partial z}\left(1 - \rho(z)\right) dz \tag{9.7}$$

In our case the 2nd integral may be evaluated numerically; a rough estimate, which is sufficient here, gives a value

$$0.11(k-1)\ \tau/2t \tag{9.8}$$

[redacted]

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 [redacted]

Consequently we find analogous to (9.5)

$$g_o^k(t) = \frac{3(k-1)}{16}\left(\frac{\tau}{t} + \frac{z\tau}{t}\right) + \frac{0.11(k-1)\tau}{2t} \qquad (9.9)$$

The functions $g_o^k(t)$ for $k = 1,2,3$ are shown in Fig. 2 which gives the velocity $(1-g)$ as a function of the age of the wave in units of the reaction zone length.

c) It is clear from the above that the deviation $g_o(t)$ is of the order $(d/\ell)^n$ ($n = 2$, plane case, $n = 1$, expanding cases), where ℓ is the age of the wave and d is the distance behind the front at which the Chapman-Jouguot conditions are approximately fulfilled. The fact that $d \sim \tau \ln t/\tau$ is a consequence of our particular model; if the reaction were complete in a finite time then d would tend to a constant. We can consider some variations of our model to see how sensitive to it the numerical values are.

First let $\gamma = 2.5$ instead of $\gamma = 3$. Then

$$\left[p + v - 1\right]_{id} = 0.230\left(\frac{t-s}{\tau}\right)^2 \qquad (9.10)$$

So (9.3) becomes

$$\frac{27}{128}\, e^{-z} = 0.230\, z/(t/\tau)^2 \qquad (9.11)$$

and instead of $g_o'(t)$

$$g_a(t) = 0.115\,(2z + z^2)/(t/\tau)^2 \qquad (9.12)$$

Next suppose $\eta_o = 0.32$ and $\gamma = 3$, corresponding to an initial overpressure of 40%; this leads to the equations

$$.0675\, e^{-z} = (3/8) \cdot z/(t/\tau)^2 \qquad (9.13)$$

and

$$g_b(t) = \left[(3/8)\, z + (3/16)\, z^2\right](t/\tau)^2 \qquad (9.14)$$

Finally consider a model in which the pressure gradient is more steep immediately behind the shock, such as

$$[illegible] \left(\sqrt{1+(t-s)/\tau}\right) \qquad (9.15)$$

[redacted]

(the factor 4 is inserted to make more comparable the values of τ in (9.16) and (9.2)). This leads to the equations

$$\frac{27}{64}\frac{e^{-\sqrt{z}}}{2\sqrt{z}} = \frac{3}{16}\frac{z}{(t/\tau)^2} \qquad (9.16)$$

and

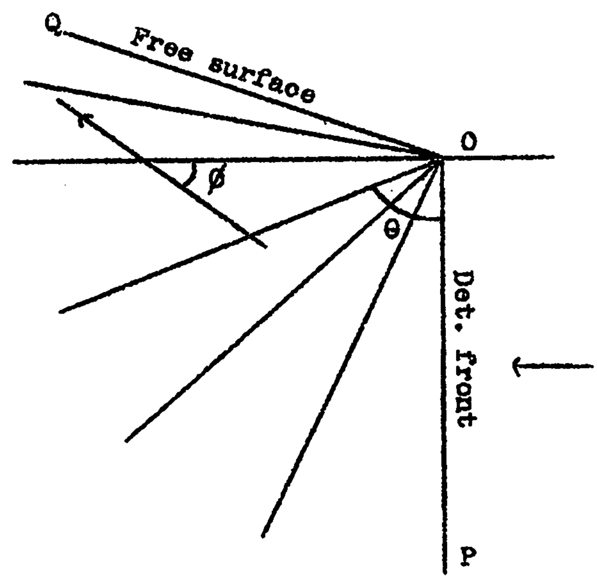
$$g_c(t) = \frac{3}{64}\left(\frac{4z^{3/2}+z^2}{(t/\tau)^2}\right) \qquad (9.17)$$

