Lecture Notes

from simple field theories to the standard model

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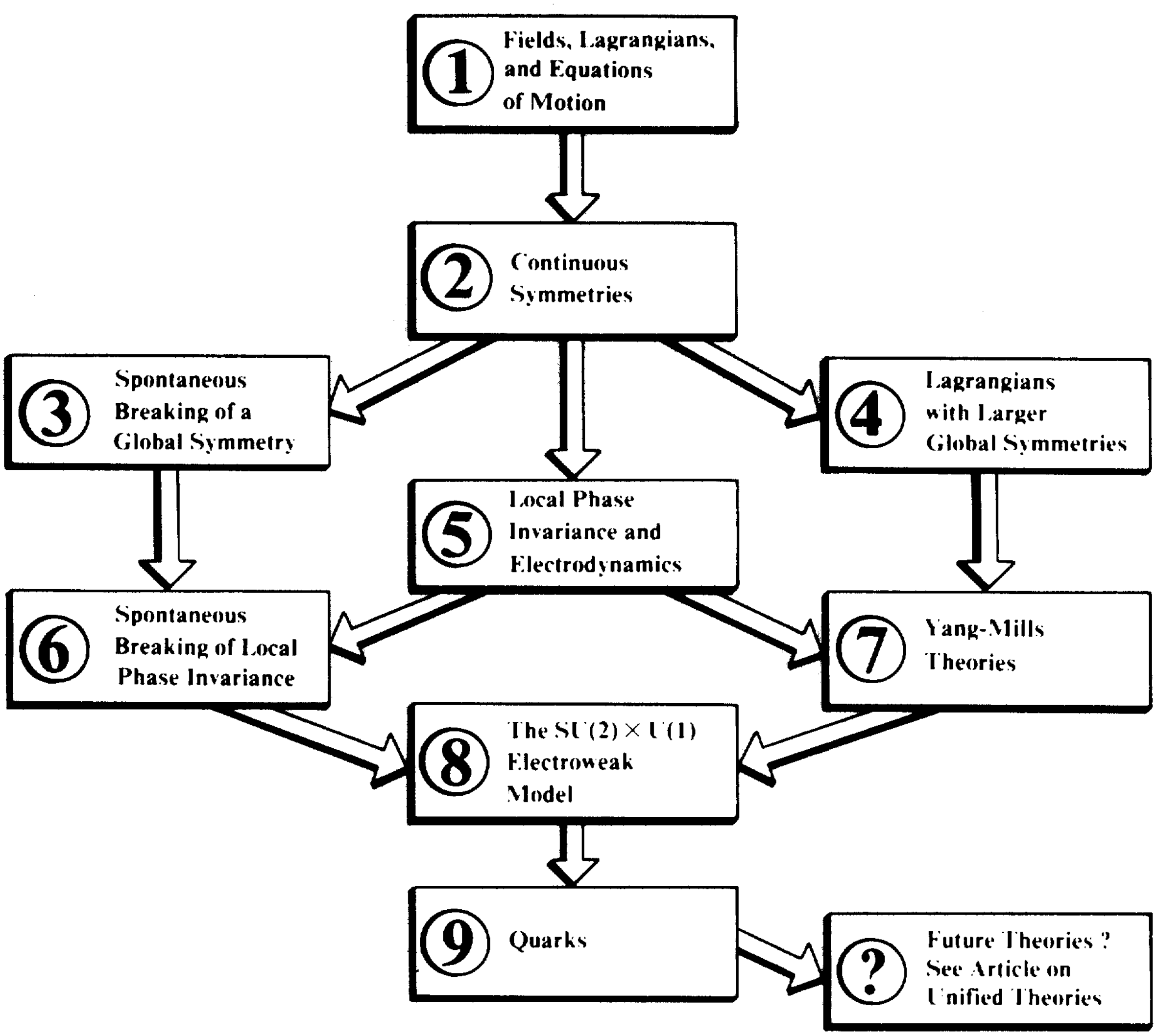
The standard model of electroweak and strong interactions consists of two relativistic quantum field theories, one to describe the strong interactions and one to describe the electromagnetic and weak interactions. This model, which incorporates all the known phenomenology of these fundamental interactions, describes spinless, spin-½, and spin-1 fields interacting with one another in a manner determined by its Lagrangian. The theory is relativistically invariant, so the mathematical form of the Lagrangian is unchanged by Lorentz transformations.

