Supersymmetry in Quantum Mechanics

I intend to develop here some of the algebra pertinent to the basic concepts of supersymmetry. I will do this by showing an analogy between the quantum-mechanical harmonic oscillator and a bosonic field and a further analogy between the quantum-mechanical spin-½ particle and a fermionic field. One result of combining the two resulting fields will be to show that a "tower" of degeneracies between the states for bosons and fermions is a natural feature of even the simplest of supersymmetry theories.

A supersymmetry operation changes bosons into fermions and vice versa, which can be represented schematically with the operators $Q_\alpha^\dagger$ and Q_α and the equations

$$Q_\alpha^\dagger|\text{boson}\rangle = |\text{fermion}\rangle_\alpha \quad \text{and} \quad Q_\alpha|\text{fermion}\rangle = |\text{boson}\rangle_\alpha . \tag{1}$$

In the simplest version of supersymmetry, there are four such operators or generators of supersymmetry (Q_α and the Hermitian conjugate $Q_\alpha^\dagger$ with $\alpha = 1, 2$). Mathematically, the generators are Lorentz spinors satisfying fermionic anticommutation relations

$$\{Q_\alpha^\dagger, Q_\beta\} = p^\mu (\sigma_\mu)_{\alpha\beta} , \tag{2}$$

where p^μ is the energy-momentum four-vector ($p^0 = H$, p^i = three-momentum) and the σ_μ are two-by-two matrices that include the Pauli spin matrices σ^i ($\sigma_\mu = (1, \sigma^i)$ where $i = 1, 2, 3$). Equation 2 represents the unusual feature of this symmetry: the supersymmetry operators combine to generate translation in space and time. For example, the operation of changing a fermion to a boson and back again results in changing the position of the fermion.

If supersymmetry is an invariance of nature, then

$$[H, Q_\alpha] = 0 , \tag{3}$$

that is, Q_α commutes with the Hamiltonian H of the universe. Also, in this case, the vacuum is a supersymmetric singlet ($Q_\alpha|\text{vac}\rangle = 0$).

Equations 1 through 3 are the basic defining equations of supersymmetry. In the form given, however, the supersymmetry is solely an external or space-time symmetry (a supersymmetry operation changes particle spin without altering any of the particle's internal symmetries). An extended supersymmetry that connects external and internal symmetries can be constructed by expanding the number of operators of Eq. 2. However, for our purposes, we need not consider that complication.

The Harmonic Oscillator. In order to illustrate the consequences of Eqs. 1 through 3, we first need to review the quantum-mechanical treatment of the harmonic oscillator.

The Hamiltonian for this system is

$$H_{\text{osc}} = \frac{1}{2}(p^2 + \omega^2 q^2) , \tag{4}$$

where p and q are, respectively, the momentum and position coordinates of a nonrelativistic particle with unit mass and a $2\pi/\omega$ period of oscillation. The coordinates satisfy the quantum-mechanical commutation relation

$$[p, q] = (pq - qp) = -i\hbar \ . \tag{5}$$

The well-known solution to the harmonic oscillator (the set of eigenstates and eigenvalues of H_{osc}) is most conveniently expressed in terms of the so-called raising and lowering operators, $a^\dagger$ and a, respectively, which are defined as

$$a^\dagger = \frac{1}{\sqrt{2\omega\hbar}}(p + i\omega q)$$

and (6)

$$a = \frac{1}{\sqrt{2\omega\hbar}}(p - i\omega q) \, ,$$

and which satisfy the commutation relation

$$[a, a^\dagger] = 1 \ . \tag{7}$$

In terms of these operators, the Hamiltonian becomes

$$H_{osc} = \hbar\omega(a^\dagger a + \tfrac{1}{2}) \, , \tag{8}$$

with eigenstates

$$|n\rangle = N_n(a^\dagger)^n|0\rangle \, , \tag{9}$$

where N_n is a normalization factor and $|0\rangle$ is the ground state satisfying

$$a|0\rangle = 0$$

and (10)

$$\langle 0|0\rangle = 1 \ .$$

It is easy to show that

$$a^\dagger|n\rangle = \sqrt{n+1}\ |n+1\rangle$$

and (11)

$$a|n\rangle = \sqrt{n}\ \ |n-1\rangle \, ,$$

hence the names raising operator for $a^\dagger$ and lowering operator for a. Also note that $a^\dagger a$ is just a counting operator since $a^\dagger\, a\, |n\rangle = n \mid n\rangle$. Finally, we find that

$$H_{osc}\,|n\rangle = \hbar\omega(n + \tfrac{1}{2})\,|n\rangle \, , \tag{12}$$

that is, the states $|n\rangle$ have energy $(n + \tfrac{1}{2})\,\hbar\omega$.