The functions g_o', g_a, g_b, g_c are given for comparison in Fig. 3. Asymptotically g_c is of higher order than the others but the absolute differences are not great; these results suggest that the absolute value of the velocity deviation is not very sensitive to the model chosen (for variations of the kind considered above).

d) With regard to the factor $1/(1-r)$ occurring in (9.4), it is difficult to make any estimate, but it seems likely that the adiabatic and Hugoniot curves will be closer than either is to the slope line ABC of Fig. 1. That is, it is likely that this factor is not much greater than unity.

10. Ideal Solution for Semi-Infinite Slab

Consider a plane detonation-wave of infinite age travelling parallel to the free surface of a semi-infinite slab of explosive. Take a coordinate system in which the detonation front OP is stationary and explosive advances toward it with velocity D.



Then the solution is known to be a Prandtl-Meier expansion about the corner O. It is convenient to use the notation of the theory of characteristics developed by Fuchs[4]. One system of characteristics (+)

4) K. Fuchs in LA-165 (section 14), Shock Hydrodynamics.

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are the rays through O; if such a ray makes an angle ϕ with the detonation front then it carries a constant value of

$$\alpha_+ \equiv \phi + F(\mu) = 2\Phi(\theta) - \pi/2 \qquad (10.1)$$

where ϕ is the angle the velocity makes with the normal to the D-front, μ is the Mach angle ($\sin \mu = c/u$), and (in the case $\gamma = 3$)

$$F(\mu) = -\mu - \sqrt{2} \tan^{-1}\left(\frac{\cot \mu}{\sqrt{2}}\right) \qquad (10.2)$$

The values of α_- are everywhere the same

$$\alpha_- \equiv \phi - F(\mu) = \pi/2 \qquad (10.3)$$

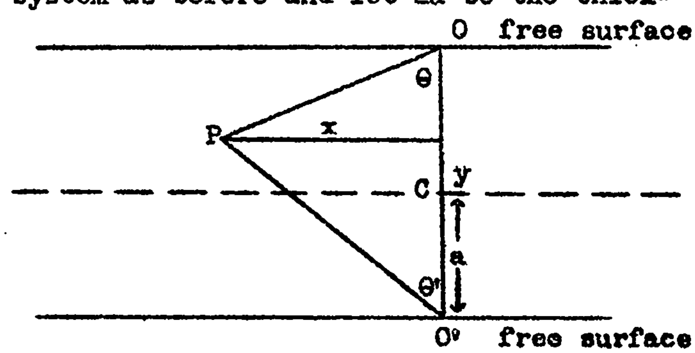
The function $\Phi(\theta)$ is

$$\Phi(\theta) = -\theta + \tan^{-1}\left(\sqrt{2} \tan \frac{\theta}{\sqrt{2}}\right) \qquad (10.4)$$

11. Ideal Solution for Finite Slab

It is not possible to find a simple analytic representation for the solution of this problem; however, all we require for the application of our previous methods is an expansion near the D-front and near the central plane of the slab. This we find as follows.

Take the stationary coordinate system as before and let 2a be the thickness of the slab; take a plane section normal to the surfaces so that OO' represents the D-front. Let P be some other point in this section, and OP, O'P makes angles ϕ, ϕ' with the D-front.



Near O, O' the solution must be the same as for a semi-infinite slab; thus from O there start + characteristics carrying values

$$a_+ = 2\Phi(\theta) - \pi/2 \qquad (11.1)$$

[redacted]

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Similarly from O' these start - characteristics carrying values

$$\alpha_- = \pi/2 - 2\Phi(\Theta') \tag{11.2}$$

Near O, $\Theta' \longrightarrow 0$ and $\alpha_- \longrightarrow \pi/2$; therefore the complete solution of the problem could be found by an iterative procedure as follows,

1) For any point P, PO, PO' define Θ, Θ'; and let at P,

$$\alpha_+ = \alpha_+(\Theta),\ \alpha_- = \alpha_-(\Theta').$$

2) With this as first approximation we may calculate the slope of the characteristics through P.

