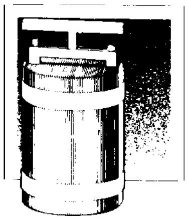

THE SHORT STACK

To calculate thermodynamic efficiency for an acoustic heat engine. we need to know the hydrodynamic heat flow and the work flow. A heuristic derivation of these two quantities and the resulting efficiency for the particular case of a short stack follow. We then briefly discuss the effects of viscosity.

Heat Flow

Consider a slack of plates in a heat engine whose length is short compared to the acoustic wavelength and to the distance from the stack to the end of the tube. If that length is short enough. wc can ignore the change in the longitudinal acoustic velocity magnitude u_1 and the change in the dynamic, or acoustic, pressure magnitude p_1 with respect to longitudinal distance x (measured from the end of the acoustically resonant tube). Further, if we ignore the effects of fluid viscosity, u_1 does not depend on *lateral* distance from the plates. Next, we can take the lateral distance between plates to be large compared to the thermal penetration depth δ_κ (the characteristic length for heat transfer in the fluid during a given cycle of the acoustic wave). Thus, and effects we estimate for a stack of plates will be the same as the a single plate having the same overall perimeter II (measured transverse to the flow).

The adiabatic temperature change T_1 accompanying the pressure change p_1 can be derived from thermodynamics and is

$$T_1 = \frac{T_m\beta}{\rho_m c_p}\, p_1 \,, \tag{1}$$

where T_m is the mean absolute temperature, β is the isobaric expansion coefficient, ρ_m is the mean density, and c_p is the specific heat at constant pressure.

The change of entropy for a parcel oscillating in the manner depicted in Fig. 8 of the main text is just the lateral heat flow from the second medium divided by T_m or $\rho_m c_p \delta T/T_m$ per unit volume, where δT is the change in the fluid temperature due to that heat flow. The volume transport rate for that part of the fluid that is thermodynamically active is $\Pi\delta_\kappa u_1$. We thus can estimate the flow of hydrodynamically transported heat $\dot{Q}$ as the product of these two quantities times T_m; that is,

$$\dot{Q} \sim \Pi\delta_\kappa u_1 \rho_m c_p \delta T \,. \tag{2}$$

Now from Fig. 8 we also see that

$$\delta T = T_1 - x_1\nabla T = T_1\left(1 - \frac{\nabla T}{T_1/x_1}\right) , \tag{3}$$

where ∇T is the temperature gradient along the plate and x_1 is the fluid displacement. The value of ∇T that makes $\delta T = 0$ *is* the critical gradient, so

$$\nabla T_{\rm crit} = \frac{T_1}{x_1} \,. \tag{4}$$

Combining these equations and defining the temperature gradient ratio parameter as $\Gamma \equiv \nabla T/\nabla T_{\rm crit}$ gives an estimate for the hydrodynamic heat flow as

$$\dot{Q} \sim -\Pi\delta_\kappa(T_m\beta)p_1u_1(\Gamma - 1) \,. \tag{5}$$

The parameter $T_m\beta$ is what we call the *heat parameter* of the fluid. The presence of the $\Pi\delta_\kappa$ factor is obvious because it is the thermodynamically active area in a plane perpendicular to the longitudinal acoustic motion. The formula shows that when $\Gamma < 1$, heat flows up the temperature gradient, as for a heat pump; when 1_= 1, there is no heat flow; when $\Gamma > 1$, heat flows down the temperature gradient. as for a prime mover.

Work Flow

Now that we have estimated the heat flow, we need to calculate the work flow, which is given by the work per cycle (the area $p_1\delta V$ enclosed by the pressure-volume diagram in Fig, 8 of the main text) times the rate at which that work occurs (the angular acoustic frequency *w),* The volumetric change δV that will contribute to the net work is just

$$\frac{\delta V}{V} = \beta\delta T \,, \tag{6}$$

where δT is the temperature change of Eq. 3. *V,* the total volume of gas that is thermodynamically active, is given by

$$V = \Pi\delta_\kappa\Delta x \,, \tag{7}$$

where Δx is plate length.

We can now simply put these pieces together and, using Eqs. 1, 3, 6. and 7. write down the work flow as

$$\dot{W} \sim p_1\delta V\omega$$

$$\sim \Pi\delta_\kappa \frac{T_m\beta^2}{\rho_m c_p}\, p_1^2\omega\Delta x(\Gamma - 1) \,. \tag{8}$$

From thermodynamics we know that

$$\gamma - 1 = \frac{T_m\beta^2 a^2}{c_p} \,, \tag{9}$$

where the quantity y — 1 is what we call the *work parameter* of the fluid, and *a* is the speed of sound, so we can rewrite the expression for work flow as

$$\dot{W} \sim \Pi\delta_\kappa(\gamma - 1)\ \frac{p_1^2}{\rho_m a}\ (\Gamma - 1)\ \frac{\Delta x}{\bar{\lambda}}\ , \quad (10)$$

where $\bar{\lambda} = a/\omega$ is the radian length of the acoustic wave.

