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ON DISCONTINUOUS INITIAL VALUE PROBLEMS FOR NONLINEAR EQUATIONS AND FINITE DIFFERENCE SCHEMES

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PHYSICS

Abstract

This paper describes a new numerical scheme for calculating hydrodynamical flows with shocks. It is similar to a scheme promulgated some years ago by von Neumann, see [9], and modified more recently by him and R. Richtmyer, see [11], inasmuch as it is a straightforward numerical scheme which ignores the presence of discontinuities. It is more closely related to the scheme described in [9] since no viscosity term is used; what is new about the method is:

(a) The difference scheme used is based on the conservation form of the hydrodynamic equations.

(b) The difference scheme is unsymmetric in time.

<u>Description of the difference equations</u>: Write the hydrodynamic equations in the form of conservation laws (mass, momentum and energy); in this form each term in the equation is a perfect x or t derivative. Replace all x derivatives by centered difference quotients, all time derivatives f_t by a <u>forward facing</u> difference quotient of this sort:

$$\frac{f_{\ell}^{n+1} - \overline{f_{\ell}^{n}}}{\Delta t},$$

where $\overline{f_{\ell}^{n}}$ is taken as the <u>arithmetic mean</u> of the values of f at all neighboring space points at time cycle n.

This scheme uses a staggered lattice, i.e., at time cycle n we use all lattice vectors ℓ with, say, even components, at the next time cycle we use odd lattice vectors.

The conjecture is that if the meshsize tends to zero, and the stability condition of Courant-Friedrichs-Lewy is satisfied, the approximate solutions computed by this method will tend to the exact solution uniformly except in neighborhoods of discontinuity lines or surfaces.

The mathematical soundness of this proposition is discussed in detail, using as an example the equation $u_t + uu_x = 0$. Test calculations performed on this equation and on the hydrodynamic equations in one dimension, both Euler and Lagrange form, show fairly conclusively that the method works. Some of the numerical results are presented at the end of the report.

ON DISCONTINUOUS INITIAL VALUE PROBLEMS FOR NONLINEAR EQUATIONS AND FINITE DIFFERENCE SCHEMES

Let $U_t + AU_x + B = 0$ be a quasilinear hyperbolic system of first order equations; U denotes a column vector of n unknown functions, A a coefficient matrix, and B a vector. A and B are assumed to be functions of x,t and U. The system is called hyperbolic if all eigenvalues of A are real and if A has n linearly independent eigenvectors.

The initial value problem for such a system is to find a solution with prescribed values on the x axis (or an interval of it), $U(x,0) = \Phi(x)$. According to the theory of hyperbolic equations this initial value problem has a (unique) solution if $\Phi(x)$ is differentiable, or is at least Lipschitz continuous (in this latter case the solution would not have continuous partial derivatives). The range of t for which the solution exists is at least as large as $c(\max|\Phi'|)^{-1}$, c being a constant depending on the coefficients A and B and their first derivatives.

The example of the simple equation $u_t + uu_x = 0$ shows that this estimate cannot be improved in general. In this case, namely, the solution of the initial value problem $u(x,0) = \varphi(x)$ is given by the implicit relation $u - \varphi(x - ut) = 0$. This relation defines u as a (differentiable) function of x and t as long as the derivative of the left hand side with respect to u, $1 + t\varphi$, does not vanish. The smallest value of t for which this quantity vanishes is

$t = (\max - \varphi')^{-1}$; this shows that the width, of the domain of existence in the t direction, does depend on a bound for the magnitude of φ' (although only on a one-sided bound).

Suppose we wish to solve an initial value problem where the initial values no longer satisfy a Lipschitz condition; say they are downright discontinuous, as in the Riemann shock tube problem. One could attempt to solve this problem by approximating the given differentiable initial values $\Phi_i(x)$, construct the corresponding solution U_i and take their limit - if it exists - in the sense of some norm or topology. This method works for linear equations but does not in general for quasi-linear equations; for if the sequence Φ_i approximates an initial vector that is not Lipschitz continuous, the first derivatives of Φ_i are not uniformly bounded, and so the range of t for which the solution of the i^{th} problem, U_i, exists shrinks to zero as i tends to infinity. This shows that the theory of discontinuous initial value-problems for nonlinear equations is not a mere appendix to the theory of differentiable initial value-problems but has to be developed independently.

There are several ways of developing such a theory. One is to generalize the concept of a function satisfying a differential equation. This leads to the notion of weak solutions and the initial value problem is to ascertain whether in the aggregate of all weak solutions there exists one with the prescribed initial data.

Another way is to define the solution of a discontinuous initial value problem directly by a limiting process of some kind. This limiting process would usually consist of approximating the equation by a sequence of equations for which the initial value problem can be solved. For the equations of hydrodynamics this is usually done by including viscous forces; what is proposed here is to use a straightforward finite difference scheme; that such a method works is of interest for the theory and for practical computations.

It would be desirable to develop an abstract theory which would include these special methods. The appropriate class of abstract equations may possibly be the ones of the form

$$U_t = A\ N\ u$$

where A is an unbounded linear, N a continuous nonlinear operation.

We shall describe now the three methods mentioned, illustrating them on the equation $u_t + u\ u_x = 0$.

1. Generalizing the concept of a solution.

Let v be some test function which is zero on the boundary of some region G of the x,t plane; G is supposed to lie within the domain of definition of the solution u. Multiply the equation $u_t + u\ u_x = 0$ by v, integrate over G, and integrate by parts. The result is that the integral

$$\iint v_t u + \frac{1}{2} v_x\ u^2 \tag{1}$$

is zero for all G test functions v and solutions u. Conversely: if u is a function with continuous derivatives for which the integral (1) vanishes for all test functions, then u is a solution of the original differential equation (this is easily seen by integrating (1) by parts and applying the so-called fundamental lemma of the calculus of variations).

We define u to be a generalized or weak solution if the integral (1) is zero for all test functions v. As stated before, a generalized solution which is differentiable is a bona fide solution. But amongst the class of non-differentiable functions we have a genuine extension of the notion of solution.

Weak solutions, for linear equations, are discussed briefly in Courant-Hilbert, vol. II, p. 469-470. They play an important role in Friedrich's work on differential operators; their theory was treated systematically by Sobolev, and L. Schwartz. In the nonlinear case - which interests us most - the concept of weak solutions is discussed, usually in connection with shock problems of hydrodynamics (see also E. Hopf, [7]).

Consider discontinuous solutions, i.e., functions u that suffer a jump discontinuity across a smooth arc C, on either side of which it has continuous derivatives and satisfies the equation. Straightforward application of the definition shows that a discontinuous solution is a weak solution if and only if U, the slope of the discontinuity line at any point on C is the arithmetic mean of the values of u on the two sides at this point (analogue of the shock relations).

This example shows (a) that there are weak solutions of our equa-

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tion which are not genuine solutions

(b) that the class of weak solutions is associated not so much with an equation but with the form in which it is written. For had we written our equation in the form $u^{-1}u_t + u_x = 0$, the criterion for discontinuous solutions to be weak solutions would have been $\bar{U} = (u_1 - u_2)(\log u_1 - \log u_2)^{-1}$, which defines an entirely different class of weak solutions. The form of the equation to be used is dictated entirely by outside physical consideration. E.G., the equations of hydrodynamics in mass coordinates can be written as four different conservation laws; namely, conservation of mass, momentum, energy, and entropy. For physical reasons we would operate with the first three of these conservation laws.

The test of usefulness of the concept of weak solutions is whether weak solutions with arbitrarily prescribed initial data of a wide class (say, the class of all piecewise continuous or all bounded, measurable functions) exist, and whether the initial values determine the solutions uniquely (a weak solution having prescribed initial data can be defined either in an almost everywhere sense or in a weak sense). It turns out that the answer to the first query is affirmative, to the second, negative.

That for the equation $u_t + uu_x = 0$ weak solutions with arbitrarily prescribed initial data exist has been shown by E. Hopf in [7] as a corollary to the theory developed there. That the solution is not in general unique is well known; it can be seen from this example: Let the initial value be

$$u(x,0) = 0 \qquad \text{for } x < 0$$
$$= 1 \qquad \text{for } x > 0.$$

The function

$$u(x,t) = 0 \text{ for } t > 2x$$
$$= 1 \text{ for } t < 2x$$

is a weak solution of our problem since it assumes the initial value and satisfies the jump condition. But so is the function

$$u(x,t) = 0 \qquad \text{for } x < 0$$
$$= \frac{x}{t} \qquad \text{for } x > t$$
$$= 1 \qquad \text{for } x < t.$$

In analogy with hydrodynamics we would exclude the first solution since it represents a rarefaction shock; whether the exclusion of rarefaction shocks would leave only one weak solution of any initial value problem, is not known.

So the problem is to characterize the physically relevant weak solutions in some systematic way, and to prove that the initial value problem has a unique physically relevant weak solution for a wide class of initial values. In connection with this problem it should be remarked that whereas the class of regular solutions of our equation displays reversibility in time; i.e., if $u(x,t)$ is a regular solution, so is $u(-x, -t)$, and the class of all weak solutions likewise, the class of physically relevant weak solutions (i.e., the ones without rarefaction shocks) no longer share this property; e.g., the weak solution

$$u(x,t) = 1 \qquad \text{for } t > 2x$$
$$\qquad = 0 \qquad \text{for } t < 2x$$

is physically relevant for it represents a compression shock, whereas $u(-x, -t)$ represents a rarefaction shock.