Although rather complicated in detail, the standard model Lagrangian is based on just two basic ideas beyond those necessary for a quantum field theory. One is the concept of local symmetry, which is encountered in its simplest form in electrodynamics. Local symmetry determines the form of the interaction between particles, or fields, that carry the charge associated with the symmetry (not necessarily the electric charge). The interaction is mediated by a spin-1 particle, the vector boson, or gauge particle. The second concept is spontaneous symmetry breaking, where the vacuum (the state with no particles) has a nonzero charge distribution. In the standard model the nonzero weak-interaction charge distribution of the vacuum is the source of most masses of the particles in the theory. These two basic ideas, local symmetry and spontaneous symmetry breaking, are exhibited by simple field theories. We begin these lecture notes with a Lagrangian for scalar fields and then, through the extensions and generalizations indicated by the arrows in the diagram below, build up the formalism needed to construct the standard model.



1 Fields, Lagrangians, and Equations of Motion

We begin this introduction to field theory with one of the simplest theories, a complex scalar field theory with independent fields $\varphi(x)$ and $\varphi^\dagger(x)$. ($\varphi^\dagger(x)$ is the complex conjugate of $\varphi(x)$ if $\varphi(x)$ is a classical field, and, if $\varphi(x)$ is generalized to a column vector or to a quantum field, $\varphi^\dagger(x)$ is the Hermitian conjugate of $\varphi(x)$.) Since $\varphi(x)$ is a complex function in classical field theory, it assigns a complex number to each four-dimensional point $x = (ct, \mathbf{x})$ of time and space. The symbol x denotes all four components. In quantum field theory $\varphi(x)$ is an operator that acts on a state vector in quantum-mechanical Hilbert space by adding or removing elementary particles localized around the space-time point x.

In this note we present the case in which $\varphi(x)$ and $\varphi^\dagger(x)$ correspond respectively to a spinless charged particle and its antiparticle of equal mass but opposite charge. The charge in this field theory is like electric charge, except it is not yet coupled to the electromagnetic field. (The word "charge" has a broader definition than just electric charge.) In Note 3 we show how this complex scalar field theory can describe a quite different particle spectrum: instead of a particle and its antiparticle of equal mass, it can describe a particle of zero mass and one of nonzero mass, each of which is its own antiparticle. Then the scalar theory exhibits the phenomenon called spontaneous symmetry breaking, which is important for the standard model.

A complex scalar theory can be defined by the Lagrangian density,

$$\mathscr{L}(\varphi,\partial_\mu\varphi,\varphi^\dagger,\partial_\mu\varphi^\dagger) = \partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi - \lambda(\varphi^\dagger\varphi)^2\,, \tag{1a}$$

where $\partial_\mu\varphi \equiv \partial\varphi/\partial x^\mu$. (Upper and lower indices are related by the metric tensor, a technical point not central to this discussion.) The Lagrangian itself is

$$\mathrm{L}(t_1,t_2) \equiv \int_{t_2}^{t_1} dt \int d^3\mathbf{x}\,\mathscr{L}\,. \tag{1b}$$

The first term in Eq. 1a is the kinetic energy of the fields $\varphi(x)$ and $\varphi^\dagger(x)$, and the last two terms are the negative of the potential energy. Terms quadratic in the fields, such as the $-m^2\varphi^\dagger\varphi$ term in Eq. 1a. are called mass terms. If $m^2 > 0$, then $\varphi(x)$ describes a spinless particle and $\varphi^\dagger(x)$ its antiparticle of identical mass. If $m^2 < 0$, the theory has spontaneous symmetry breaking.

The equations of motion are derived from Eq. 1 by a variational method. Thus, let us change the fields and their derivatives by a small amount $\delta\varphi(x)$ and $\delta\partial_\mu\varphi(x) = \partial_\mu\delta\varphi(x)$. Then,

$$\delta\mathrm{L}(t_1,t_2) = \int_{t_2}^{t_2}\left[\frac{\partial\mathscr{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathscr{L}}{\partial\varphi^\dagger}\delta\varphi^\dagger + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi)}\partial_\mu\delta\varphi + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi^\dagger)}\partial_\mu\delta\varphi^\dagger\right]d^4x\,, \tag{2}$$

where the variation is defined with the restrictions $\delta\varphi(\mathbf{x},t_1) = \delta\varphi(\mathbf{x},t_2) = \delta\varphi^\dagger(\mathbf{x},t_1) = \delta\varphi^\dagger(\mathbf{x},t_2) = 0$, and $\delta\varphi(x)$ and $\delta\varphi^\dagger(x)$ are independent. The last two terms are integrated by parts, and the surface term is dropped since the integrand vanishes on the boundary. This procedure yields the Euler-Lagrange equations for $\varphi^\dagger(x)$,

$$\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial\mathscr{L}}{\partial\varphi} = 0\,, \tag{3}$$

and for $\varphi(x)$. (The Euler-Lagrange equation for $\varphi(x)$ is like Eq. 3 except that $\varphi^\dagger$ replaces φ. There are two equations because $\delta\varphi(x)$ and $\delta\varphi^\dagger(x)$ are independent.) Substituting the Lagrangian density, Eq. 1a, into the Euler-Lagrange equations, we obtain the equations of motion,

$$\partial^\mu\partial_\mu\varphi + m^2\varphi + 2\lambda(\varphi^\dagger\varphi)\varphi = 0\,, \tag{4}$$

plus another equation of exactly the same form with $\varphi(x)$ and $\varphi^\dagger(x)$ exchanged.