The formulas for estimating $\dot{Q}$ and $\dot{W}$ (Eqs. 5 and 10) have a very similar structure, which is expected since they are closely related thermodynamically. The heat parameter $T_m\beta$ appears in the formula for *Q*, and the work parameter *y* – 1 appears in the formula for *W*. Both *Q* and *W* are quadratic in the acoustic amplitude p_1 or u_1; both change sign as Γ passes through unity.

Efficiency

A quantitative evaluation of *W* and *Q* for this case of the short stack but for sinusoidal p_1 and u_1 would give the same results except each formula has a numerical coefficient of 1/4. Thus the efficiency n of a short stack with no viscous or longitudinal conduction losses is

$$\eta = \frac{\dot{W}}{\dot{Q}} = \frac{\gamma - 1}{T_m\beta}\ \frac{\omega\Delta x p_1}{\rho_m a^2 u_1}\ . \quad (11)$$

For our standing acoustic wave, $u_1 = u_0 \sin x/\bar{\lambda}$ and $p_1 = \rho_m a u_0 \cos x/\bar{\lambda}$, where x is the distance of the stack from the end of the tube. Then the efficiency can be rewritten simply as

$$\eta = \frac{\gamma - 1}{T_m\beta}\ \frac{\Delta x}{\bar{\lambda}\tan x/\bar{\lambda}}\ . \quad (12)$$

In the important limit of $x \ll \bar{\lambda}$, the efficiency is simply

$$\eta = \frac{\gamma - 1}{T_m\beta}\ \frac{\Delta x}{x} \quad (13)$$

Thus, in either case, efficiency depends only on geometry and fluid parameters, just as for the Brayton and Otto cycles discussed in the text. The temperatures T_h and T_c do not enter.

As the *actual* temperature gradient approaches the critical temperature gradient ∇T_{crit}, the temperature difference δT approaches zero, so that even at the acoustic angular frequency *w* the heat transfer rate and the power output approach zero, just what is needed to give the Carnot efficiency in the Brayton and Otto cycles. What happens in this engine? We use Eqs. 1, 4, and 9 and the fact that $u_1 = x_1 w$ to rewrite the efficiency formula (Eq. 11) in general as

$$\eta = \frac{\Delta x \nabla T_{crit}}{T_m} \quad (14)$$

Because $\Delta T = \Delta x \nabla T_{crit}$ when $\nabla T = \nabla T_{crit}$, we have at the critical temperature gradient

$$\eta = \frac{\Delta T}{T_m}\ . \quad (15)$$

The Carnot efficiency is $\eta_C = 1 - T_c/T_h$. But if $T_c = T_h - \Delta T$, and if $\Delta T/T_h$ is small so that T_h can be replaced by T_m, we get, with our approximations, the same formula for η_C as Eq. 15. So the acoustic engine approaches Carnot's efficiency as the power output and heat transfer rates approach zero, just like the Otto and Brayton cycles.

What About Viscosity?

So far we have assumed that the working fluid is inviscid. What if it is not? We know how to do the theory quantitatively for this more general case, but the resulting expressions for *Q* and *W* are terribly complicated and opaque. We can simplify them by assuming that the Prandtl number (the square of the ratio of the viscous penetration depth δ_ν to the thermal penetration depth δ_κ) is small. In that case we obtain

$$\dot{Q} = \frac{1}{4}\ \Pi\delta_\kappa(T_m\beta)p_1u_1(\Gamma - 1) - \frac{1}{4}\ \Pi\delta_\nu(T_m\beta)p_1u_1 \quad (16)$$

$$\dot{W} = \frac{1}{4}\ \Pi\delta_\kappa(\gamma - 1)\ \frac{p_1^2}{\rho_m a}\ \frac{\Delta x}{\bar{\lambda}}\ (\Gamma - 1) - \frac{1}{4}\ \Pi\delta_\nu\rho_m a u_1^2\ \frac{\Delta x}{\bar{\lambda}}\ . \quad (17)$$

To lowest order, then, the effect of viscosity on heat flow is just to decrease *Q* by a term proportional to the viscous penetration depth. This simply means that viscosity prevents a layer of fluid of thickness δ_ν adjacent to the plate from moving acoustically and contributing to the acoustically stimulated heat transport. Similarly, the work flow is decreased by a term proportional to δ_ν; this term is simply the energy lost from the acoustic wave due to viscous drag on the plate.

For simplicity in Eqs. 16 and 17 we have kept our old definition of ∇T_{crit}, even though another effect of viscosity is to make the concept of a critical temperature gradient less well defined. In fact, with viscosity present there is a lower critical gradient below which the engine pumps heat and a higher critical gradient above which the engine is a prime mover. Between these two gradients the engine is in a useless state. using work to pump heat from *hot* to *cold*.

The Prandtl number for helium gas is about 0.67, so that viscous effects are very significant for our gas acoustic engines (and, in fact, Eqs. 16 and 17 are rather poor approximations). On the other hand. the Prandtl number for liquid sodium is about 0.004, so that viscous effects are much smaller. ■