One systematic method of introducing physically relevant weak solutions is to take those solutions which are limits of "viscous flows". I.e., consider the augmented equation

$$u_{,t} + u\, u_x = \lambda\, u_{xx} \tag{2}$$

with some positive constant λ, solve the initial value problem $u_\lambda(x,t) = u_o$, and let λ tend to zero, Equation (2), and the above limiting process, was introduced into the literature, by Burgers; an especially elegant and rigorous treatment of it is due to E. Hopf [7]. This procedure was conceived as a simple analogue of the process of obtaining this discontinuous solution of the hydrodynamic equations as limits of viscous flows, see Becker [1], L. H. Thomas, Gilbarg [10], Grad [6], and Courant-Friedrichs [2], pp. 134-138.

Equation (2) is a semi-linear parabolic equation; the introduction of a new unknown φ, related to u by $u = -2\lambda\, \varphi_x/\varphi$ reduces it, as E. Hopf has observed, to a linear parabolic equation $\varphi_t = \lambda\varphi_{xx}$ whose solution can be written down explicitly. This in turn gives an explicit representation of any solution of (2) in terms of its initial values; this representation enabled Hopf to prove that for fixed ini-

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tial values u_o the solution $u_\lambda(x,t)$ tends to a limit as λ tends to zero, for almost all x and t. This limit can be called the generalized solution of the initial value problem $u(x,0) = u_o$ of the original equation (1).

It is easy to show that these generalized solutions are weak solutions; just multiply equation (2) by any twice differentiable test function v and integrate by parts:

$$\iint v_t\, u + \frac{1}{2} v_x\, u^2 = \lambda \iint v_{xx} u \ ;$$

u remains uniformly bounded for λ , and so, v being held fixed, the right side tends to zero with λ .

This class of generalized solutions is irreversible in t; there is nothing surprising in this, for the process whereby they were defined is openly biased in favor of the positive t direction, i.e., the initial value problem for the parabolic equation (2) can be solved for positive t but not for negative t.

A different limiting procedure for constructing weak solutions is by a straightforward finite difference scheme; the conjecture is that this process furnishes the same class of physically relevant weak solutions as the viscosity method. Several arguments will be presented which make the conjecture plausible, or at least possible; the numerical evidence in favor of it is very strong but there is no rigorous proof for it yet.

First the description of the scheme itself: Since the concept of weak solutions is linked not to the equation itself but the form in which it is written, it is important that the difference scheme should be linked to the distinguished form of the equation. Secondly, the possibility of defining weak solutions rests on the fact that the given equation is in divergence form, i.e., each term is a pure x or t derivative. This feature should be preserved as much as possible in the difference scheme too. Both requirements are fulfilled by this scheme: replace space derivatives by difference quotients:

f_x by $\dfrac{f^n_{\ell+1} - f^n_{\ell-1}}{2\Delta x}$, and t derivatives u_t by a forward difference quotient of this kind:

$$\frac{1}{\Delta t}\left(u^{n+1}_{\ell} - \frac{u^n_{\ell+1} + u^n_{\ell-1}}{2}\right.\,.$$

Here superscripts refer to time cycle, subscripts to position in space.

This scheme, when applied to any hyperbolic system, is stable in the sense of von Neumann if $\frac{\Delta x}{\Delta t}$ satisfies the classical Courant-Friedrichs - Lewy condition, see [5], of being greater than the slope of the steepest characteristic. The equation $u_t + u\,u_x = 0$ has one characteristic, with slope u, so the stability condition is

$\frac{\Delta x}{\Delta t} \geqslant \max |u|$. Now if we choose $\frac{\Delta x}{\Delta t}$ so that this inequality is satisfied initially, the function generated by the difference scheme will never exceed its largest value initially, and so the stability

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condition is satisfied for all future times.

Solutions constructed by the difference scheme are defined only at the lattice points; imagine them extended to the whole relevant portion of the x,t plane by defining u inside any lattice square to have the same value as, say, at the upper left corner. Diminish the size of the lattice and suppose that the corresponding solutions, thus extended, converge in the $\mathcal{L}_2$ sense to some limit function u. This limit function u is a weak solution of the original differential equation as may be easily proved by multiplying the difference equation at each lattice point by the value of a test function v there, summing over all lattice points and summing by parts. A passage to the limit leads to an integral relation between u and v that characterizes u as a weak solution. What is not at all clear is

(i) Whether the sequence of solutions of the difference equations converges in the $\mathcal{L}_2$ sense.

(ii) Whether the sequence converges uniformly except in a neighborhood of the discontinuity lines.

(iii) Whether the weak solutions obtained in this manner are the physically relevant ones.

Experimental evidence, presented below, indicates that the answer to all three questions is yes. Concerning (iii) it should be pointed out that, just as in the case of the passage to the limit through viscous flows, the class of weak solutions obtainable by this finite difference method is not likely to be invariant under replacement of x by minus x and t by minus t, because the difference scheme distinguishes between the positive and negative t direction. I mention this as a possible guide to finding other adequate difference schemes.

In case of regular solutions, i.e., ones with continuous first derivatives, the difference scheme described here furnishes a uniformly convergent sequence of approximations to the true solutions. This has been proved, for arbitrary quasilinear hyperbolic systems, by Keller and Lax in [8] and for a slightly different scheme by Courant, Isaacson and Rees [4].

It should be pointed out that if the sequence of solutions of the difference equations or a subsequence of them converges only weakly, the weak limit is not a weak solution. For in this case the weak limit of u_n^2 is not the square of the weak limit of u_n and so the procedure of multiplying the difference equations by v, summing by parts and passing to the limit leads to an equation in which the role of u^2 is taken by the weak limit of u_n^2 .

Experimental calculations were performed using IBM Card Programmed Calculators; the problem was coded by Mr. Stewart Schlesinger. The first case considered was the initial values $u(x, 0) = 1$ for $x < 0$, $= 0$ for $x > 0$, taking $\Delta t/\Delta x$ to be one. The initial values were deliberately chosen to be homogeneous, so that carrying the calculations further in time would have the effect of refining the meshsize; the idea was to carry out the calculations until it became evident that the scheme was converging, diverging or oscillating. It turned out that the scheme was converging, and with astonishing rapidity. After 44 steps in time the calculated values of u were

x	u
17	1.00000
19	.99548
21	.76818
23	.21061
25	.02343
27	.00018
29	.00018

The values of u not listed differ from one or zero by at most 10^{-5}. The theoretical position of the discontinuity, propagating with speed 1/2, is at x = 22; this is precisely the center of zone of transition; the zone is, roughly speaking, spread over three intervals.

Four steps later, at t = 48, the calculated values of u were:

x	u
19	1.00000
21	.99548
23	.76817
25	.21061
27	.02344
29	.00210
31	.00018

The theoretical position of the discontinuity line is at x = 24; the figures show that relative to this discontinuity line the profile of the solution has changed by at most one figure in the last decimal; this suggests that not only does the solution of the difference scheme converge to the true discontinuous solution uniformly in every subset not containing the line of discontinuity, but that the shape of the transition tends to a definite limit. This limiting shape can be characterized as the steady state solution of the difference equations. The difference equation is

$$u_{\ell}^{n+1} = (u_{\ell+1}^{n} + u_{\ell-1}^{n})/2 + \frac{1}{4} \, ({u_{\ell-1}^{n}}^{2} - {u_{\ell+1}^{n}}^{2}) \;;$$

here the superscript n refers to time cycle, ℓ to space position. The equation satisfied by the steady state solution would be

$$\frac{f(x-1) + f(x+1)}{2} + \frac{f^2(x-1 - f^2(x+1)}{4} = f(x + \tfrac{1}{2}) \qquad (3)$$

and the boundary conditions are:

$$f(-\infty) = 1, \qquad f(\infty) = 0 \;. \qquad (4)$$

More precisely, the state of affairs is probably as follows: <u>The difference equation</u> (3), <u>subject to the boundary conditions</u> (4), <u>has a continuous, monotonic solution as function of the real variable x; this solution is unique except for an arbitrary phase shift</u>. Furthermore, starting with any function g(x) defined over the odd integers, repeated application of the transformation **T g = g'** defined by

$$\frac{g(x - 1) + g(x + 1)}{2} + \frac{g^2(x-1) - g^2(x+1)}{4} = g'(x + \tfrac{1}{2})$$

leads to the steady state solution f(x). I.E., if we denote T^n_g by

$g_n(x)$, then $g_n(x)$ tends uniformly to $f(x + \alpha)$, where $f(x)$ is the steady state solution*; the phase shift α depends only on the initial distribution g.

Observe that the function $g_n(x)$ is defined only at points by n/2. Thus the $g_n(x)$ are defined either at the integers of halfway in between, and consequently we need the values of $f(x + \alpha)$ at these points only. This is however an exceptional situation which arose because $\Delta t/\Delta x$ was chosen to be commensurable to the speed of the propagation of the discontinuity.

The numerical evidence presented before for the verity of this theorem is very strong. The calculations cited refer to the initial values $g(x) = 1$ for x a negative odd integer, $= 0$ for x a positive odd integer; as a further check the values: $g(x) = 1$ for x an odd integer less than 0 minus one, $g(-1) = .9$, $g(x) = 0$ for x a positive odd integer were tried. The results were the same as with the original choice of initial $g(x)$; the tables below give the values of u at $t = 44$ and 48; these differ by less than one figure in the fifth decimal.

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*Fixed, say, uniquely by picking $f(0)$ to be 1/2.

$t = 44$		$t = 48$	
x	u	x	u
17	1.00000	19	1.00000
19	.99195	21	.99195
21	.71566	23	.71566
23	.17449	25	.17449
25	.01858	27	.01859
27	.00165	29	.00165
29	.00014	31	.00014

Table I, appended to this paper, gives the values of $g_{45}(x)$, $g_{46}(x)$, $g_{47}(x)$, $g_{48}(x)$ corresponding to the first choice of $g_0(x)$ over those values of x where the deviation from the constant values 0 or 1 is significant; (for all subsequent values of n, $g_n(x)$ coincides in the first five figures with one of the four listed). Table II contains the same information referring to the second choice for initial g.