This method for finding the equations of motion can be easily generalized to more fields and to fields with spin. For example, a field theory that is incorporated into the standard model is electrodynamics. Its list of fields includes particles that carry spin. The electromagnetic vector potential $A_\mu(x)$ describes a "vector" particle with a spin of 1 (in units of the quantum of action $\hbar = 1.0546 \times 10^{-27}$ erg second), and its four spin components are enumerated by the space-time vector index μ (= 0, 1, 2, 3, where 0 is the index for the time component and 1, 2, and 3 are the indices for the three space components). In electrodynamics only two of the four components of $A_\mu(x)$ are independent. The electron has a spin of ½, as does its antiparticle, the positron. Electrons and positrons of both spin projections, ±½, are described by a field $\psi(x)$, which is a column vector with four entries. Many calculations in electrodynamics are complicated by the spins of the fields.

There is a much more difficult generalization of the Lagrangian formalism: if there are constraints among the fields, the procedure yielding the Euler-Lagrange equations must be modified, since the field variations are not all independent. This technical problem complicates the formulation of electrodynamics and the standard model, especially when computing quantum corrections. Our examination of the theory is not so detailed as to require a solution of the constraint problem.

2 Continuous Symmetries

It is often possible to find sets of fields in the Lagrangian that can be rearranged or transformed in ways described below without changing the Lagrangian. The transformations that leave the Lagrangian unchanged (or invariant) are called symmetries. First, we will look at the form of such transformations, and then we will discuss implications of a symmetrical Lagrangian. In some cases symmetries imply the existence of conserved currents (such as the electromagnetic current) and conserved charges (such as the electric charge), which remain constant during elementary-particle collisions. The conservation of energy, momentum, angular momentum, and electric charge are all derived from the existence of symmetries.

Let us consider a continuous linear transformation on three real spinless fields $\varphi_i(x)$ (where $i = 1, 2, 3$) with $\varphi_i(x) = \varphi_i^\dagger(x)$. These three fields might correspond to the three pion states. As a matter of notation, $\varphi(x)$ is a column vector, where the top entry is $\varphi_1(x)$, the second entry is $\varphi_2(x)$, and the bottom entry is $\varphi_3(x)$. We write the linear transformation of the three fields in terms of a 3-by-3 matrix $U(\varepsilon)$, where

$$\varphi'(x') = U(\varepsilon)\varphi(x)\,, \tag{5a}$$

or in component notation

$$\varphi_i'(x') = U_{ij}(\varepsilon)\varphi_j(x)\,. \tag{5b}$$

The repeated index is summed from 1 to 3, and generalizations to different numbers or kinds of fields are obvious. The parameter ε is continuous, and as ε approaches zero, $U(\varepsilon)$ becomes the unit matrix. The dependence of x' on x and ε is discussed below. The continuous transformation $U(\varepsilon)$ is called linear since $\varphi_j(x)$ occurs linearly on the right-hand side of Eq. 5. (Nonlinear transformations also have an important role in particle physics, but this discussion of the standard model will primarily involve linear transformations except for the vector-boson fields, which have a slightly different transformation law, described in Note 5.) For N independent transformations, there will be a set of parameters ε_a, where the index a takes on values from 1 to N.

For these continuous transformations we can expand $\varphi'(x')$ in a Taylor series about $\varepsilon_a = 0$; by keeping only the leading term in the expansion, Eq. 5 can be rewritten in infinitesimal form as

$$\delta\varphi(x) \equiv \varphi'(x) - \varphi(x) = i\varepsilon^a T_a \varphi(x)\,, \tag{6a}$$

where T_a is the first term in the Taylor expansion,

$$i\varepsilon^a T_a = \varepsilon^a \left[\frac{\partial U(\varepsilon)}{\partial \varepsilon_a}\right]_{\varepsilon=0} - \delta x^\mu \partial_\mu\,, \tag{6b}$$

with $\delta x = x' - x$. The T_a are the "generators" of the symmetry transformations of $\varphi(x)$. (We note that $\delta\varphi(x)$ in Eq. 6a is a small symmetry transformation, not to be confused with the field variations $\delta\varphi$ in Eq. 2.)

