LA-234

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March 6, 1945

This document contains 14 pages
and 2 pages of unbound curves

A GRAPHICAL REPRESENTATION OF CRITICAL MASSES AND MULTIPLICATION RATES

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ABSTRACT

An approximate method of calculating critical masses and multiplication rates is described which gives the results in a very convenient form. In this approximation the efficacy of the tamper in returning neutrons to the core can be described by a single number, the tamper's reflection coefficient, which depends only on the properties of the tamper, and not at all on what the core material may be. The core can similarly be described by a reflection coefficient: the reflection coefficient a tamper would need in order to make the assembly critical. For given core and tamper materials the critical radius is determined simply by the condition that core and tamper reflection coefficients be equal.

The economy introduced by this formulation is so great that all results on the critical radius can be represented by a pair of graphs, which, except for extreme values of the constants, are accurate to 1 per cent. A second pair of graphs serves to give multiplication rates.

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A GRAPHICAL REPRESENTATION OF CRITICAL MASSES AND MULTIPLICATION RATES

In this note we shall present an approximate method of finding critical masses and multiplication rates which, while not as accurate as other methods that have been developed, has the advantage of being simple to understand and easy to apply. It has, in addition, the very convenient feature that the results come out in a form which lends itself readily to a graphical presentation.

The method is based on a fact learned from the more elaborate treatments of the problem, that the neutron distributions in core and tamper can be approximated in a rather simple way. The neutron density in the core is taken to be of sinusoidal form, and in the tamper an exponential is used. The true distribution shows a transition effect near the core-tamper interface, the density dropping rapidly in passing from core to tamper. We represent this by allowing a discontinuity in density at the interface. The magnitude of the discontinuity can be determined from the conservation law: the rate of production of neutrons in the core of a critical assembly must just equal their rate of absorption in the tamper (supposed infinitely thick). Once the density is fixed in this way, the critical radius can be decided by examining the neutron balance at the center of the core, that is, by demanding that the integral equation which governs the diffusion of neutrons be satisfied at $r = 0$. The extension of the method to non-critical assemblies is immediate, since a multiplying system is equivalent to a critical one with time absorption.

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The core can be characterized by two constants in addition to its radius a: its reciprocal mean free path $\sigma = N\sigma_{tr}$, where N is the nuclear density and σ_{tr} is the transport cross section, and its reactivity $f = \left[(\nu - 1)\sigma_f - \sigma_r\right]/\sigma_{tr}$, where ν is the number of neutrons per fission, σ_f is the fission cross section, and σ_r is the (radiative) capture cross section. The number of neutrons emerging from a collision is just 1 + f. For the tamper, $\sigma' = N'\sigma'_{tr}$, $f' = -\sigma'_r/\sigma'_{tr}$, the primes denoting tamper quantities. The neutron flux density n (true density times velocity) will be taken to be

$$n = \sin k\sigma r/k\sigma r \text{ in the core,}$$
$$n = Be^{-K\sigma' r}/K\sigma' r \text{ in the tamper,} \tag{1}$$

with k and K given by[1)]

$$k/\tan^{-1}k = 1 + f \quad , \quad K/\tanh^{-1}K = 1 + f' \tag{2}$$

The conservation law, which determines the discontinuity at the surface, i.e. the magnitude of the constant B, is

$$\sigma f \int_{core} n\, d\tau = -\sigma' f' \int_{tamper} n\, d\tau . \tag{3}$$

The integral equation for the density is

$$n(\underline{r}) = (1 + f)\sigma \int_{core} \frac{e^{-\int_r^{r'} \sigma_1 d\rho}}{4\pi|\underline{r} - \underline{r}'|^2}\, n(r')\, d\tau$$
$$+ (1 + f')\sigma' \int_{tamper} \frac{e^{-\int_r^{r'} \sigma_1 d\rho}}{4\pi|\underline{r} - \underline{r}'|^2}\, n(r')\, d\tau . \tag{4}$$