3) Use this to find the deviations of the characteristics from straight lines and so to find corrected angles Θ, Θ' belonging to an arbitrary point.

We shall carry out this procedure near the point C. With coordinates x, y measured from C as shown,

$$\tan\Theta = x/(a-y),\ \tan\Theta' = x/(a+y) \tag{11.3}$$

Near C, Θ, Θ' are small, so approximately

$$\left.\begin{aligned}(\alpha_+)_p &= -\pi/2 - 1/3\,\Theta^3 \ldots\\ (\alpha_-)_p &= \pi/2 + 1/3\,\Theta'^3 \ldots\end{aligned}\right\} \tag{11.4}$$

So at P,

$$F(\mu) = -\pi/2 - 1/6(\Theta^3 + \Theta'^3) \tag{11.5}$$

and

$$\mu = \pi/2 - \left[\Theta^3 + \Theta'^3\right]^{1/3} \tag{11.6}$$

also

$$\phi = 1/6(\Theta'^3 - \Theta^3) \tag{11.7}$$

Consider a fixed small Θ; as P varies from O to the central line, Θ' goes from O to Θ. Since along a characteristic,

$$dy/dx = \pi/2 \mp (\phi \pm \mu) \tag{11.8}$$

and since Θ, Θ' are small we have approximately that along the Θ-characteristic

[redacted]

 [redacted]

$$\frac{dx}{d\xi} = a\left[\left(\frac{x}{a\xi}\right)^3 + \left(\frac{x}{(2-\xi)a}\right)^3\right]^{1/3} \qquad (11.9)$$

where

$$a - y = a\xi \qquad (11.10)$$

So that

$$x = \theta(a-y)\exp\int_0^{1-y/a}\left\{\left[1+\left(\frac{\xi}{2-\xi}\right)^3\right]^{1/3} - 1\right\}\frac{d\xi}{\xi} \qquad (11.11)$$

and on the central line y = 0

$$\theta = 0.942\ x/a \qquad (11.12)$$

Now by Bernoulli's theorem

$$c^2 + u^2 = 9/8$$

so that

$$c^2 = \frac{9}{16(1+\frac{1}{2}\cot^2\psi)}$$

or approximately

$$c = (3/4)(1-\theta^2/2^{4/3}) \qquad (11.13)$$

on the line y = 0. This gives

$$p + v - 1 = \frac{3.\theta^4}{8.2^{2/3}} = 0.186\left(\frac{x}{a}\right)^4 \qquad (11.14)$$

12. Effect of Reaction Zone for Finite Slab

1) In Cartesian coordinates the hydrodynamical equations of a steady-state motion are

$$\left.\begin{aligned} \underline{u}\cdot\frac{\partial \underline{u}}{\partial s} &= -\frac{1}{\rho}\frac{\partial p}{\partial s} \\ \mathrm{div}\,(\rho\underline{u}) &= 0 \end{aligned}\right\} \qquad (12.1)$$

Let us introduce coordinates z, t into our problem where z is defined by

$$(\rho u_x) = (\partial z/\partial y)_x;\ \rho u_y = (\partial z/\partial x)_y \qquad (12.2)$$

so that at the shock front

$$(\partial z/\partial y)_{shock} = \text{constant} = \text{shock velocity at center} \qquad (12.3)$$

[redacted]

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and t is the time since a particle crossed the D-front. Since $z_x/z_y = -u_y/u_x$ z is constant along a stream-line.