The space-time point x' is, in general, a function of x. In the case where $x' = x$, Eq. 5 is called an internal transformation. Although our primary focus will be on internal transformations, space-time symmetries have many applications. For example, all theories we describe here have Poincaré symmetry, which means that these theories are invariant under transformations in which $x' = \Lambda x + b$, where Λ is a 4-by-4 matrix representing a Lorentz transformation that acts on a four-component column vector x consisting of time and the three space components, and b is the four-component column vector of the parameters of a space-time translation. A spinless field transforms under Poincaré transformations as $\varphi'(x') = \varphi(x)$ or $\delta\varphi = -b^\mu\partial_\mu\varphi(x)$. Upon solving Eq. 6b, we find the infinitesimal translation is represented by $i\partial_\mu$. The components of fields with spin are rearranged by Poincaré transformations according to a matrix that depends on both the ε's and the spin of the field.

We now restrict attention to internal transformations where the space-time point is unchanged; that is, $\delta x^\mu = 0$. If ε_a is an infinitesimal, arbitrary function of x, $\varepsilon_a(x)$, then Eqs. 5 and 6a are called local transformations. If the ε_a are restricted to being constants in space-time, then the transformation is called global.

Before beginning a lengthy development of the symmetries of various Lagrangians, we give examples in which each of these kinds of linear transformations are, indeed, symmetries of physical theories. An example of a global, internal symmetry is strong isospin, as discussed briefly in "Particle Physics and the Standard Model." (Actually, strong isospin is not an exact symmetry of Nature, but it is still a good example.) All theories we discuss here have global Lorentz invariance, which is a space-time symmetry. Electrodynamics has a local phase symmetry that is an internal symmetry. For a charged spinless field the infinitesimal form of a local phase transformation is $\delta\varphi(x) = i\varepsilon(x)\varphi(x)$ and $\delta\varphi^\dagger(x) = -i\varepsilon(x)\varphi^\dagger(x)$, where $\varphi(x)$ is a complex field. Larger sets of local internal symmetry transformations are fundamental in the standard model of the weak and strong interactions. Finally, Einstein's gravity makes essential use of local space-time Poincaré transformations. This complicated case is not discussed here. It is quite remarkable how many types of transformations like Eqs. 5 and 6 are basic in the formulation of physical theories.

Let us return to the column vector of three real fields $\varphi(x)$ and suppose we have a Lagrangian that is unchanged by Eqs. 5 and 6, where we now restrict our attention to internal transformations. (One such Lagrangian is Eq. 1a, where $\varphi(x)$ is now a column vector and $\varphi^\dagger(x)$ is its transpose.) Not only the Lagrangian, but the Lagrangian density, too, is unchanged by an internal symmetry transformation.

Let us consider the infinitesimal transformation (Eq. 6a) and calculate $\delta\mathcal{L}$ in two different ways. First of all, $\delta\mathcal{L}=0$ if $\delta\varphi$ is a symmetry identified from the Lagrangian. Moreover, according to the rules of partial differentiation,

$$\delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\varphi_i}\,\delta\varphi_i+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\,\partial_\mu\delta\varphi_i\,. \tag{7}$$

Then, using the Euler-Lagrange equations (Eq. 3) for the first term and collecting terms, Eq. 7 can be written in an interesting way:

$$\delta\mathcal{L}=\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\right)\delta\varphi_i\,. \tag{8}$$

The next step is to substitute Eq. 6a into Eq. 8. Thus, let us define the current $J_\mu^a(x)$ as

$$J_\mu^a(x)=i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\,T_{ij}^a\varphi_j\,. \tag{9}$$

Then Eq. 8 plus the requirement that $\delta\varphi$ is a symmetry imply the continuity equation,

$$\partial^\mu J_\mu^a(x)=0\,. \tag{10}$$

We can gain intuition about Eq. 10 from electrodynamics, since the electromagnetic current satisfies a continuity equation. It says that charge is neither created nor destroyed locally: the change in the charge density, $J_0(x)$, in a small region of space is just equal to the current $\mathbf{J}(x)$ flowing out of the region. Equation 10 generalizes this result of electrodynamics to other kinds of charges, and so $J_\mu^a(x)$ is called a current. In particle physics with its many continuous symmetries, we must be careful to identify which current we are talking about.

Although the analysis just performed is classical, the results are usually correct in the quantum theory derived from a classical Lagrangian. In some cases, however, quantum corrections contribute a nonzero term to the right-hand side of Eq. 10; these terms are called anomalies. For global symmetries these anomalies can improve the predictions from Lagrangians that have too much symmetry when compared with data because the anomaly wrecks the symmetry (it was never there in the quantum theory, even though the classical Lagrangian had the symmetry). However, for local symmetries anomalies are disastrous. A quantum field theory is locally symmetric only if its currents satisfy the continuity equation, Eq. 10. Otherwise local symmetry transformations simply change the theory. (Some care is needed to avoid this kind of anomaly in the standard model.) We now show that Eq. 10 can imply the existence of a conserved quantity called the global charge and defined by

$$Q^a(t)=\int d^3\mathbf{x}\,J_0^a(x)\,, \tag{11}$$

provided the integral over all space in Eq. 11 is well defined; that is, $J_0^a(x)$ must fall off rapidly enough as $|\mathbf{x}|$ approaches infinity that the integral is finite.