The Bosonic Field. There is a simple analogy between the quantum oscillator and the scalar quantum field needed to represent bosons (scalar particles). A free scalar field is quite rigorously described by an infinite set of noninteracting harmonic oscillators $\{a_p^\dagger, a_p\}$, where p is an index labeling the set. The Hamiltonian of the free field can be written as

$$H_{scalar} = \sum_p \hbar\omega_p\left(a_p^\dagger\, a_p + \tfrac{1}{2}\right) \, , \tag{13}$$

with the summation taken over the individual oscillators p.

The ground state of the free scalar quantum field is called the vacuum (it contains no scalar particles) and is described mathematically by the conditions

$$a_p\,|\mathrm{vac}\rangle = 0$$

and (14)

$$\langle \mathrm{vac}|\mathrm{vac}\rangle = 1 \ .$$

The $a_p^\dagger$ and a_p operators create or annihilate, respectively, a single scalar particle with energy $\hbar\omega_p$ ($\hbar\omega_p = \sqrt{p^2+m^2}$, where p is the momentum carried by the created particle and m is the mass). A scalar particle is thus an excitation of one particular oscillator mode.

The Fermionic Field. The simple quantum-mechanical analogue of a spin-½ field needed to represent fermions is just a quantum particle with spin ½. This is necessary because, whereas bosons can be represented by scalar particles satisfying commutation relations, fermions must be represented by spin-½ particles satisfying anticommutation relations.

A spin-½ particle has two spin states: $|0\rangle$ for spin down and $|1\rangle$ for spin up. Once again we define raising and lowering operators, here $b^\dagger$ and b, respectively. These operators satisfy the anticommutation relations

$$\{b, b^\dagger\} = (bb^\dagger + b^\dagger b) = 1$$

and (15)

$$\{b^\dagger, b^\dagger\} = \{b, b\} = 0 \,.$$

If $b|0\rangle = 0$, it is easy to show that

$$b^\dagger\,|0\rangle = |1\rangle$$

and (16)

$$b^\dagger\,|1\rangle = |0\rangle \,,$$

where $b^\dagger b$ is again a counting operator satisfying

$$b^\dagger b\,|1\rangle = |1\rangle$$

and (17)

$$b^\dagger b\,|0\rangle = 0 \,.$$

We may define a Hamiltonian

$$H_{\text{spin}} = \hbar\omega(b^\dagger b - ½) \,, \tag{18}$$

so that states $|1\rangle$ and $|0\rangle$ will have energy equal to $½\hbar\omega$ and $-½\hbar\omega$, respectively.

The analogy between the free quantum-mechanical fermionic field and the simple quantum-mechanical spin-½ particle is identical to the scalar field case. For example, once again we may define an infinite set $\{b_p^\dagger, b_p\}$ of noninteracting spin-½ particles labeled by the index p. The vacuum state satisfies

$$b_p\,|\text{vac}\rangle = 0$$

and (19)

$$\langle \text{vac}|\text{vac}\rangle = 1 \,.$$

Here $b_p^\dagger$ and b_p are identified as creation and annihilation operators, respectively, of a single fermionic particle. Note that since $\{b_p^\dagger, b_p^\dagger\} = 0$, it is only possible to create *one* fermionic particle in the state p. This is the Pauli exclusion principle.

Supersymmetry. Let us now construct a simple supersymmetric quantum-mechanical system that includes the bosonic oscillator degrees of freedom ($a^\dagger$ and a) and the fermionic spin-½ degrees of freedom ($b^\dagger$ and b). We define the anticommuting charges

$$Q = a^\dagger b(\hbar\omega)^{1/2}$$

and (20)

$$Q^\dagger = a b^\dagger(\hbar\omega)^{1/2} \,.$$

It is then easy to verify that

$$\{Q^\dagger, Q\} = H = H_{\text{osc}} + H_{\text{spin}}$$
$$= \hbar\omega(a^\dagger a + b^\dagger b) \,, \tag{21}$$

and

$$[H, Q] = 0 \,. \tag{22}$$

Equations 21 and 22 are the direct analogues of Eqs. 2 and 3, respectively. We see that the anticommuting charges Q combine to form the generator of time translation, namely, the Hamiltonian H. The ground state of this system is the state $|0\rangle_{\text{osc}}|0\rangle_{\text{spin}} = |0{,}0\rangle$, where both the oscillator and the spin-½ degrees of freedom are in the lowest energy state. This state is a unique one, satisfying

$$Q|0{,}0\rangle = Q^\dagger|0{,}0\rangle = 0 \,. \tag{23}$$

The excited states form a tower of degenerate levels (see figure) with energy $(n + ½)\hbar\omega \pm ½\hbar\omega$, where the sign of the second term is determined by whether the spin-½ state is $|1\rangle$ (plus) or $|0\rangle$ (minus).