If we now treat z, t as <u>independent</u> variables and $x, y, p, v,$ as dependent we find the relations

$$v = \frac{\partial(x,y)}{\partial(t,z)} \qquad (12.4)$$

$$\left.\begin{aligned} \left(\frac{\partial^2 x}{\partial t^2}\right)_z &= \left(\frac{\partial y}{\partial t}\right)_z \left(\frac{\partial p}{\partial z}\right)_t - \left(\frac{\partial y}{\partial z}\right)_t \left(\frac{\partial p}{\partial t}\right)_z \\ \left(\frac{\partial^2 y}{\partial t^2}\right)_z &= -\left(\frac{\partial x}{\partial t}\right)_z \left(\frac{\partial p}{\partial z}\right)_t + \left(\frac{\partial x}{\partial z}\right)_t \left(\frac{\partial p}{\partial t}\right)_z \end{aligned}\right\} \qquad (12.5)$$

From these we get

$$\frac{\partial}{\partial t}(p+v) = (y_t\, y_z + x_t\, x_z)\frac{\partial p}{\partial z} + (x_t\, y_{zt} - y_t\, x_{zt}) - (x_z^{\,2} + y_z^{\,2} - 1)\frac{\partial p}{\partial t} \qquad (12.6)$$

Here subscripts denote partial derivatives.

2) Now for the ideal solution the shock velocity = 1 for all y, so we may take $z = y$ at the D-front. Then x, y, p, v are known functions X, Y, P, V of $z, t,$ satisfying (12.6).

When we take into account the reaction zone let the steady state velocity be S so that $z = Sy$ at the shock front. Then the solution to which we must approximate behind the front is $X^* = X(z/s,t), \ldots\ldots$ First since $X, Y, \ldots\ldots$ satisfy (12.6)

$$\begin{aligned} \frac{\partial}{\partial t}(P^* + V^*) &= S^2(Y_t^*\, Y_z^* + X_t^*\, X_z^*)\frac{\partial p^*}{\partial z} + S(X_t^*\, Y_{zt}^* - Y_t^*\, X_{zt}^*) \\ &\quad - \left\{S(X_z^{*2} + Y_z^{*2}) - 1\right\}\frac{\partial p^*}{\partial t} \end{aligned} \qquad (12.7)$$

Then as before we substitute the approximate solution P^*, V^* on the right of (12.6) getting

$$\begin{aligned} \frac{\partial}{\partial t}(p+v) &= (Y_t^*\, Y_z^* + X_t^*\, X_z^*)\frac{\partial p^*}{\partial z} + (X_t^*\, Y_{zt}^* - Y_t^*\, X_{zt}^*) \\ &\quad - (X_z^{*2} + Y_z^{*2} - 1)\frac{\partial p^*}{\partial t} \end{aligned} \qquad (12.8)$$

So that eliminating $\partial p^*/\partial z$ from these two equations

$$\frac{\partial}{\partial t}(p+v) = \frac{1}{S^2}\frac{\partial}{\partial t}(P^*+V^*) + \left(1-\frac{1}{S}\right)(X_t^* Y_{zt}^* - Y_t^* X_{zt}^*) + \left(1-\frac{1}{S^2}\right)\frac{\partial p^*}{\partial t} \qquad (12.9)$$

and by integration with respect to t

$$\begin{aligned} p+v = {} & p_s + v_s + (1/S^2)(P^* + V^* - 1) \\ & + (1-1/S)\int_0^t (X_t^* Y_{zt}^* - Y_t^* X_{zt}^*)\, dt \\ & + (1-1/S^2)\int_0^t \frac{\partial p^*}{\partial t}\, dt \end{aligned} \qquad (12.10)$$

The last two terms will be of small order at the critical point, so that as before we get an equation

$$v + \mu v^{-\gamma} - 1 = \mu e^{-2t/\tau} - 2(1-v_s)\, g(z) + \frac{81}{256} \times 0.186\, (t/z)^4 \qquad (12.11)$$

where z is to be equal to its value at the center of the slab, $z = a$ neglecting higher powers of $(1 - S) = g(a)$. This may be treated exactly as (7.3); the curve for $g(a)$ as a function of a/τ is shown in Fig. 4.