If $Q^a(t)$ is indeed a conserved quantity, then its value does not change in time, which means that its first time derivative is zero. We can compute the time derivative of $Q^a(t)$ with the aid of Eq. 10:

$$\frac{d}{dt}\,Q^a(t)=\int d^3\mathbf{x}\,\frac{\partial J_0^a(x)}{\partial t}=\int d^3\mathbf{x}\,\nabla\cdot\mathbf{J}^a(x)=\int\mathbf{J}^a\cdot d\mathbf{S}=0\,. \tag{12}$$

The next to the last step is Gauss's theorem, which changes the volume integral of the divergence of a vector field into a surface integral. If $\mathbf{J}^a(x)$ falls off more rapidly than $1/|\mathbf{x}|^2$ as $|\mathbf{x}|$ becomes very large, then the surface integral must be zero. It is not a always true that $\mathbf{J}^a(x)$ falls off so rapidly, but when it does, $Q^a(t)=Q^a$ is a constant in time. One of the most important experimental tests of a Lagrangian is whether the conserved quantities it predicts are, indeed, conserved in elementary-particle interactions.

The Lagrangian for the complex scalar field defined by Eq. 1 has an internal global symmetry, so let us practice the above steps and identify the conserved current and charge. It is easily verified that the global phase transformation

$$\varphi'(x)=e^{i\varepsilon}\varphi(x) \tag{13}$$

leaves the Lagrangian density invariant. For example, the first term of Eq. 1 by itself is unchanged: $\partial_\mu\varphi^\dagger\partial^\mu\varphi$ becomes $\partial_\mu(e^{-i\varepsilon}\varphi^\dagger)\partial^\mu(e^{i\varepsilon}\varphi)=\partial_\mu\varphi^\dagger\partial^\mu\varphi$, where the last equality follows only if ε is constant in space-time. (The case of local phase transformations is treated in Note 5.) The next step is to write the infinitesimal form of Eq. 13 and substitute it into Eq. 9. The conserved current is

$$J_\mu(x)=i[(\partial_\mu\varphi^\dagger)\varphi-(\partial_\mu\varphi)\varphi^\dagger]\,, \tag{14}$$

where the sum in Eq. 9 over the fields $\varphi(x)$ and $\varphi^\dagger(x)$ is written out explicitly.

If $m^2>0$ in Eq. 1, then all the charge can be localized in space and time and made to vanish as the distance from the charge goes to infinity. The steps in Eq. 12 are then rigorous, and a conserved charge exists. The calculation was done here for classical fields, but the same results hold for quantum fields: the conservation law implied by Eq. 12 yields a conserved global charge equal to the number of φ particles minus the number of φ antiparticles. This number must remain constant in any interaction. (We will see in Note 3 that if $m^2<0$, the charge distribution is spread out over all space-time, so the global charge is no longer conserved even though the continuity equation remains valid.)

Identifying the transformations of the fields that leave the Lagrangian invariant not only satisfies our sense of symmetry but also leads to important predictions of the theory without solving the equations of motion. In Note 4 we will return to the example of three real scalar fields to introduce larger global symmetries, such as SU(2), that interrelate different fields.

3 Spontaneous Breaking of a Global Symmetry

It is possible for the vacuum or ground state of a physical system to have less symmetry than the Lagrangian. This possibility is called spontaneous symmetry breaking, and it plays an important role in the standard model. The simplest example is the complex scalar field theory of Eq. 1a with $m^2 < 0$.

In order to identify the classical fields with particles in the quantum theory, the classical field must approach zero as the number of particles in the corresponding quantum-mechanical state approaches zero. Thus the quantum-mechanical vacuum (the state with no particles) corresponds to the classical solution $\varphi(x) = 0$. This might seem automatic, but it is not. Symmetry arguments do not necessarily imply that $\varphi(x) = 0$ is the lowest energy state of the system. However, if we rewrite $\varphi(x)$ as a function of new fields that do vanish for the lowest energy state, then the new fields may be directly identified with particles. Although this prescription is simple, its justification and analysis of its limitations require extensive use of the details of quantum field theory.