The tower of states illustrates the boson-fermion degeneracy for exact supersymmetry. The bosonic states $|n{+}1{,}0\rangle$ (called bosonic in the field theory analogy because they contain no fermions) have the same energy as their fermionic partners $|n{,}1\rangle$.

Moreover, it is easy to see that the charges Q and $Q^\dagger$ satisfy the relations

$$Q|n{,}1\rangle = \sqrt{n+1}\,|n+1{,}0\rangle$$

and

Energy	States	
	Boson	Fermion
0	|0,0>	
$\hbar\omega$	|1,0>	|0,1>
$2\hbar\omega$	|2,0>	|1,1>
$3\hbar\omega$	|3,0>	|2,1>
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.	.	.
.	.	.

The boson-fermion degeneracy for exact supersymmetry in which the first number in $|\mathbf{n,m}\rangle$ corresponds to the state for the oscillator degree of freedom (the scalar, or bosonic, field) and the second number to that for the spin-½ degree of freedom (the fermionic field).

$$Q^\dagger|n+1,0\rangle = \sqrt{n+1}\ |n,1\rangle\,, \tag{24}$$

which are analogous to Eq. 1 because they represent the conversion of a fermionic state to a bosonic state and vice versa.

The above example is a simple representation of supersymmetry in quantum mechanics. It is, however, trivial since it describes non-interacting bosons (oscillators) and fermions (spin-½ particles). Non-trivial *interacting* represntations of supersymmetry may also be obtained. In some of these representations it is possible to show that the ground state is not supersymmetric even though the Hamiltonian is. This is an example of spontaneous supersymmetry breaking.

Symmetry Breaking. If supersymmetry were an exact symmetry of nature, then bosons and fermions would come in degenerate pairs. Since this is not the case, the symmetry must be broken. There are two inequivalent ways in which to do this and thus to have the degeneracy removed.

First we may add a small symmetry breaking term to the Hamiltonian, that is, $H \rightarrow H + \varepsilon H'$, where ε is a small parameter and

$$[H', Q] \neq 0\,. \tag{25}$$

This mechanism is called *explicit symmetry breaking*. Using it we can give scalars a mass that is larger than that of their fermionic partners, as is observed in nature. Although this breaking mechanism may be perfectly self-consistent (even this is in doubt when one includes gravity), it is totally ad hoc and lacks predictive power.

The second symmetry breaking mechanism is termed *spontaneous symmetry breaking*. This mechanism is characterized by the fact that the Hamiltonian remains supersymmetric,

$$[Q,H] = 0\,, \tag{26}$$

but the ground state does not,

$$Q|\mathrm{vac}\rangle \neq 0\,. \tag{27}$$

Supersymmetry can either be a global symmetry, such as the rotational invariance of a ferromagnet, or a local symmetry, such as a phase rotation in electrodynamics. Spontaneous breaking of a *global* symmetry leads to a massless Nambu-Goldstone particle. In supersymmetry we obtain a massless fermion $\tilde{G}$, the goldstino.

Spontaneous breaking of a *local* symmetry, however, results in the gauge particle becoming massive. (In the standard model, the W bosons obtain a mass $M_W = gV$ by "eating" the massless Higgs bosons, where g is the SU(2) coupling constant and V is the vacuum expectation value of the neutral Higgs boson.) The gauge particle of local supersymmetry is called a gravitino. It is the spin-3/2 partner of the graviton; that is, local supersymmetry incorporates Einstein's theory of gravity. When supersymmetry is spontaneously broken, the gravitino obtains a mass

$$m_G = G_N^{1/2}\Lambda_{ss}^2 \tag{28}$$

by "eating" the goldstino (here G_N is Newton's gravitational constant and Λ_{ss} is the vacuum expectation of some field that spontaneously breaks supersymmetry).

Thus, if the ideas of supersymmetry are correct, there is an underlying symmetry connecting bosons and fermions that is "hidden" in nature by spontaneous symmetry breaking. ■