1) See Frankel and Nelson, LA-53

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Here σ_1 is σ or σ' depending on whether we are in core and tamper, and the integral $\int_r^{r'} \sigma_1 d\rho$ is to be taken along a straight line between $\underline{r}$ and $\underline{r}'$. For arbitrary $\underline{r}$ and $\underline{r}'$ the geometry of the intersection of this line with the surface of the core is quite unpleasant, and leads to complicated angular integrations in (4). However, for the center of the sphere, $\underline{r} = 0$, the geometry is simple, and (4) reduces to

$$n(o) = (1 + f)\sigma \int_o^a e^{-\sigma r'} n(r')dr' + (1 + f)\sigma' \int_a^\infty e^{-\sigma'(r' - a) - \sigma a} n(r')dr' \quad . \tag{5}$$

Our procedure is to obtain a formula for the critical radius by substituting n, as given by (1), on both sides of this equation.

The integral over the core on the right hand side of (5) can be split into two terms $\sigma(1 + f)\int_o^a = \sigma(1 + f)\left[\int_o^\infty - \int_a^\infty\right]$. The first of these terms would give the number of neutrons the core returns to the center for very large radius $(a \rightarrow \infty)$. It is

$$\sigma(1 + f)\int_o^\infty \frac{\sin k\sigma r'}{k\sigma r'} dr' = \frac{(1 + f)\tan^{-1}k}{k} = 1 \quad ,$$

in virtue of (2), which equals the $n(o) = 1$ which appears on the left hand side of (5). This, of course, is just the reason for the choice of the relation (2) between k and f. The left-over term, $\sigma(1 + f)\int_a^\infty$, thus represents the deficiency in neutrons returned to the center by a finite core, and (5) tells us simply that this deficiency must be made up by the

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neutrons returned by the tamper.

Eq. (5) now takes the form

$$(1 + f)\sigma e^{\sigma a} \int_a^\infty e^{-\sigma r'} n_c(r')dr' = (1 + f')\sigma' e^{\sigma' a} \int_a^\infty e^{-\sigma' r'} n(r')dr' \qquad (6)$$

The density function which appears on the left hand side is the core density. To avoid confusion this has been indicated by the subscript c. In writing (6) we have multiplied (5) by a factor $e^{\sigma a}$, which takes out the effect of the attenuation of neutrons in travelling from the edge of the core to its center. The right hand side of (6) is thus (aside from a factor 4π) just the flux of neutrons per unit solid angle directed radially inward across the core surface. In order to put our relation in a dimensionless form, let us divide (6) by the net outward flux of neutrons from the core surface. According to (3), this may be written in either of the forms of $\int_{core} n d\tau / 4\pi a^2$ or $-\sigma' f' \int_{tamper} n d\tau / 4\pi a^2$.

We obtain

$$\frac{4\pi a^2 (1 + f) e^{\sigma a} \int_a^\infty e^{-\sigma r'} n_c(r')dr'}{f \int_{core} n d\tau} = \frac{4\pi a^2 (1 + f') e^{\sigma' a} \int_a^\infty e^{-\sigma' r'} n(r')dr'}{-f' \int_{tamper} n d\tau}$$

The integrals which appear are readily evaluated:

$$\int_a^\infty e^{-\sigma r'} n_c(r')dr' = \int_a^\infty e^{-\sigma r'} (\sin k\sigma r' / k\sigma r')dr'$$

$$= -\mathrm{I}E_1 \,(\sigma a + ik\sigma a)/k\sigma \quad ,$$

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$$\int_a^\infty e^{-\sigma' r'} n(r')\,dr' = B\int_a^\infty (e^{-(1+K)\sigma' r'}/K\sigma r')\,dr'$$

$$= BE_1(\sigma' a + K\sigma' a)/K\sigma' \quad ,$$

$$\int_{core} n\,d\tau = (4\pi/k\sigma)\int_0^a \sin(k\sigma r)\,r\,dr$$

$$= (4\pi/k^3\sigma^3)(\sin k\sigma a - k\sigma a\cos k\sigma a) \quad ,$$

$$\int_{tamper} n\,d\tau = (4\pi B/K\sigma')\int_a^\infty e^{-K\sigma' r}\,r\,dr$$

$$= 4\pi B(1 + K\sigma' a)/K^3\sigma'^3$$

Here $E_1(z)$ is the exponential integral function,

$$E_1(z) = \int_z^\infty (e^{-z}/z)\,dz \quad ,$$

and IE_1 is the imaginary part of the function. Extensive tables of $E_1(z)$ are available (AM 509).