The energy of the complex scalar theory is the sum of kinetic and potential energies of the $\varphi(x)$ and $\varphi^\dagger(x)$ fields, so the energy density is

$$\mathcal{H} = \partial^\mu\varphi^\dagger\partial_\mu\varphi + m^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2 , \tag{15}$$

with $\lambda > 0$. Note that $\partial^\mu\varphi^\dagger\partial_\mu\varphi$ is nonnegative and is zero if φ is a constant. For $\varphi = 0$, $\mathcal{H} = 0$. However, if $m^2 < 0$, then there are nonzero values of $\varphi(x)$ for which $\mathcal{H} < 0$. Thus, there is a nonzero field configuration with lowest energy. A graph of $\mathcal{H}$ as a function of $|\varphi|$ is shown in Fig. 1. In this example $\mathcal{H}$ is at its lowest value when both the kinetic and potential energies ($V = m^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2$) are at their lowest values. Thus, the vacuum solution for $\varphi(x)$ is found by solving the equation $\partial V/\partial\varphi = 0$, or

$$\varphi^\dagger(x)\varphi(x) = -\frac{m^2}{2\lambda} = \frac{1}{2}|\varphi_0|^2 > 0 . \tag{16}$$

Next we find new fields that vanish when Eq. 16 is satisfied. For example, we can set

$$\varphi(x) = \frac{1}{\sqrt{2}}\left[\rho(x) + \varphi_0\right]\exp\left[i\pi(x)/\varphi_0\right] , \tag{17}$$

where the real fields $\rho(x)$ and $\pi(x)$ are zero when the system is in the lowest energy state. Thus $\rho(x)$ and $\pi(x)$ may be associated with particles. Note, however, that φ_0 is not completely specified; it may lie at any point on the circle in field space defined by Eq. 16, as shown in Fig. 2.

Suppose φ_0 is real and given by

$$\varphi_0 = (-m^2/\lambda)^{1/2} . \tag{18}$$

Then the Lagrangian is still invariant under the phase transformations in Eq. 13, but the choice of the vacuum field solution is changed by the phase transformation. Thus, the vacuum solution is not invariant under the phase transformations, so the phase symmetry is spontaneously broken. The symmetry of the Lagrangian is *not* a symmetry of the vacuum. (For $m^2 > 0$ in Eq. 1, the vacuum and the Lagrangian both have the phase symmetry.)

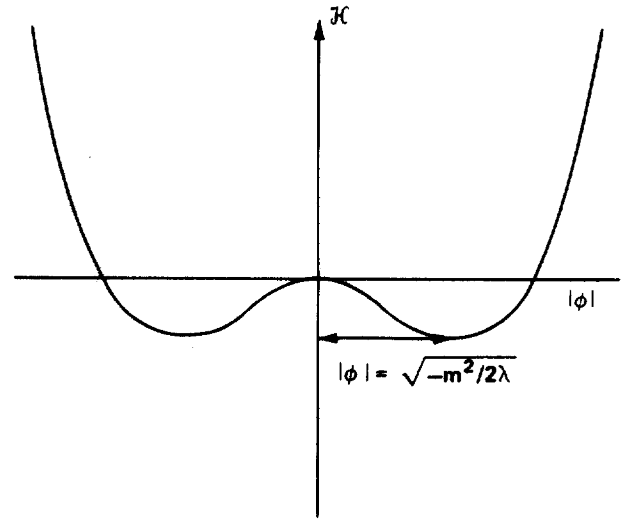


Fig. 1. The Hamiltonian $\mathcal{H}$ defined by Eq. 15 has minima at nonzero values of the field φ.

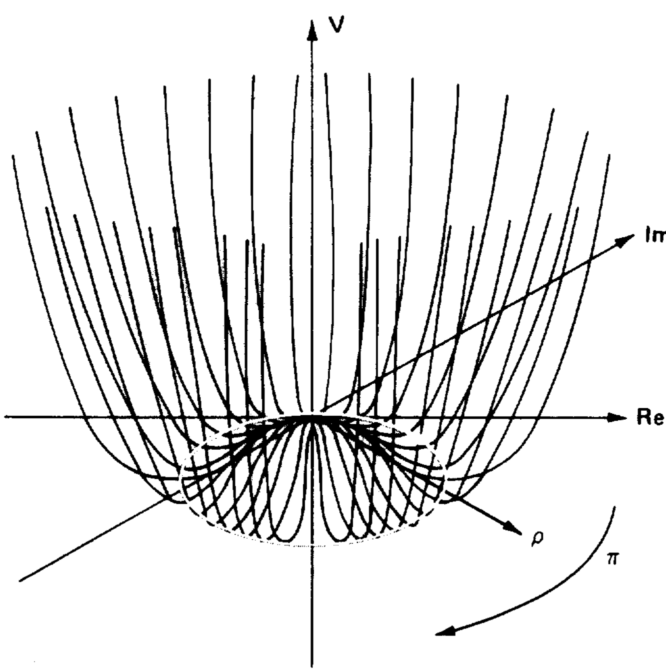


Fig. 2. The blue curve is the location of the minimum of V in the field space φ.