Graphs I and II show a plot of these values; they lie on smooth curves, and these curves indeed appear to have the same shape.

Returning to the difference equation (3), it should be remarked that if the boundary values of f are switched, i.e. $f(-\infty) = 0$, $f(\infty) = 1$ or, more generally, are replaced by values for which $f(-\infty)$ is less than $f(\infty)$, then no solution would exist. This result, for which I have no proof at present, expresses the fact that the finite difference method furnishes solutions with compression shocks but not with rarefaction shocks. Mathematically, it is an analogue of a well-known result on steady viscous flows (see [1], [6], [10], [11]), which I shall present for the simplified equation $u_t + u\, u_x = \lambda\, u_{xx}$.

Let $u_o(x,t)$ be a steady state solution of the equation $u_t + u\,u_x = \lambda\,u_{xx}$, i.e. u_o is a function of $x - c\,t$ only, $u_o = u(x - ct)$. Then $u(\xi)$ satisfies the ordinary differential equation

$$c\,u' + u\,u' = \lambda\,u'' .$$

Integrate both sides with respect to ξ :

$$K + c\,u + \frac{1}{2}u^2 = \lambda\,u' ,$$

so

$$\frac{d\xi}{du} = \frac{2\lambda}{u^2 + 2c\,u + 2K} . \tag{5}$$

We are interested in solutions which at $\xi = -\infty$ and $\xi = \infty$ have prescribed values u_i (initial) and u_f (final). From equation (5) it is clear that ξ will approach infinity only if u approaches one of the roots of the quadratic function in the denomination of the right hand side; these roots must be then just the initial and final values of u, u_i and u_f, and c, the propagation speed, must be their arithmetic mean. Furthermore, $u^2 + 2\,cu + 2K$ is <u>negative</u> between the two roots u_i and u_f, and so, λ being positive, $\frac{d\xi}{du}$ is negative, i.e., u is a decreasing function of ξ . So we conclude that an initial and final state can be connected through a solution of (5) only if $u_i \geqslant u_f$. If this inequality is fulfilled, then they can be connected and the explicit formula

$$\xi = \frac{2\lambda}{u_i - u_f} \log \frac{u_i - u}{u - u_f}$$

gives the shape of the connecting curve.

Numerical calculations were carried out for the initial values $u(x,0) = 0$ for $x < 0$, $u(x,0) = 1$ for $x > 0$, using the same difference scheme as before; the results after 48 steps, are tabulated in Table III and plotted in Graph III. The dashed line in Graph III refers to the exact solution.

The same problem was run with $\Delta t/\Delta x = 1/2$; the results after steps in time, are tabulated in Table IV, plotted in Graph IV.

So far, only the equation $u_t + u\,u_x = 0$ has been discussed; the question is, how much of what was said before can be generalized to quasilinear systems. The first observation is that weak solutions are defined only for systems in which all first order terms are perfect x or t derivatives (or at most combinations of such terms with coefficients which are functions of independent variables only); for such systems I propose the same finite difference scheme, i.e. replace all x derivatives by centered difference quotients, and replace all t derivatives v_t by $\left(v_{\ell}^{n+1} - \frac{v_{\ell+1}^{n} + v_{\ell-1}^{n}}{2}\right) / \Delta t$.

This was tried on the hydrodynamic equations of one dimensional time dependent flow; the equations were written in the form of conservation laws. They are, in Eulerian coordinates,

$$\rho_t + (u\rho)_x = 0, \qquad \text{Cons. mass}$$

$$(u\rho)_t + (u^2\rho)_x + p_x = 0, \qquad \text{Cons. of momentum}$$

$$\left(\rho e + \frac{u^2}{2}\right)_t + \left(\rho e u + \frac{u^3\rho}{2}\right)_x + (up)_x = 0, \qquad \text{Cons. of energy.}$$

Here ρ, u,p and e denote density, velocity, pressure and energy per unit mass. The equation of state expresses e as a function of p and ρ, e.g. for an ideal gas $e = \frac{p}{\rho(\lambda-1)}$.

In the computations we will operate with the quantities ρ, $u\,p = m$ and $e\rho + u^2\rho/2 = E$, the mass, momentum and total energy per unit volume. In terms of these the equations are:

$$\rho_t + m_x = 0$$

$$m_t + \left[(\gamma-1)E + \frac{3-\gamma}{2}\frac{m^2}{\rho}\right]_x = 0$$

$$E_t + \left[\gamma\frac{m}{\rho}E - \frac{\gamma-1}{2}\frac{m^3}{\rho^2}\right]_x = 0$$

To these equations the difference scheme described before was applied. Several calculations were made, with different choice of the initial values and γ, and in all cases the answer agreed fairly well with the theoretically calculated flow. The calculations were performed on the Los Alamos MANIAC. The flow diagram for the calculations was prepared by Stewart Schlesinger and the problem was coded by Lois Cook.

In the first problem γ was chosen equal to 1.5, and

$u = 2$	for $x < 0$,
$= 0$	for $x > 0$
$p = 50$	for $x < 0$
0	for $x > 0$
$\rho = 50$	for $x < 0$
$= 10$	for $x > 0$.

The two constant states chosen can be connected by a shock (notice that compression is five-fold at the shock, the value corresponding to $\gamma = 1.5$): $\Delta t/\Delta x$ was chosen to be .25.

The results after 49 time cycles are given in Table V. The fourth column, ν, gives the label of the lattice point in hexadecimal notation; the Eulerian position x is related to the label ν by $x - 4(2\nu - 52)$ (taking t to be one). There is a rapid transition from one state to another around $\nu = 41$; this corresponds to $x - 124$, and gives for the speed of propagation of the discontinuity $\frac{122}{49} = 2.48$; this agrees pretty well with the theoretical value of the shock speed which is 2.5.

The values of ρ, u and p after 99 time cycles are given in Table VI; the position of the ν^{th} subdivision now is given by $x = 4\ (2\nu - 102)$. Again there is a rapid transition from one set of values to the other, around $\nu = 82$; so the speed of propagation is $\frac{248}{99} = 2.50$.

Notice that the <u>width</u> of the zone of transition is approximately the same in both calculations.

The stability constant, i.e. the reciprocal of the ratio of $\Delta x/\Delta t$ to the maximum of the true propagation speed is .863.

A second calculation started with the initial states $u = 2$, $p = 50$, $\rho = 50$ to the left, $u = 0$, $p = 0$, $\rho = 10$ to the right of $x = 0$. $\Delta t/\Delta x$ was taken to be .25. These two constant states can be connected to each other through a rarefaction wave, a contact discontinuity, a constant state and a shock (going from left to right). According to theory, the constant state behind the shock is $u = 1.47$, $p = 27.1$, $\rho = 50$, and shock speed is $U = 1.84$.

The results after 49 time cycles are given in Table VII, after 99 time cycles in Table VIII. In Table VII there is a rapid transition around $\nu = 37$ which corresponds to a shock speed of $\frac{88}{49} = 1.79$, which is in fair agreement with the calculated value. In Table VIII the transition occurs around $\nu = 74$ which gives for shockspeed $184/99 = 1.86$, in even better agreement with the calculated value.

In Table VIII, u and p appear to be fairly constant for a while behind the shock, the value of p being around $27 \pm .3$, and of u around $(.184 \pm .001)8 = 1.47 \pm .01$. These are in fair agreement with the theoretically calculated values, in spite of the fact that the value of ρ is way off (only around 39 at the shock front, whereas the correct value is 50).

A third calculation was done for the case $\gamma = 2$, and initial states $u = 2$, $\rho = 50$, $p = 100$ to the left of $x = 0$, $u = 0$, $\rho = 10$, $p = 0$ to the right. $\frac{\Delta t}{\Delta x}$ was chosen as .25 which turned out larger than permissible by the Courant-Friedrichs-Lewy criterion. Consequently, instability occurred near the shock front, but not enough to make the calculations meaningless, as the listings in Table IX and X show; these present the calculated values of the unknowns after 49, resp. 99 steps.

The exact solution connects the two states through a rarefaction wave, a contact discontinuity, a constant state and a shock. The theoretically calculated value of u, ρ and p behind the shock front are: u = 2.26, ρ = 30, p = 76.5; these compare favorably with the calculated values of u and p.

Two general features of these calculations are:

(i) The width of the transition shock in the shock is narrowest if $\Delta t / \Delta x$ is chosen as large as possible.

(ii) The values of u and p converge to the exact value more rapidly than the value of ρ.

The method can be set up in Lagrange coordinates as well. Denoting specific volume by V and by ξ unit mass along the x axis, the conservation equations are:

$V_t = u_\xi$ Conservation of mass

$u_t = p_\xi$ Conservation of momentum

$(e + {}^1/2u^2) = -(up)_\xi$ Conservation of energy

Introduce as unknowns V, u and $E = e + 1/2\ u^2$, mass, momentum and energy per unit volume. In terms of these, the equations for a perfect gas ($e = \frac{pV}{\gamma - 1}$) can be written as

$$V_t = u_\xi$$

$$u_t = \left[(\gamma - 1) \frac{E - 1/2\ u^2}{V} \right]$$

$$E_t = \left[(\gamma - 1) \frac{uE - 1/2u^3}{V} \right]$$

Experimental calculations in this setup are being carried out by Lester Baumhoff. Results so far are encouraging.