Substituting these integrals in (7) gives us our formula for the critical radius,

$$-\frac{(1+f)(k\sigma a)^2 e^{\sigma a} IE_1(\sigma a + ik\sigma a)}{f(\sin k\sigma a - k\sigma a\cos k\sigma a)} = \frac{(1+f')(K\sigma' a)^2 e^{\sigma' a} E_1(\sigma' a + K\sigma' a)}{-f'(1 + K\sigma' a)\,e^{-K\sigma' a}} \equiv T. \qquad (8)$$

This formula has previously been obtained by A. H. Wilson[2], by a somewhat different argument.

Eq. (8) has a remarkably simple structure, in that the left-hand side involves only core properties, and the right hand side only tamper properties.

2) A. H. Wilson, BM 441. I am indebted to K. M. Case for calling Wilson's result to my attention. While Wilson's procedure, which is based on a consideration of the angular distribution of neutrons at the interface, does not seem as simple as ours for the problem in hand, it has certain advantages for more complicated problems where the choice of neutron density is not so unambiguous.

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This means that, in our approximation, the effectiveness of a tamper of given material surrounding a sphere of radius a can be characterized by a single number, which depends in no way on the nature of the core material. As our derivation has shown, this number, which we call T, is the ratio of the inward radial flux returning from the tamper to the net outward flux entering it. It may therefore appropriately be called the reflection coefficient of the tamper[3]. In a similar way, a core of given radius is also characterized by a reflection coefficient, namely, the reflection coefficient a tamper would need in order to make the system critical. Expressed in these terms, the critical condition (8) is simply the requirement that core and tamper reflection coefficients be equal.

The fact that the core reflection coefficient is a function only of f and σa, and the tamper reflection coefficient of f' and $\sigma' a$, leads to a convenient graphical representation of our results. One plots, on semi-log paper, curves of T vs $\ln(\sigma a)$ for a series of values of f. On a transparent sheet one plots T vs $\sigma' a$ for a series of f''s. To find the critical radius for given core and tamper materials, i.e. for given σ, σ', f, f', one lays the transparent tamper sheet over the core sheet and displaces it to the right by an amount $\ln(\sigma/\sigma')$. The value of σa at which the curve for the correct f crosses the curve for the correct f' gives the critical radius.

For the determination of the multiplication rate of a non-critical system another form of plotting may be used. One now knows σ, σ', f, f' and a, and wishes to find the multiplication rate α (defined by $n \sim n_0 e^{\alpha t}$).

3) This formulation of the result is contained in a note by H.A. Bethe, LA-206, to which the reader is referred for a more detailed discussion.

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The non-critical system is equivalent to a critical system with σ and σ' replaced by $\bar{\sigma} = \sigma + \alpha/v$ and $\bar{\sigma}' = \sigma' + \alpha/v$ (v is the neutron velocity), but with $\sigma(1 + f)$ and $\sigma'(1 + f')$ left unaltered. The invariance of the latter quantities is evident if they are written out explicitly: $\sigma(1 + f) = N(\nu\sigma_f + \sigma_{e\ell})$; $\sigma'(1 + f') = N'\sigma'_{e\ell}$. They are independent of the absorption cross section, and thus are unaffected by the time absorption. The scheme of graphing is to plot curves of T vs $\bar{\sigma}a$ for a series of values of $\sigma(1 + f)a$, and, on a transparent sheet, curves of T vs $\bar{\sigma}'a$ for a series of values of $\sigma'(1 + f')a$. The transparent tamper sheet is laid over the core sheet and shifted to the right by an amount $\sigma a - \sigma' a$. The point at which the curve of the appropriate value of $\sigma(1 + f)a$ crosses the curve of appropriate $\sigma'(1 + f')a$ gives the value of $\bar{\sigma}a$. The multiplication rate is then found from the relation $\alpha a/v = \bar{\sigma}a - \sigma a$. The multiplication rate is often expressed in terms of a quantity $\gamma = \alpha/\sigma v$. This quantity is given by $1 + \gamma = \bar{\sigma}a/\sigma a$.