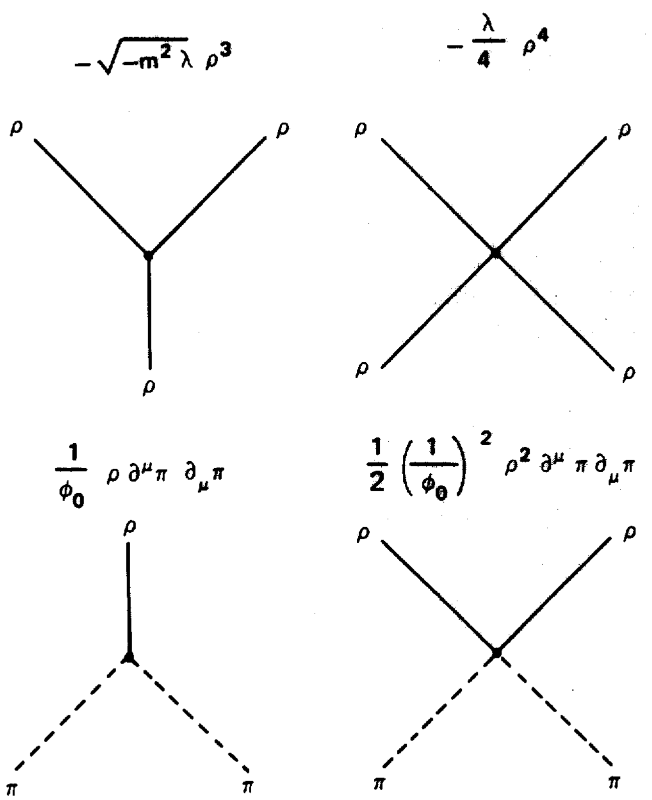


Fig. 3. A graphic representation of the last four terms of Eq. 20, the interaction terms. Solid lines denote the ρ field and dotted lines the π field. The interaction of three ρ(x) fields at a single point is shown as three solid lines emanating from a single point. In perturbation theory this so-called vertex represents the lowest order quantum-mechanical amplitude for one particle to turn into two. All possible configurations of these vertices represent the quantum-mechanical amplitudes defined by the theory.

We now rewrite the Lagrangian in terms of the particle fields $\rho(x)$ and $\pi(x)$ by substituting Eq. 17 into Eq. 1. The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial^\mu\rho\partial_\mu\rho + \frac{1}{2}(1+\rho/\varphi_0)^2\partial^\mu\pi\partial_\mu\pi - \frac{m^2}{2}(\rho+\varphi_0)^2 - \frac{\lambda}{4}(\rho+\varphi_0)^4 . \qquad (19)$$

To estimate the masses associated with the particle fields $\rho(x)$ and $\pi(x)$, we substitute Eq. 18 for the constant φ_0 and expand $\mathcal{L}$ in powers of the fields $\pi(x)$ and $\rho(x)$, obtaining

$$\mathcal{L} = \frac{1}{2}\partial^\mu\rho\partial_\mu\rho + \frac{1}{2}\partial^\mu\pi\partial_\mu\pi + \frac{m^4}{4\lambda} + m^2\rho^2 - (-\lambda m^2)^{1/2}\rho^3 - \frac{\lambda}{4}\rho^4 + \frac{1}{\varphi_0}\rho\partial^\mu\pi\partial_\mu\pi + \frac{1}{2\varphi_0^2}\rho^2\partial^\mu\pi\partial_\mu\pi . \qquad (20)$$

This Lagrangian has the following features.

- The fields $\rho(x)$ and $\pi(x)$ have standard kinetic energy terms.
- Since $m^2 < 0$, the term $m^2\rho^2$ can be interpreted as the mass term for the $\rho(x)$ field. The $\rho(x)$ field thus describes a particle with mass-squared equal to $|m^2|$, *not* $-|m^2|$.
- The $\pi(x)$ field has no mass term. (This is obvious from Fig. 2, which shows that $\mathcal{L}(\rho,\pi)$ has no curvature (that is, $\partial^2\mathcal{L}/\partial\pi^2 = 0$) in the $\pi(x)$ direction.) Thus, $\pi(x)$ corresponds to a massless particle. This result is unchanged when all the quantum effects are included.
- The phase symmetry is hidden in $\mathcal{L}$ when it is written in terms of $\rho(x)$ and $\pi(x)$. Nevertheless, $\mathcal{L}$ has phase symmetry, as is proved by working backward from Eq. 20 to Eq. 16 to recover Eq. 1a.
- In theories without gravity, the constant term $V \propto m^4/\lambda$ can be ignored, since a constant overall energy level is not measurable. The situation is much more complicated for gravitational theories, where terms of this type contribute to the vacuum energy-momentum tensor and, by Einstein's equations, modify the geometry of space-time.
- The ρ field interacts with the π field only through derivatives of π. The interaction terms in Eq. 20 may be pictured as in Fig. 3.