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x	u
-8	.99991
-7	.99921
-6	.99548
-5	.98205
-4	.94713
-3	.87779
-2	.76816
-1	.62581
0	.47071
1	.32661
2	.21061
3	.12798
4	.07450
5	.04216
6	.02344
7	.01291
8	.00707
9	.00386
10	.00210
11	.00114
12	.000621

TABLE I

x	u
-9	.99998
-8	.99978
-7	.99836
-6	.99195
-5	.97176
-4	.92476
-3	.83976
-2	.71566
-1	.56551
0	.41203
1	.27747
2	.17449
3	.10407
4	.05982
5	.03359
6	.01859
7	.01021
8	.00558
9	.00304
10	.00166
11	.00090
12	.00049

TABLE II

APPROVED FOR PUBLIC RELEASE

x	u
47	.92695
45	.88187
43	.83994
41	.79948
39	.7599
37	.7209
35	.6825
33	.6444
31	.6066
29	.5692
27	.5321
25	.4954
23	.4590
21	.4229
19	.3873
17	.3523
15	.3177
13	.2839
11	.2509
9	.2189
7	.1881
5	.1587
3	.1310
1	.1055
-1	.0823
-3	.0619
-5	.0447
-7	.0306
-9	.0198
-11	.0120

TABLE III

Rarefaction wave, $t = 48$, $\frac{\Delta t}{\Delta x} = 1$

APPROVED FOR PUBLIC RELEASE

x	u
64	.8553
60	.8170
56	.7758
52	.7322
48	.6869
44	.6405
40	.5933
36	.5457
32	.4980
28	.4506
24	.4039
20	.3580
16	.3134
12	.2704
8	.2295
4	.1911
0	.1555
-4	.1234
-8	.0949
-12	.0706
-16	.0505
-20	.0345
-24	.0225
-28	.0139

TABLE IV

Rarefaction wave, $t = 63$, $\frac{\Delta t}{\Delta x} = 1/2$

ρ	u/8	p	ν
0004999999	0249999996	0005000000	0000100001
0004999999	0249999996	0005000000	0000200002
0004999999	0249999996	0005000000	0000300003
0004999999	0249999996	0005000000	0000400004
0004999999	0249999996	0005000000	0000500005
0004999999	0249999996	0005000000	0000600006
0004999999	0249999995	0005000000	0000700007
0004999999	0249999995	0005000000	0000800008
0004999999	0249999995	0005000000	0000900009
0004999999	0249999995	0005000000	0000A0000A
0005000000	0249999995	0005000000	0000B0000B
0005000000	0249999994	0005000000	0000C0000C
0005000000	0249999992	0005000000	0000D0000D
0004999999	0249999997	0005000000	0000E0000E
0004999999	0250000003	0004999999	0000F0000F
0004999998	0250000034	0004999998	0001000010
0004999995	0250000130	0004999993	0001100011
0004999985	0250000448	0004999977	0001200012
0004999954	0250001372	0004999932	0001300013
0004999874	0250003843	0004999811	0001400014
0004999676	0250009892	0004999515	0001500015
0004999233	0250023472	0004998850	0001600016
0004998317	0250051481	0004997478	0001700017
0004996581	0250104557	0004994879	0001800018
0004993556	0250196950	0004990358	0001900019
0004988711	0250344587	0004983141	0001A0001A
0004981579	0250560809	0004972585	0001B0001B
0004971936	0250850137	0004958489	0001C0001C
0004959944	0251201791	0004941400	0001D0001D
0004946159	0251585849	0004922792	0001E0001E
0004931292	0251954895	0004904964	0001F0001F
0004915703	0252252703	0004890617	0002000020
0004898791	0252428575	0004882157	0002100021
0004878665	0252452643	0004880985	0002200022
0004852601	0252325585	0004887019	0002300023
0004818585	0252077313	0004898683	0002400024
0004777518	0251751784	0004913220	0002500025
0004734457	0251369295	0004926485	0002600026
0004695222	0250805127	0004928633	0002700027
0004648115	0249273378	0004880761	0002800028
0004497307	0243178521	0004620551	0002900029
0003919992	0219682262	0003672908	0002A0002A
0002622113	0152416320	0001753900	0002B0002B
0001424088	0049060026	0000289912	0002C0002C
0001047003	0004295607	0000012999	0002D0002D
0001002900	0000179506	0000000347	0002E0002E
0001000129	0000005885	0000000007	0002F0002F
0001000004	0000000161	F000000000	0003000030
0001000000	0000000001	F000000000	0003100031
0001000000	0000000000	F000000000	0003200032

TABLE V $\gamma = 1.5$

ρ	u/8	p	ν
0004999999	0249999996	0005000000	0000100001
0004999999	0249999996	0005000000	0000200002
0004999999	0249999996	0005000000	0000300003
0004999999	0249999996	0005000000	0000400004
0004999999	0249999996	0005000000	0000500005
0004999999	0249999996	0005000000	0000600006
0004999999	0249999995	0005000000	0000700007
0004999999	0249999995	0005000000	0000800008
0004999999	0249999995	0005000000	0000900009
0004999999	0249999995	0005000000	0000A0000A
0004999999	0249999995	0005000000	0000B0000B
0004999999	0249999995	0005000000	0000C0000C
0004999999	0249999995	0005000000	0000D0000D
0004999999	0249999995	0005000000	0000E0000E
0004999999	0249999996	0005000000	0000F0000F
0004999999	0249999996	0005000000	0001000010
0004999999	0249999995	0005000000	0001100011
0004999999	0249999995	0005000000	0001200012
0004999999	0249999995	0005000000	0001300013
0004999999	0249999996	0005000000	0001400014
0004999999	0249999996	0005000000	0001500015
0004999999	0249999995	0005000000	0001600016
0004999999	0249999995	0005000000	0001700017
0004999999	0249999995	0005000000	0001800018
0004999999	0249999995	0005000000	0001900019
0004999999	0249999995	0005000000	0001A0001A
0004999999	0249999995	0005000000	0001B0001B
0004999999	0249999995	0005000000	0001C0001C
0004999999	0249999995	0005000000	0001D0001D
0004999999	0249999996	0005000000	0001E0001E
0004999999	0249999996	0005000000	0001F0001F
0004999999	0249999995	0005000000	0002000020
0005000000	0249999993	0005000000	0002100021
0005000000	0249999992	0005000000	0002200022
0004999999	0249999992	0005000000	0002300023
0004999999	0249999996	0005000000	0002400024
0004999999	0249999999	0004999999	0002500025
0004999999	0250000004	0004999999	0002600026
0004999999	0250000019	0004999999	0002700027
0004999997	0250000058	0004999997	0002800028
0004999995	0250000141	0004999993	0002900029
0004999989	0250000327	0004999983	0002A0002A
0004999975	0250000728	0004999964	0002B0002B
0004999949	0250001550	0004999924	0002C0002C
0004999896	0250003166	0004999844	0002D0002D
0004999797	0250006207	0004999695	0002E0002E
0004999617	0250011702	0004999426	0002F0002F
0004999306	0250021217	0004998960	0003000030
0004998791	0250037014	0004998186	0003100031
0004997975	0250062145	0004996956	0003200032

TABLE VI - $\gamma = 1.5$

ρ	u/8	p	ν
0004996719	0250100453	0004995080	0003300033
0004994892	0250156385	0004992343	0003400034
0004992341	0250234525	0004988521	0003500035
0004988933	0250338921	0004983418	0003600036
0004984586	0250472142	0004976912	0003700037
0004979298	0250634225	0004969007	0003800038
0004973182	0250821754	0004959873	0003900039
0004966478	0251027272	0004949879	0003A0003A
0004959555	0251239273	0004939588	0003B0003B
0004952874	0251443000	0004929714	0003C0003C
0004946936	0251621990	0004921054	0003D0003D
0004942193	0251760277	0004914371	0003E0003E
0004938950	0251844846	0004910288	0003F0003F
0004937269	0251867837	0004909178	0004000040
0004936891	0251827979	0004911101	0004100041
0004937197	0251730811	0004915790	0004200042
0004937229	0251587657	0004922704	0004300043
0004935771	0251413526	0004931121	0004400044
0004931526	0251224559	0004940261	0004500045
0004923350	0251035561	0004949424	0004600046
0004910556	0250858225	0004957977	0004700047
0004893217	0250700262	0004965597	0004800048
0004872400	0250565399	0004972077	0004900049
0004850238	0250454031	0004977390	0004A0004A
0004829740	0250364137	0004981618	0004B0004B
0004814306	0250291814	0004984875	0004C0004C
0004806966	0250229836	0004987155	0004D0004D
0004809290	0250157723	0004987796	0004E0004E
0004819311	0249994894	0004983197	0004F0004F
0004824868	0249395725	0004956963	0005000050
0004777634	0246902908	0004840299	0005100051
0004508862	0236906638	0004388264	0005200052
0003646730	0202778528	0003094194	0005300053
0002208321	0120726590	0001150186	0005400054
0001250264	0027848932	0000131838	0005500055
0001024238	0001956039	0000005078	0005600056
0001001503	0000084023	0000000146	0005700057
0001000074	0000003133	0000000003	0005800058
0001000003	0000000107	F000000000	0005900059
0001000000	0000000001	F000000000	0005A0005A
0001000000	0000000000	F000000000	0005B0005B
0001000000	0000000000	F000000000	0005C0005C
0001000000	0000000000	F000000000	0005D0005D
0001000000	0000000000	F000000000	0005E0005E
0001000000	0000000000	F000000000	0005F0005F
0001000000	0000000000	F000000000	0006000060
0001000000	0000000000	F000000000	0006100061
0001000000	0000000000	F000000000	0006200062
0001000000	0000000000	F000000000	0006300063
0001000000	0000000000	F000000000	0006400064

TABLE VI - $\gamma = 1.5$ (Continued)