Comparison of critical radii calculated from (8) with the more exact calculations of Glauber (LA-174) and of Frankel and Nelson (LA-53a) show that (8) gives a critical radius which is consistently a little too low. Over the ranges of constants which are of most interest the discrepancy is nearly a constant 2 per cent. Such a tendency of our results is easy to understand; it is a consequence of replacing the transition effect at the interface by a discontinuity. This makes the neutron density near the interface a little too high in the core and a little too low in the tamper, gives a little too much fission and a smaller critical radius. The effect

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is partly compensated for by the fact that, for the same reason, B, as determined by (3), will be somewhat too small, so the tamper will be counted as slightly less effective than it should be in returning neutrons to the core. Thus for highly absorbing tampers, where the return of neutrons is less important, a larger discrepancy might be expected. This is borne out by the results for the untamped critical radius, which turns out to be more nearly 4 per cent low. In plotting the graphs a 2 per cent correction has been applied to the radius. In the following table a comparison is given between critical radii read from the graphs and those calculated by Glauber, for values of f approximately right for 25 and 49, and a value of f' near that for a U tamper.

$\frac{\sigma}{\sigma'}$	f = .39, f' = -.03		f = .72, f' = -.03	
	σa graph	σa Glauber	σa graph	σa Glauber
∞	1.98	2.021	1.25	1.284
2	1.69	1.685	1.11	1.095
1	1.50	1.496	0.99	0.984
1/2	1.29	1.300	0.85	0.858
1/5	1.06	1.060	0.69	0.701

A comparison between multiplication rates from the graphs and those

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quoted by Glauber is given for the constants $f = .39$, $f' = -.03$, $\sigma/\sigma' = 1$:

M/M_c	γ graph	γ Glauber
1	0	0
1.5	.046	.047
2	.080	.081
3	.129	.129
4	.164	.161
5	.184	.184

The graphical method is seen to be quite good.

Illustration of Use of Graphs

Critical Mass

To find the critical radius for $f = 0.39$, $f' = -0.03$, $\sigma/\sigma' = 0.5$. It will be noted that there is a mark, labeled σ/σ', on the transparent sheet, which lies on the line $\sigma a = 1$ when the sheets are superimposed. To adjust to the mean free path ratio $\sigma/\sigma' = 0.5$, shift the transparent sheet to the left until this mark lies on the line $\sigma a = 0.5$. Follow the line on the transparent tamper sheet marked $f' = -0.03$ till it crosses the line $f = 0.39$ on the core sheet (interpolate between $f = 0.40$ and $f = 0.375$). Read the value of σa at the crossing point, $\sigma a = 1.29$. This is the critical radius in units of the core mean free path.

Multiplication Rate

To find the multiplication rate for $f = 0.3$, $f' = -0.03$, $\sigma a = 3.0$, $\sigma' a = 3.3$. Compute the values of $\sigma a(1 + f)$, $\sigma' a(1 + f')$ and $\sigma' a - \sigma a$. One

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obtains

$$\sigma a(1 + f) = 3.9, \quad \sigma' a(1 + f') = 3.2, \quad \sigma' a - \sigma a = 0.3.$$

To adjust for the mean free path difference, $\sigma' a - \sigma a = 0.30$, shift the transparent sheet to the left by 0.3, i.e. so the right hand margin lies on the line $\overline{\sigma a} = 4.7$. The curves for $\sigma a(1 + f) = 3.9$ and $\sigma' a(1 + f') = 3.2$ cross at $\overline{\sigma a} = 3.427$. The multiplication rate, γ, is then $\gamma = (\overline{\sigma a}/\sigma a) - 1 = (3.427/3.0) - 1 = 0.142$.