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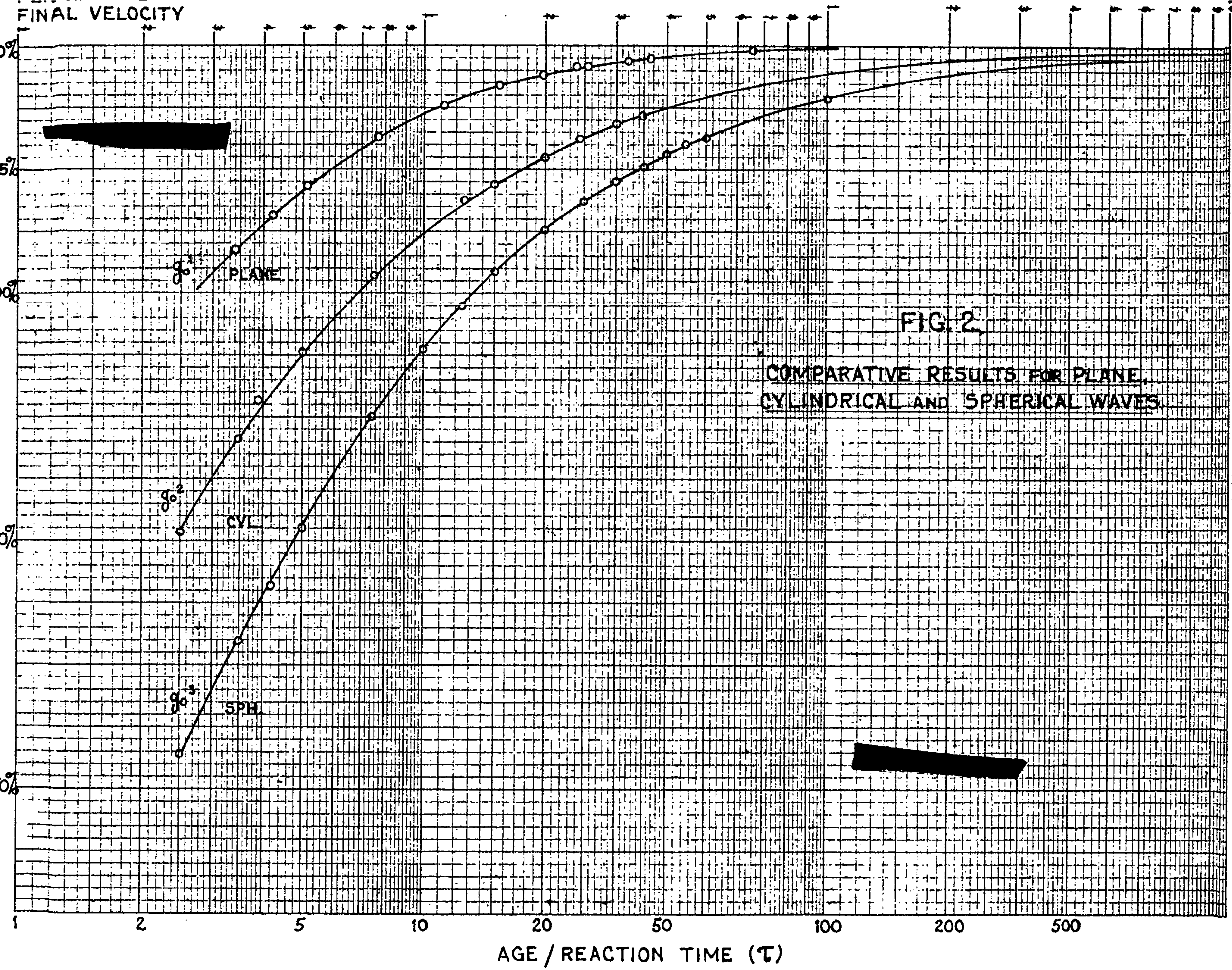

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FIG. 2.

COMPARATIVE RESULTS FOR PLANE, CYLINDRICAL AND SPHERICAL WAVES.