ρ	u/8	p	ν
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ1ØØØØ1
ØØØ4999999	Ø124999999	ØØØ5ØØØØØ2	ØØØØ2ØØØØ2
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ3ØØØØ3
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ4ØØØØ4
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ5ØØØØ5
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ6ØØØØ6
ØØØ4999999	Ø125ØØØØØØ	ØØØ4999999	ØØØØ7ØØØØ7
ØØØ4999999	Ø125ØØØØØ5	ØØØ4999999	ØØØØ8ØØØØ8
ØØØ4999999	Ø125ØØØØ25	ØØØ4999998	ØØØØ9ØØØØ9
ØØØ4999995	Ø125ØØØ128	ØØØ4999993	ØØØØAØØØØA
ØØØ4999981	Ø125ØØØ558	ØØØ4999972	ØØØØBØØØØB
ØØØ499993Ø	Ø125ØØ2124	ØØØ4999895	ØØØØCØØØØC
ØØØ4999766	Ø125ØØ7154	ØØØ4999649	ØØØØDØØØØD
ØØØ4999293	Ø125Ø21613	ØØØ4998941	ØØØØEØØØØE
ØØØ4998Ø75	Ø125Ø589Ø9	ØØØ4997114	ØØØØFØØØØF
ØØØ4995235	Ø125145787	ØØØ4992861	ØØØ1ØØØØ1Ø
ØØØ4989231	Ø125329351	ØØØ4983884	ØØØ11ØØØ11
ØØØ4977661	Ø125682875	ØØØ4966631	ØØØ12ØØØ12
ØØØ4957242	Ø1263Ø6583	ØØØ49363Ø6	ØØØ13ØØØ13
ØØØ4924Ø62	Ø12732Ø271	ØØØ4887333	ØØØ14ØØØ14
ØØØ4874146	Ø128847595	ØØØ4814278	ØØØ15ØØØ15
ØØØ48Ø4189	Ø13Ø995577	ØØØ4713Ø26	ØØØ16ØØØ16
ØØØ4712274	Ø133835291	ØØØ4581827	ØØØ17ØØØ17
ØØØ4598364	Ø137388475	ØØØ4421882	ØØØ18ØØØ18
ØØØ4464475	Ø141621783	ØØØ4237352	ØØØ19ØØØ1S
ØØØ4314571	Ø146447161	ØØØ4Ø3485Ø	ØØØ1AØØØ1A
ØØØ4154283	Ø151725428	ØØØ3822668	ØØØ1BØØØ1B
ØØØ399Ø574	Ø157271Ø68	ØØØ36Ø994Ø	ØØØ1CØØØ1C
ØØØ3831415	Ø162858252	ØØØ34Ø5844	ØØØ1DØØØ1D
ØØØ3685479	Ø16823Ø818	ØØØ3218856	ØØØ1EØØØ1E
ØØØ3561819	Ø17312ØØ97	ØØØ3Ø55951	ØØØ1FØØØ1F
ØØØ3469376	Ø17727Ø73Ø	ØØØ2921594	ØØØ2ØØØØ2Ø
ØØØ3415949	Ø18Ø459794	ØØØ281626Ø	ØØØ21ØØØ21
ØØØ34Ø5562	Ø182462672	ØØØ2733818	ØØØ22ØØØ22
ØØØ3431719	Ø18287Ø1Ø4	ØØØ2655878	ØØØ23ØØØ23
ØØØ3461991	Ø18Ø617433	ØØØ2539879	ØØØ24ØØØ24
ØØØ3412297	Ø173138931	ØØØ23Ø29Ø8	ØØØ25ØØØ25
ØØØ3138476	Ø155484943	ØØØ183536Ø	ØØØ26ØØØ26
ØØØ2541275	Ø121241Ø99	ØØØ1127898	ØØØ27ØØØ27
ØØØ1795728	ØØ7Ø47ØØ25	ØØØØ444845	ØØØ28ØØØ28
ØØØ127Ø824	ØØ244Ø33Ø5	ØØØØØ95873	ØØØ29ØØØ29
ØØØ1Ø62314	ØØØ4718241	ØØØØØ12112	ØØØ2AØØØ2A
ØØØ1Ø1Ø716	ØØØØ632853	ØØØØØØ1182	ØØØ2BØØØ2B
ØØØ1ØØ1496	ØØØØØ7Ø634	ØØØØØØØ1Ø4	ØØØ2CØØØ2G
ØØØ1ØØØ176	ØØØØØØ6932	ØØØØØØØØØ8	ØØØ2DØØØ2D
ØØØ1ØØØØ17	ØØØØØØØ596	ØØØØØØØØØØ	ØØØ2EØØØ2E
ØØØ1ØØØØØ1	ØØØØØØØØ41	FØØØØØØØØØ	ØØØ2FØØØ2F
ØØØ1ØØØØØØ	ØØØØØØØØØ1	FØØØØØØØØØ	ØØØ3ØØØØ3Ø
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ31ØØØ31
ØØØ1ØØØØØØ	ØØØØØØØØØØ	FØØØØØØØØØ	ØØØ32ØØØ32

TABLE VII - $\gamma = 1.5$ $\qquad \frac{\Delta t}{\Delta x} = .25$

ρ	u/8	p	ν
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ1ØØØØ1
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ2ØØØØ2
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ3ØØØØ3
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ4ØØØØ4
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ5ØØØØ5
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØØ6ØØØØ6
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ7ØØØØ7
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ8ØØØØ8
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØ9ØØØØ9
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØAØØØØA
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØBØØØØB
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØCØØØØC
ØØØ4999999	Ø124999999	ØØØ5ØØØØØØ	ØØØØDØØØØD
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØØEØØØØE
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØØFØØØØF
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØ1ØØØØ1Ø
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØ11ØØØ11
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØ12ØØØ12
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØ13ØØØ13
ØØØ4999999	Ø125ØØØØØØ	ØØØ5ØØØØØØ	ØØØ14ØØØ14
ØØØ4999999	Ø125ØØØØØØ	ØØØ4999999	ØØØ15ØØØ15
ØØØ4999999	Ø125ØØØØØ1	ØØØ4999999	ØØØ16ØØØ16
ØØØ4999999	Ø125ØØØØØ3	ØØØ4999999	ØØØ17ØØØ17
ØØØ4999999	Ø125ØØØØØ7	ØØØ4999999	ØØØ18ØØØ18
ØØØ4999999	Ø125ØØØØ17	ØØØ4999999	ØØØ19ØØØ19
ØØØ4999998	Ø125ØØØØ47	ØØØ4999997	ØØØ1AØØØ1A
ØØØ4999993	Ø125ØØØ14Ø	ØØØ4999993	ØØØ1BØØØ1B
ØØØ4999987	Ø125ØØØ37Ø	ØØØ4999981	ØØØ1CØØØ1C
ØØØ4999969	Ø125ØØØ935	ØØØ4999954	ØØØ1DØØØ1D
ØØØ4999927	Ø125ØØ223Ø	ØØØ499989Ø	ØØØ1EØØØ1E
ØØØ4999833	Ø125ØØ5Ø81	ØØØ4999751	ØØØ1FØØØ1F
ØØØ4999639	Ø125Ø11Ø43	ØØØ4999458	ØØØ2ØØØØ2Ø
ØØØ499925Ø	Ø125Ø22938	ØØØ4998876	ØØØ21ØØØ21
ØØØ4998511	Ø125Ø45559	ØØØ4997768	ØØØ22ØØØ22
ØØØ4997169	Ø125Ø86644	ØØØ4995756	ØØØ23ØØØ23
ØØØ4994841	Ø125157913	ØØØ4992268	ØØØ24ØØØ24
ØØØ499Ø98Ø	Ø125276131	ØØØ4986486	ØØØ25ØØØ25
ØØØ4984851	Ø1254638Ø9	ØØØ4977318	ØØØ26ØØØ26
ØØØ497553Ø	Ø1257493Ø3	ØØØ4963398	ØØØ27ØØØ27
ØØØ4961933	Ø126166Ø37	ØØØ4943134	ØØØ28ØØØ28
ØØØ4942878	Ø12675Ø718	ØØØ4914817	ØØØ29ØØØ29
ØØØ4917178	Ø12754Ø732	ØØØ4876765	ØØØ2AØØØ2A
ØØØ4883749	Ø128571Ø28	ØØØ4827499	ØØØ2BØØØ2B
ØØØ4841726	Ø129871Ø56	ØØØ4765915	ØØØ2CØØØ2C
ØØØ479Ø548	Ø131462178	ØØØ469142Ø	ØØØ2DØØØ2D
ØØØ473ØØ22	Ø133355935	ØØØ46Ø4ØØ5	ØØØ2EØØØ2E
ØØØ466Ø345	Ø135553271	ØØØ45Ø4264	ØØØ2FØØØ2F
ØØØ4582Ø87	Ø138Ø44579	ØØØ4393347	ØØØ3ØØØØ3Ø
ØØØ4496156	Ø14Ø81Ø4ØØ	ØØØ4272869	ØØØ31ØØØ31
ØØØ44Ø374Ø	Ø143822382	ØØØ41448Ø6	ØØØ32ØØØ32