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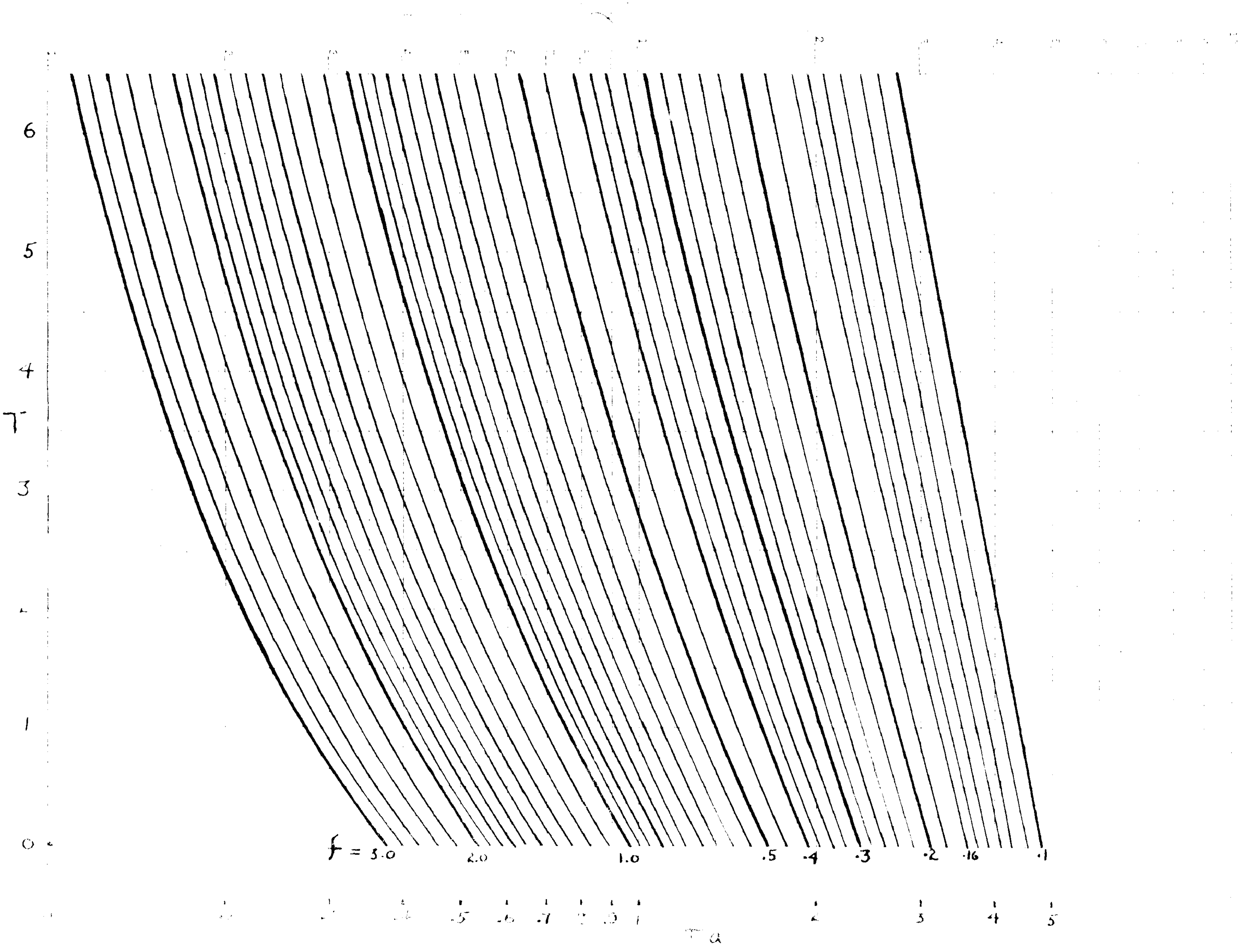
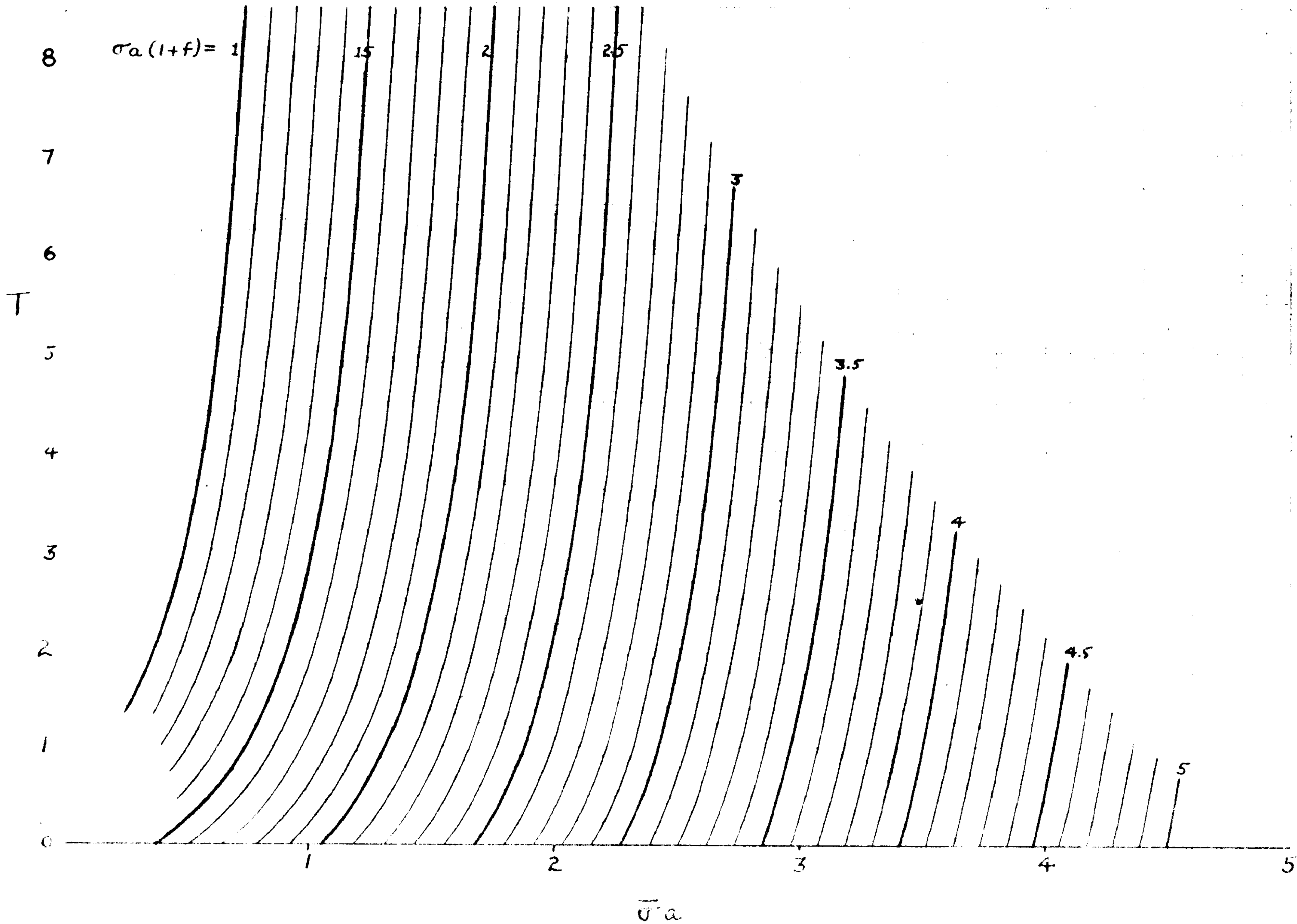