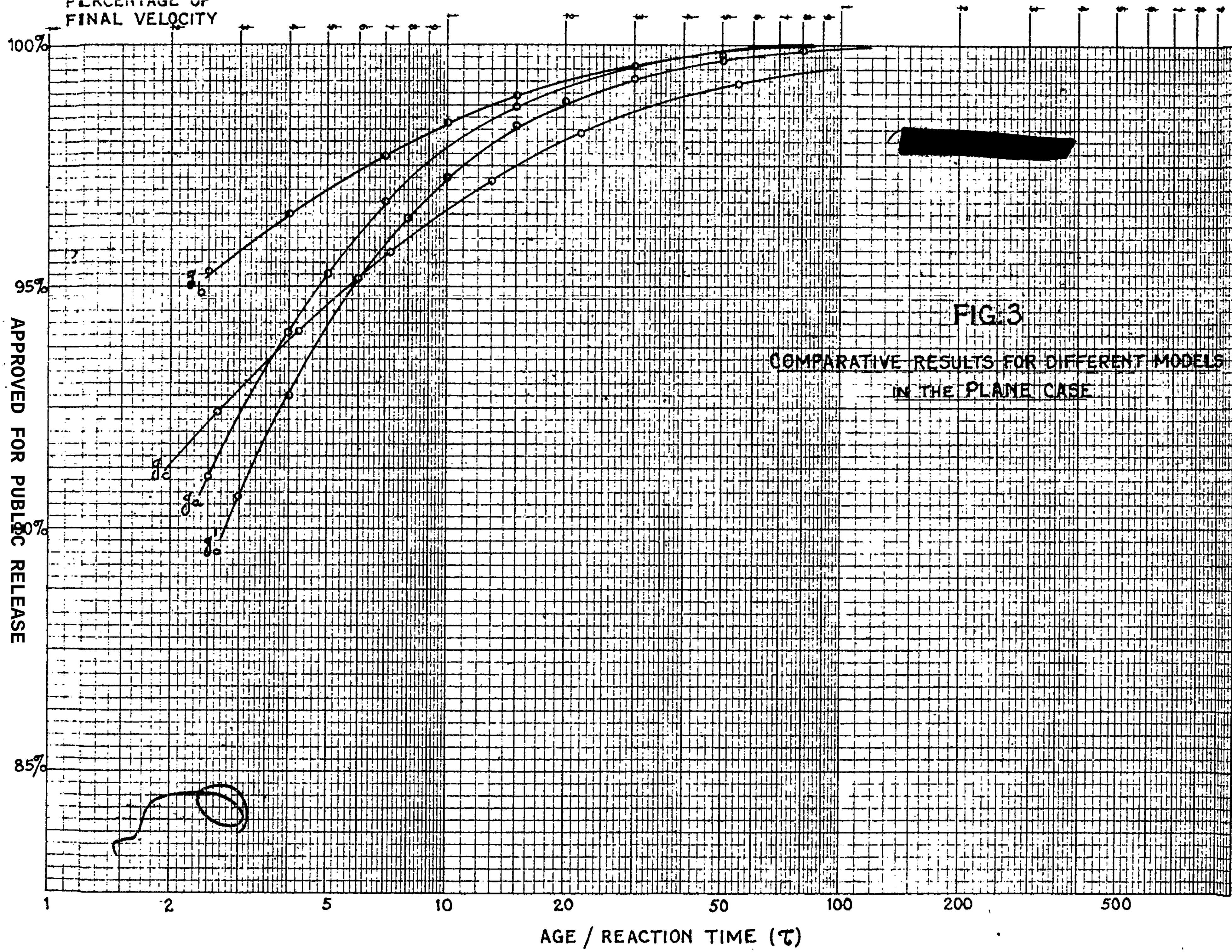

FIG. 3

COMPARATIVE RESULTS FOR DIFFERENT MODELS IN THE PLANE CASE

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