TABLE VIII - $\gamma = 1.5$ $\quad \frac{\Delta t}{\Delta x} = .25$

ρ	u/8	p	ν
0004306247	0147044333	0004011372	0003300033
0004205248	0150433162	0003874919	0003400034
0004102431	0153939577	0003737847	0003500035
0003999555	0157508701	0003602530	0003600036
0003898419	0161080611	0003471265	0003700037
0003800835	0164591203	0003346216	0003800038
0003708590	0167973545	0003229370	0003900039
0003623420	0171160248	0003122482	0003A0003A
0003546974	0174086953	0003026994	0003B0003B
0003480794	0176697087	0002943962	0003C0003C
0003426304	0178947370	0002873963	0003D0003D
0003384857	0180813099	0002817020	0003E0003E
0003357805	0182291846	0002772572	0003F0003F
0003346618	0183404060	0002739506	0004000040
0003352972	0184189790	0002716260	0004100041
0003378734	0184701446	0002700968	0004200042
0003425735	0184993077	0002691590	0004300043
0003495190	0185105149	0002685864	0004400044
0003586560	0185039430	0002680821	0004500045
0003695391	0184709371	0002671317	0004600046
0003809114	0183835998	0002646609	0004700047
0003898991	0181737768	0002583639	0004800048
0003907030	0176945998	0002437055	0004900049
0003735202	0166624061	0002134915	0004A0004A
0003273272	0146121181	0001613665	0004B0004B
0002525437	0110381908	0000932564	0004C0004C
0001747644	0061828936	0000350267	0004D0004D
0001253448	0021147218	0000075605	0004E0004E
0001061906	0004342123	0000010365	0004F0004F
0001012028	0000663114	0000001167	0005000050
0001002018	0000089157	0000000124	0005100051
0001000307	0000011252	0000000012	0005200052
0001000043	0000001358	0000000000	0005300053
0001000005	0000000156	F000000000	0005400054
0001000000	0000000014	F000000000	0005500055
0001000000	0000000000	F000000000	0005600056
0001000000	0000000000	F000000000	0005700057
0001000000	0000000000	F000000000	0005800058
0001000000	0000000000	F000000000	0005900059
0001000000	0000000000	F000000000	0005A0005A
0001000000	0000000000	F000000000	0005B0005B
0001000000	0000000000	F000000000	0005C0005C
0001000000	0000000000	F000000000	0005D0005D
0001000000	0000000000	F000000000	0005E0005E
0001000000	0000000000	F000000000	0005F0005F
0001000000	0000000000	F000000000	0006000060
0001000000	0000000000	F000000000	0006100061
0001000000	0000000000	F000000000	0006200062
0001000000	0000000000	F000000000	0006300063
0001000000	0000000000	F000000000	0006400064

TABLE VIII - $\gamma = 1.5$ $\quad \frac{\Delta t}{\Delta x} = .25$ (Continued)

ρ	u/8	p	ν
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ1ØØØØ1
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ2ØØØØ2
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ3ØØØØ3
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ4ØØØØ4
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ5ØØØØ5
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ6ØØØØ6
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ7ØØØØ7
ØØØ4999999	Ø249999997	ØØ1ØØØØØØØ	ØØØØ8ØØØØ8
ØØØ4999999	Ø249999999	ØØ1ØØØØØØØ	ØØØØ9ØØØØ9
ØØØ4999999	Ø25ØØØØØ1Ø	ØØØ9999999	ØØØØAØØØØA
ØØØ4999998	Ø25ØØØØØ63	ØØØ9999994	ØØØØBØØØØB
ØØØ4999994	Ø25ØØØØ28Ø	ØØØ9999977	ØØØØCØØØØC
ØØØ4999978	Ø25ØØØ1Ø64	ØØØ9999914	ØØØØDØØØØD
ØØØ4999927	Ø25ØØØ3597	ØØØ9999712	ØØØØEØØØØE
ØØØ499978Ø	Ø25ØØ1Ø946	ØØØ9999124	ØØØØFØØØØF
ØØØ4999396	Ø25ØØ3Ø185	ØØØ9997585	ØØØ1ØØØØ1Ø
ØØØ4998482	Ø25ØØ75868	ØØØ9993932	ØØØ11ØØØ11
ØØØ49965Ø6	Ø25Ø174632	ØØØ9986Ø37	ØØØ12ØØØ12
ØØØ49926Ø1	Ø25Ø369757	ØØØ997Ø454	ØØØ13ØØØ13
ØØØ4985525	Ø25Ø7233Ø7	ØØØ9942269	ØØØ14ØØØ14
ØØØ4973717	Ø251313185	ØØØ9895382	ØØØ15ØØØ15
ØØØ4955497	Ø252223496	ØØØ9823365	ØØØ16ØØØ16
ØØØ4929374	Ø2535293Ø9	ØØØ972Ø769	ØØØ17ØØØ17
ØØØ48944ØØ	Ø255279613	ØØØ9584547	ØØØ18ØØØ18
ØØØ485Ø461	Ø257483122	ØØØ9415144	ØØØ19ØØØ19
ØØØ4798426	Ø26Ø1ØØ471	ØØØ9216911	ØØØ1AØØØ1A
ØØØ474Ø133	Ø263Ø43837	ØØØ89978Ø2	ØØØ1BØØØ1B
ØØØ4678215	Ø266183326	ØØØ8768477	ØØØ1CØØØ1C
ØØØ4615789	Ø26935ØØ18	ØØØ8541Ø38	ØØØ1DØØØ1D
ØØØ4556Ø14	Ø272398224	ØØØ8327573	ØØØ1EØØØ1E
ØØØ45Ø15Ø7	Ø275137689	ØØØ8138632	ØØØ1FØØØ1F
ØØØ4453626	Ø277448822	ØØØ7981774	ØØØ2ØØØØ2Ø
ØØØ4411678	Ø27926Ø357	ØØØ786Ø467	ØØØ21ØØØ21
ØØØ4372245	Ø28Ø569345	ØØØ7773773	ØØØ22ØØØ22
ØØØ4328883	Ø281433279	ØØØ77171Ø1	ØØØ23ØØØ23
ØØØ4272568	Ø281945537	ØØØ7683856	ØØØ24ØØØ24
ØØØ41932Ø1	Ø2822Ø7565	ØØØ7667157	ØØØ25ØØØ25
ØØØ4Ø82348	Ø282312942	ØØØ766Ø756	ØØØ26ØØØ26
ØØØ3936881	Ø282344899	ØØØ7659168	ØØØ27ØØØ27
ØØØ3762335	Ø282371915	ØØØ7657992	ØØØ28ØØØ28
ØØØ35739ØØ	Ø282427Ø81	ØØØ7655241	ØØØ29ØØØ29
ØØØ3393169	Ø282482978	ØØØ7652372	ØØØ2AØØØ2A
ØØØ3241468	Ø28249ØØ72	ØØØ7654535	ØØØ2BØØØ2B
ØØØ313ØØ88	Ø2823155Ø5	ØØØ7656295	ØØØ2CØØØ2C
ØØØ3Ø64247	Ø282478443	ØØØ7679252	ØØØ2DØØØ2D
ØØØ3ØØ4559	Ø28Ø53934Ø	ØØØ757356Ø	ØØØ2EØØØ2E
ØØØ3Ø64186	Ø28822Ø456	ØØØ7971564	ØØØ2FØØØ2F
ØØØ1741213	Ø137682163	ØØØ2197261	ØØØ3ØØØØ3Ø
ØØØ1ØØ7659	ØØØØ74722Ø	ØØØØØØ2379	ØØØ31ØØØ31
ØØØ1ØØØØØ4	ØØØØØØØ194	FØØØØØØØØØ	ØØØ32ØØØ32

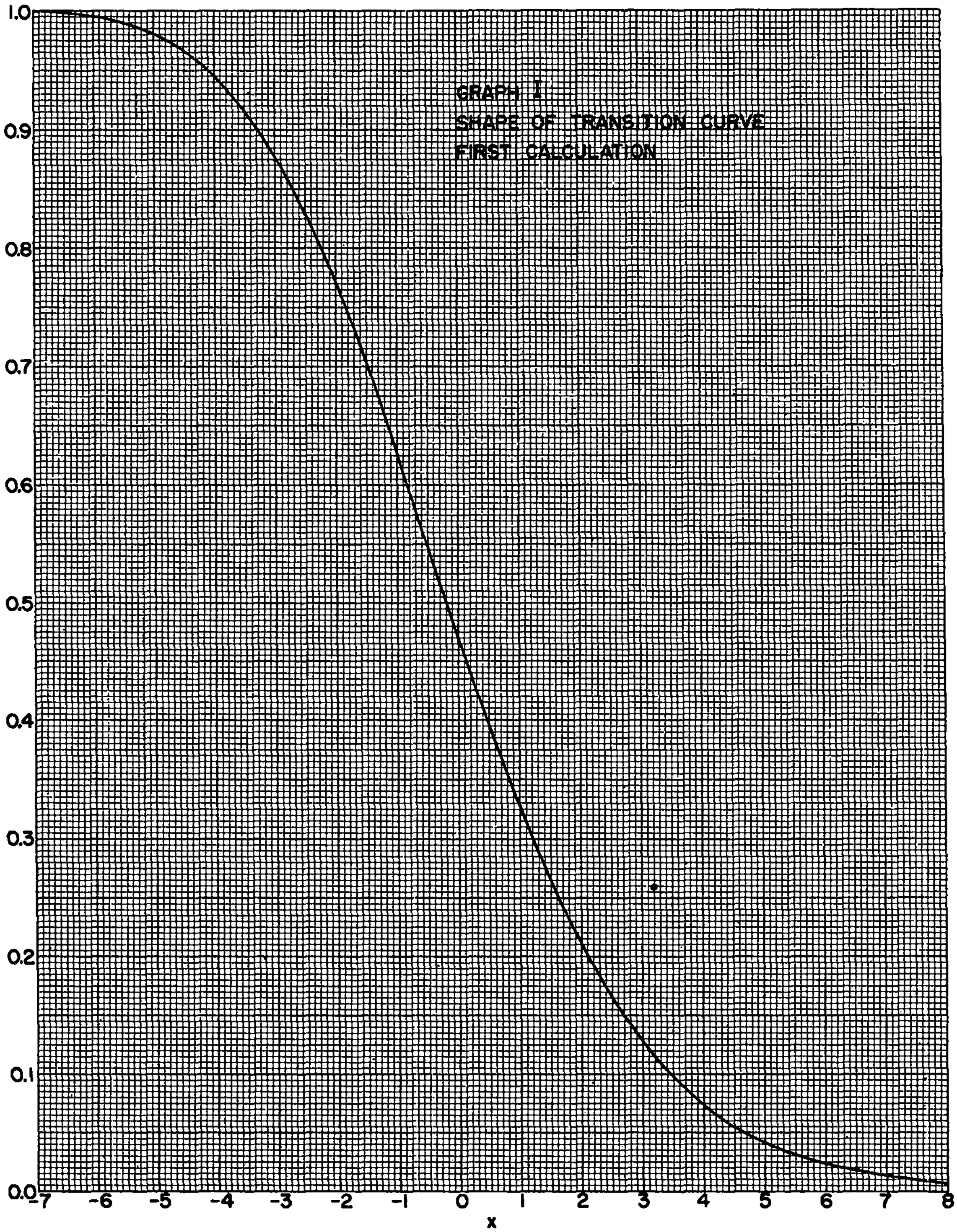
TABLE IX - $\gamma = 2 \quad \frac{\Delta t}{\Delta x} = .25$