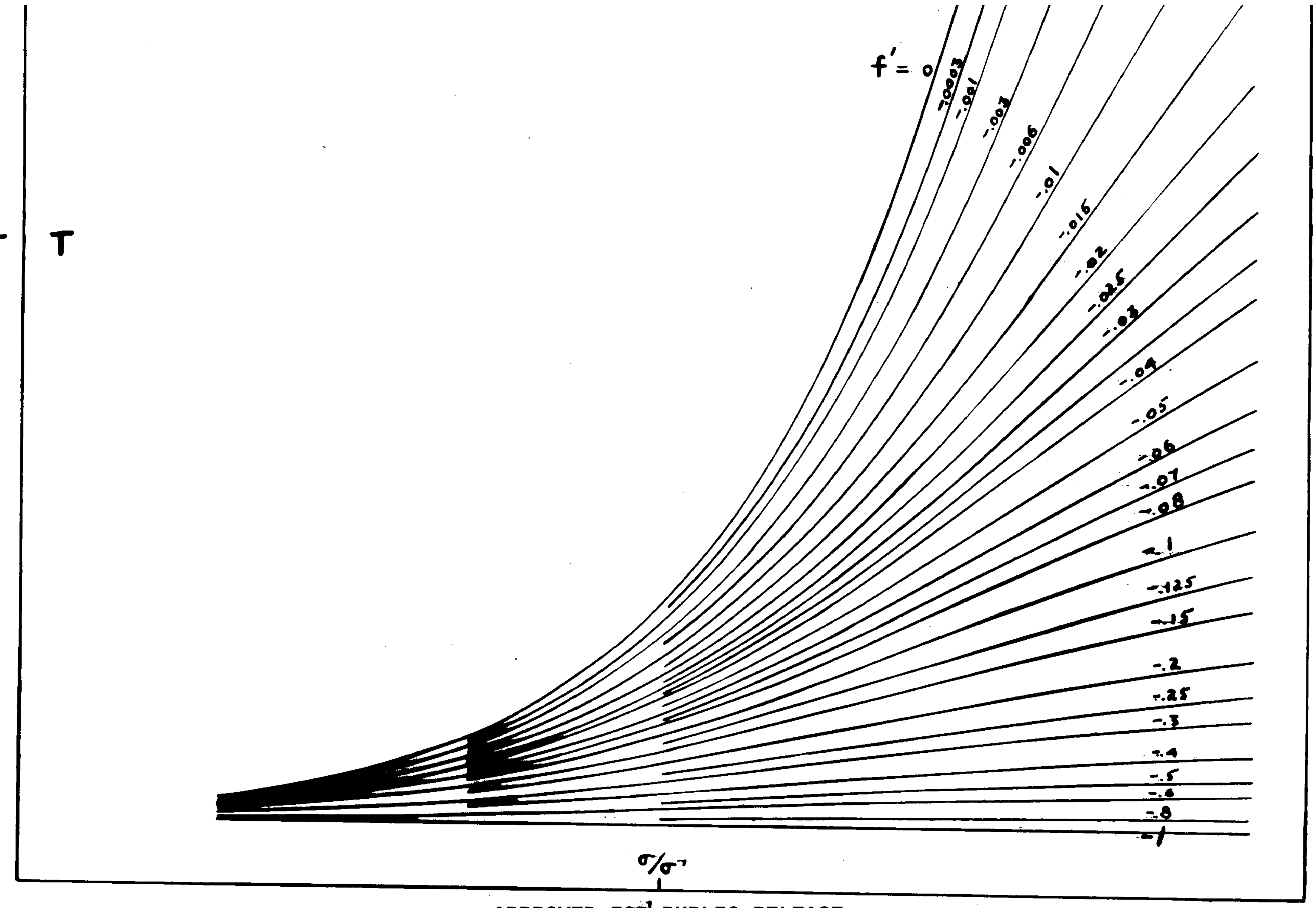
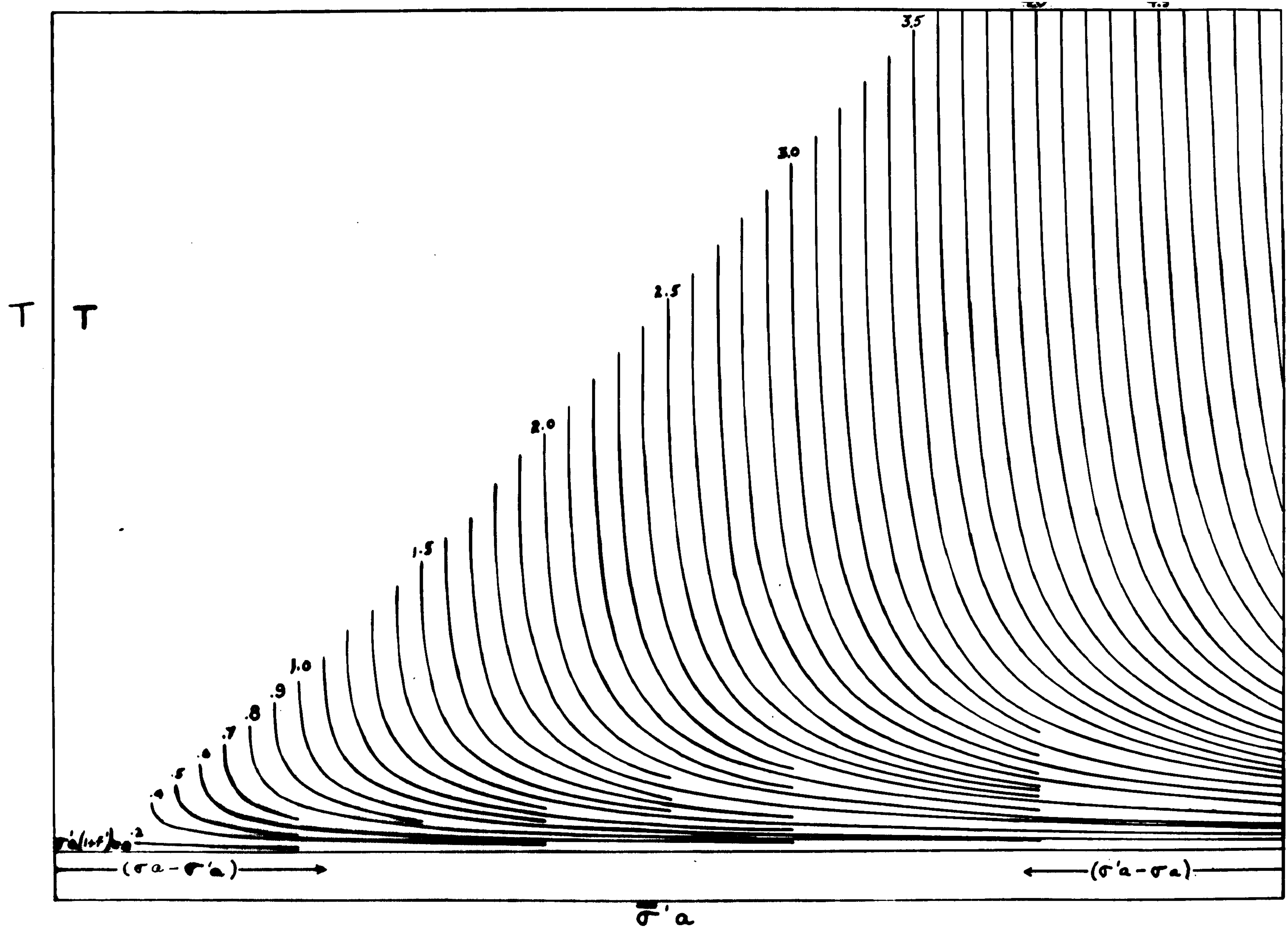

6
5
4
T
3
1
0
f = 3.0
2.0
1.0
.5
.4
.3
.2
.16
.1
.5
.6
.7
1
2
3
4
5


σa(1+f)= 1
1.5
2
2.5
3
3.5
4
4.5
5
T
8
7
6
5
4
3
2
1
0
1
2
3
4
5
σ̄a


T
f' = 0
-.0003
-.001
-.003
-.006
-.01
-.016
-.02
-.025
-.03
-.04
-.05
-.06
-.07
-.08
-.1
-.125
-.15
-.2
-.25
-.3
-.4
-.5
-.6
-.8
-1
σ/σ'


T T
3.5
3.0
2.5
2.0
1.5
1.0
.9
.8
.7
.6
.5
.4
.2
(σa - σ'a)
(σ'a - σa)
σ'a

OR PUBLIC RELEA