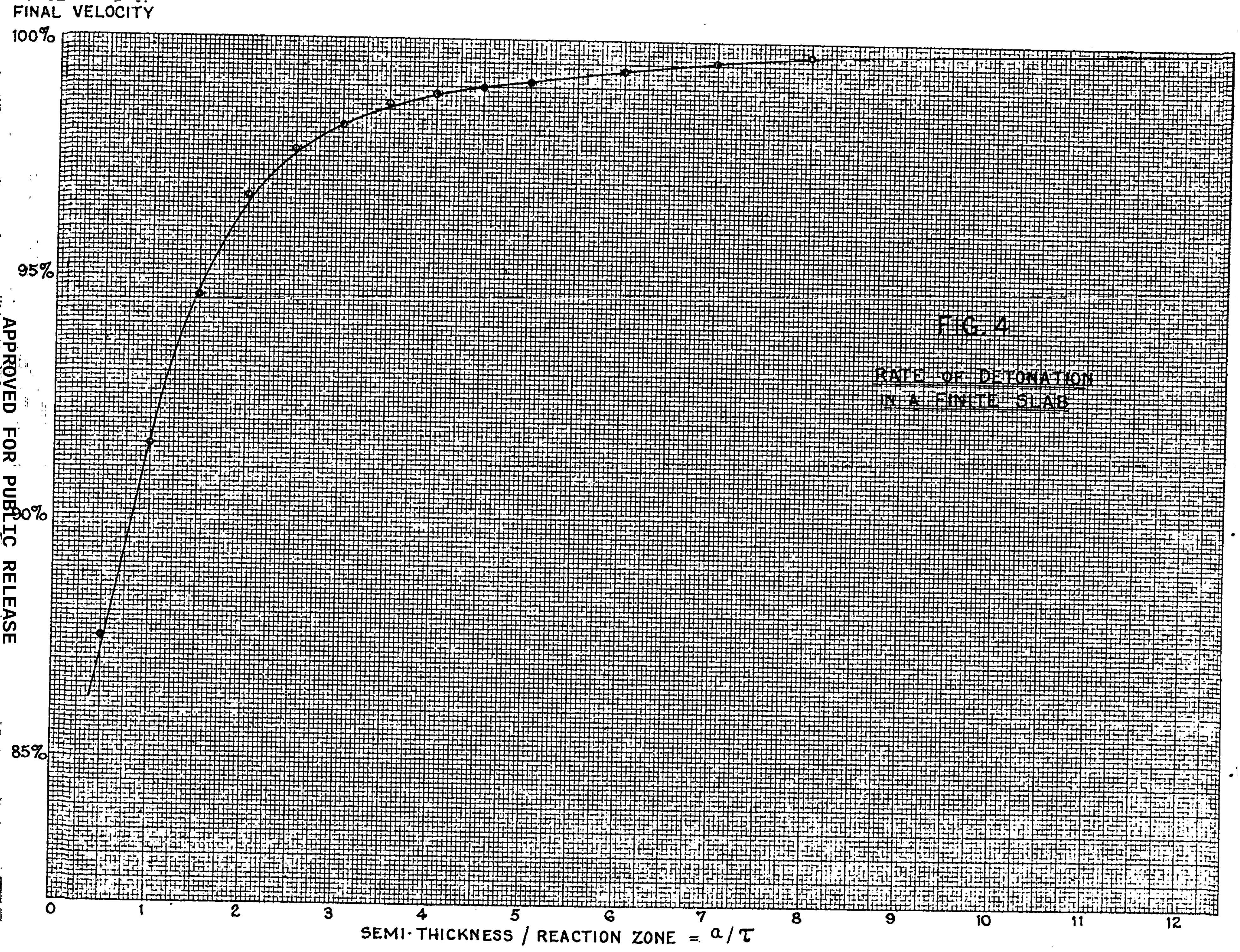

FIG. 4

RATE OF DETONATION IN A FINITE SLAB

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