ρ	u/8	p	ν
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ1ØØØØ1
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ2ØØØØ2
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ3ØØØØ3
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ4ØØØØ4
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ5ØØØØ5
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ6ØØØØ6
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ7ØØØØ7
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ8ØØØØ8
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØ9ØØØØ9
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØAØØØØA
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØBØØØØB
ØØØ4999999	Ø249999997	ØØ1ØØØØØØØ	ØØØØCØØØØC
ØØØ4999999	Ø249999997	ØØ1ØØØØØØØ	ØØØØDØØØØD
ØØØ4999999	Ø249999997	ØØ1ØØØØØØØ	ØØØØEØØØØE
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØØFØØØØF
ØØØ4999999	Ø249999997	ØØ1ØØØØØØØ	ØØØ1ØØØØ1Ø
ØØØ4999999	Ø249999997	ØØ1ØØØØØØØ	ØØØ11ØØØ11
ØØØ4999999	Ø249999997	ØØ1ØØØØØØØ	ØØØ12ØØØ12
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØ13ØØØ13
ØØØ4999999	Ø249999997	ØØ1ØØØØØØØ	ØØØ14ØØØ14
ØØØ4999999	Ø249999996	ØØ1ØØØØØØØ	ØØØ15ØØØ15
ØØØ4999999	Ø249999997	ØØ1ØØØØØØØ	ØØØ16ØØØ16
ØØØ4999999	Ø249999995	ØØØ9999999	ØØØ17ØØØ17
ØØØ4999999	Ø249999998	ØØ1ØØØØØØØ	ØØØ18ØØØ18
ØØØ4999999	Ø249999997	ØØØ9999999	ØØØ19ØØØ19
ØØØ4999999	Ø25ØØØØØØØ	ØØØ9999999	ØØØ1AØØØ1A
ØØØ4999999	Ø25ØØØØØØ2	ØØØ9999999	ØØØ1BØØØ1B
ØØØ4999999	Ø25ØØØØØ18	ØØØ9999998	ØØØ1CØØØ1C
ØØØ4999998	Ø25ØØØØØ5Ø	ØØØ9999996	ØØØ1DØØØ1D
ØØØ4999997	Ø25ØØØØ143	ØØØ9999988	ØØØ1EØØØ1E
ØØØ4999992	Ø25ØØØØ365	ØØØ999997Ø	ØØØ1FØØØ1F
ØØØ4999982	Ø25ØØØØ891	ØØØ9999928	ØØØ2ØØØØ2Ø
ØØØ4999958	Ø25ØØØ2Ø68	ØØØ9999834	ØØØ21ØØØ21
ØØØ49999Ø8	Ø25ØØØ4596	ØØØ9999633	ØØØ22ØØØ22
ØØØ49998Ø4	Ø25ØØØ9742	ØØØ999922Ø	ØØØ23ØØØ23
ØØØ49996Ø3	Ø25ØØ198Ø2	ØØØ9998416	ØØØ24ØØØ24
ØØØ4999228	Ø25ØØ3857Ø	ØØØ9996914	ØØØ25ØØØ25
ØØØ4998558	Ø25ØØ72Ø83	ØØØ9994235	ØØØ26ØØØ26
ØØØ4997412	Ø25Ø129346	ØØØ9989656	ØØØ27ØØØ27
ØØØ4995537	Ø25Ø223Ø95	ØØØ9982165	ØØØ28ØØØ28
ØØØ4992595	Ø25Ø37Ø174	ØØØ997Ø42Ø	ØØØ29ØØØ29
ØØØ4988167	Ø25Ø5916ØØ	ØØØ995276Ø	ØØØ2AØØØ2A
ØØØ4981764	Ø25Ø911746	ØØØ9927267	ØØØ2BØØØ2B
ØØØ4972863	Ø251356947	ØØØ98919Ø2	ØØØ2CØØØ2C
ØØØ496Ø946	Ø251953263	ØØØ9844684	ØØØ2DØØØ2D
ØØØ4945553	Ø252723999	ØØØ9783913	ØØØ2EØØØ2E
ØØØ4926338	Ø253687Ø58	ØØØ97Ø838Ø	ØØØ2FØØØ2F
ØØØ49Ø311Ø	Ø25485282Ø	ØØØ9617547	ØØØ3ØØØØ3Ø
ØØØ4875866	Ø256222551	ØØØ9511649	ØØØ31ØØØ31
ØØØ48448Ø3	Ø257787762	ØØØ9391729	ØØØ32ØØØ32

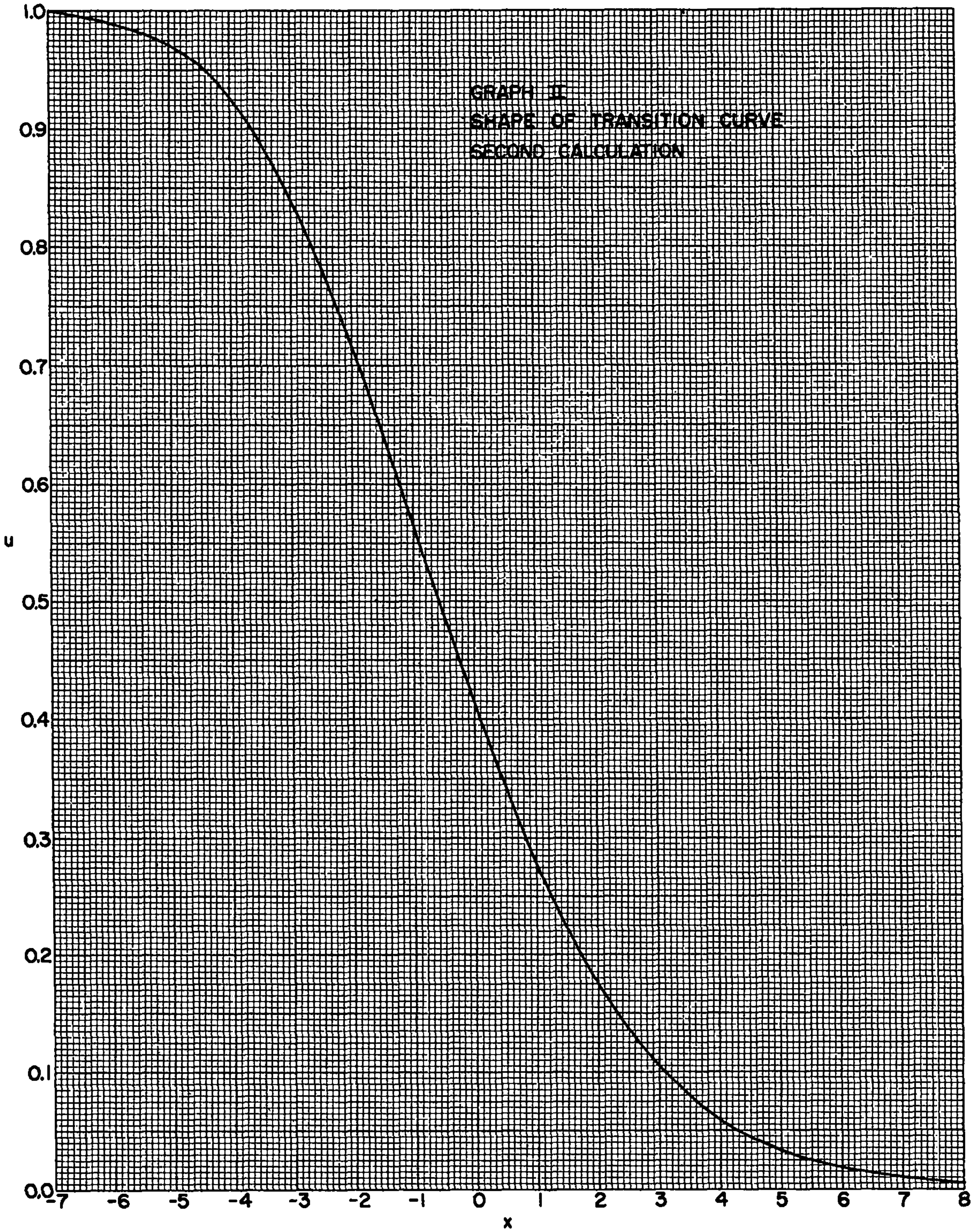
TABLE X - $\gamma = 2$ $\frac{\Delta t}{\Delta x} = .25$

ρ	u/8	p	ν
0004810314	0259530190	0009259583	0003300033
0004772972	0261422580	0009117667	0003400034
0004733503	0263429792	0008968945	0003500035
0004692748	0265510342	0008816732	0003600036
0004651630	0267618123	0008664523	0003700037
0004611103	0269704409	0008515824	0003800038
0004572114	0271720151	0008373984	0003900039
0004535546	0273618584	0008242028	0003A0003A
0004502164	0275358149	0008122489	0003B0003B
0004472566	0276905522	0008017254	0003C0003C
0004447127	0278238225	0007927441	0003D0003D
0004425973	0279346590	0007853323	0003E0003E
0004408970	0280234256	0007794344	0003F0003F
0004395731	0280917019	0007749211	0004000040
0004385657	0281420129	0007716090	0004100041
0004377964	0281774514	0007692839	0004200042
0004371703	0282012618	0007677262	0004300043
0004365751	0282164977	0007667326	0004400044
0004358748	0282257657	0007661312	0004500045
0004349008	0282311012	0007657876	0004600046
0004334432	0282339725	0007656059	0004700047
0004312454	0282353673	0007655211	0004800048
0004280111	0282359599	0007654892	0004900049
0004234284	0282362486	0007654776	0004A0004A
0004172170	0282366262	0007654610	0004B0004B
0004091933	0282373164	0007654251	0004C0004C
0003993404	0282382916	0007653725	0004D0004D
0003878614	0282393174	0007653185	0004E0004E
0003751937	0282402078	0007652749	0004F0004F
0003619728	0282410796	0007652348	0005000050
0003489498	0282422071	0007651814	0005100051
0003368764	0282434529	0007651223	0005200052
0003263850	0282442141	0007651192	0005300053
0003178571	0282418127	0007650968	0005400054
0003115717	0282506165	0007656845	0005500055
0003066646	0282077019	0007628480	0005600056
0003061654	0284160967	0007723382	0005700057
0002981044	0277856067	0007342687	0005800058
0003175166	0294981876	0008228542	0005900059
0002734976	0251060594	0005891079	0005A0005A
0003497571	0313753600	0009270922	0005B0005B
0001993561	0151592162	0002139653	0005C0005C
0003892201	0351489025	0012160273	0005D0005D
0001853143	0148033033	0002426583	0005E0005E
0001035241	0004553368	0000020656	0005F0005F
0001000016	0000000802	0000000000	0006000060
0001000000	0000000000	F000000000	0006100061
0001000000	0000000000	F000000000	0006200062
0001000000	0000000000	F000000000	0006300063
0001000000	0000000000	F000000000	0006400064

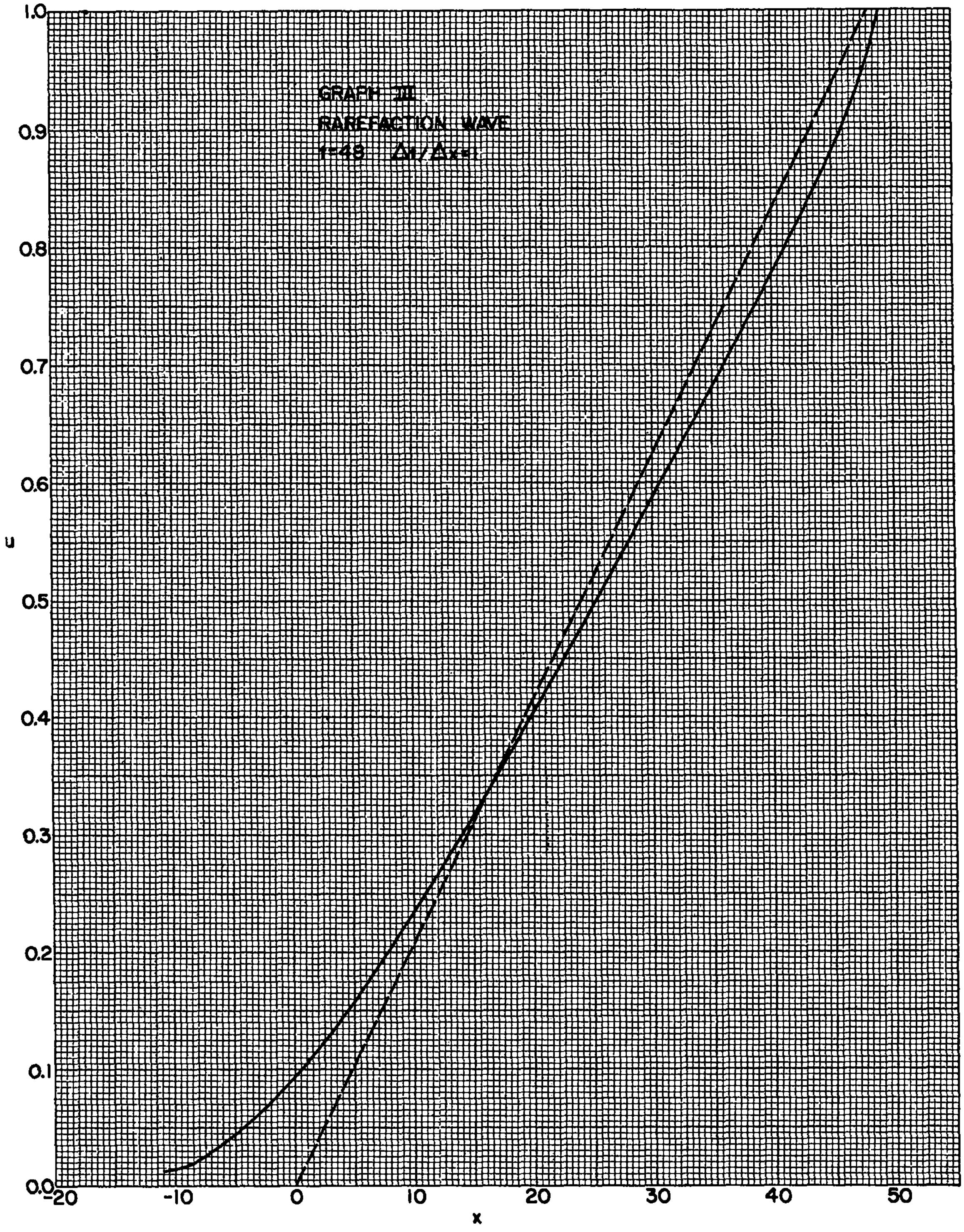
TABLE X - $\gamma = 2$ $\quad \frac{\Delta t}{\Delta x} = .25$ (Continued)



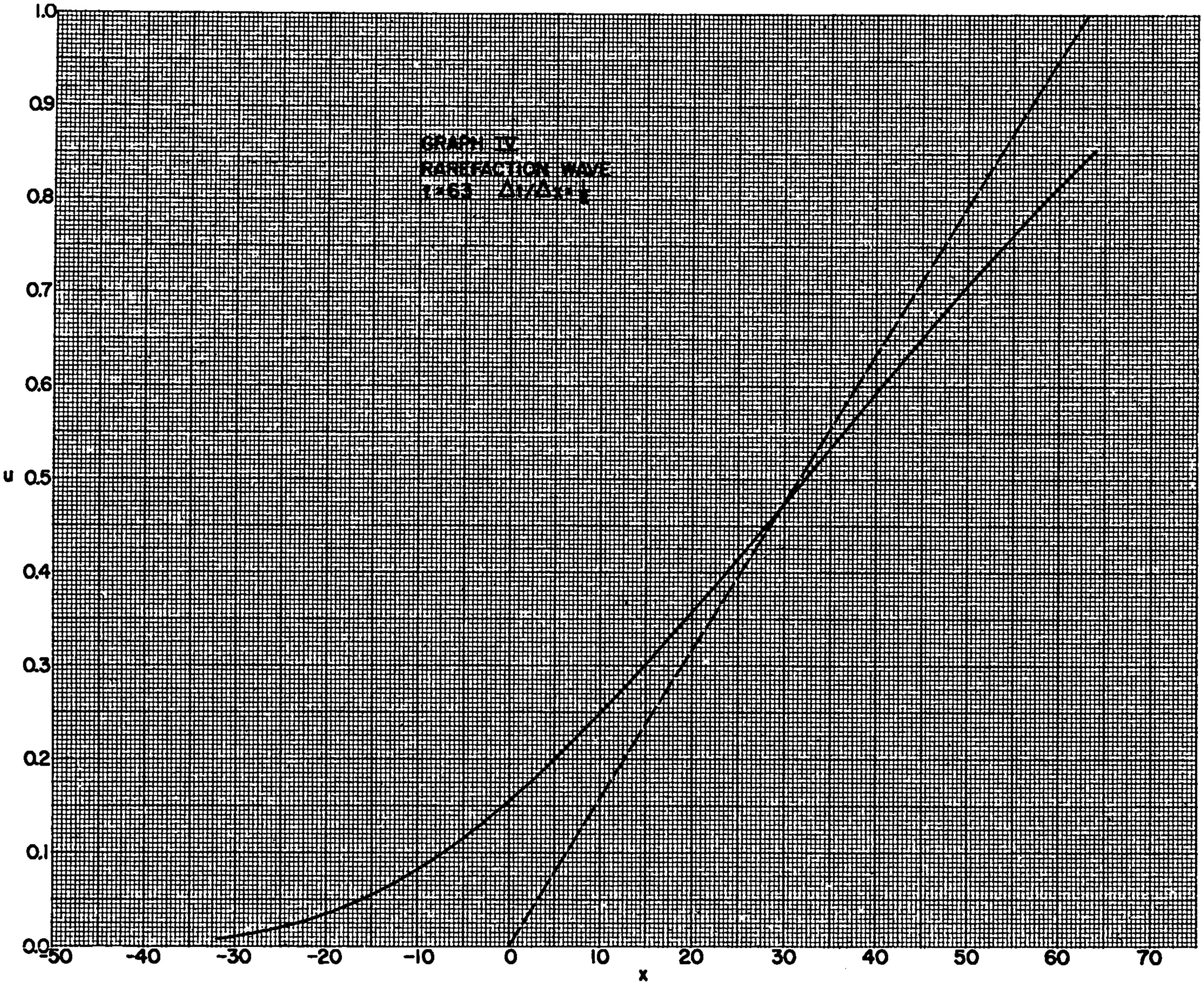
APPROVED FOR PUBLIC RELEASE



APPROVED FOR PUBLIC RELEASE



APPROVED FOR PUBLIC RELEASE



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