Although this model might appear to be an idle curiosity, it is an example of a very general result known as Goldstone's theorem. This theorem states that in any field theory there is a zero-mass spinless particle for each independent global continuous symmetry of the Lagrangian that is spontaneously broken. The zero-mass particle is called a Goldstone boson. (This general result does not apply to local symmetries, as we shall see.)

There has been one very important physical application of spontaneously broken global symmetries in particle physics, namely, theories of pion dynamics. The pion has a surprisingly small mass compared to a nucleon, so it might be understood as a zero-mass particle resulting from spontaneous symmetry breaking of a global symmetry. Since the pion mass is not exactly zero, there must also be some small but explicit terms in the Lagrangian that violate the global symmetry. The feature of pion dynamics that justifies this procedure is that the interactions of pions with nucleons and other pions are similar to the interactions (see Fig. 3) of the $\pi(x)$ field with the $\rho(x)$ field and with itself in the Lagrangian of Eq. 20. Since the pion has three (electric) charge states, it must be associated with a larger global symmetry than the phase symmetry, one where three independent symmetries are spontaneously broken. The usual choice of symmetry is global SU(2) × SU(2) spontaneously broken to the SU(2) of the strong-interaction isospin symmetry (see Note 4 for a discussion of SU(2)). This description accounts reasonably well for low-energy pion physics.

Perhaps we should note that only spinless fields can acquire a vacuum value. Fields carrying spin are not invariant under Lorentz transformations, so if they acquire a vacuum value, Lorentz invariance will be spontaneously broken, in disagreement with experiment. Spinless particles trigger the spontaneous symmetry breaking in the standard model.

4 Lagrangians with Larger Global Symmetries

In a theory with a single complex scalar field the phase transformation in Eq. 13 defines the "largest" possible internal symmetry since the only possible symmetries must relate $\varphi(x)$ to itself. Here we will discuss global symmetries that interrelate different fields and group them together into "symmetry multiplets." Strong isospin, an approximate symmetry of the observed strongly interacting particles, is an example. It groups the neutron and the proton into an isospin doublet, reflecting the fact that the neutron and proton have nearly the same mass and share many similarities in the way that they interact with other particles. Similar comments hold for the three pion states (π^+, π^0, and π^-), which form an isospin triplet.

We will derive the structure of strong isospin symmetry by examining the invariance of a specific Lagrangian for the three real scalar fields $\varphi_i(x)$ already described in Note 2. (Although these fields could describe the pions, the Lagrangian will be chosen for simplicity, not for its capability to describe pion interactions.)

We are about to discover a symmetry by deriving it from a Lagrangian; however, in particle physics the symmetries are often discovered from phenomenology. Moreover, since there can be many Lagrangians with the same symmetry, the predictions following from the symmetry are viewed as more general than the predictions of a specific Lagrangian with the symmetry. Consequently, it becomes important to abstract from specific Lagrangians the general features of a symmetry; see the comments later in this note.

A general linear transformation law for the three real fields can be written

$$\varphi_i'(x) = [\exp(i\varepsilon^a T_a)]_{ij}\varphi_j(x)\,, \tag{21}$$

where the sum on j runs from 1 to 3. One reason for choosing this form of $U(\varepsilon)$ is that it explicitly approaches the identity as ε approaches zero.

To identify the generators T_a with matrix elements $(T_a)_{ij}$, we use a specific Lagrangian,

$$\mathscr{L} = \frac{1}{2}\partial^\mu\varphi_i\partial_\mu\varphi_i - \frac{1}{2}\,m^2\varphi_i\varphi_i - \lambda\,(\varphi_i\varphi_i)^2\,. \tag{22}$$

Let us place primes on the fields in Eq. 22 and substitute Eq. 21 into it. Then $\mathscr{L}$ written in terms of the new $\varphi(x)$ is exactly the same as Eq. 22 if

$$[\exp(i\varepsilon^a T_a)]_{ij}\,[\exp(i\varepsilon^b T_b)]_{ik} = \delta_{jk}\,, \tag{23}$$

where δ_{jk} are the matrix elements of the 3-by-3 identity matrix. (In the notation of Eq. 5a, Eq. 23 is $U(\varepsilon)U^T(\varepsilon) = I$.) Equation 23 can be expanded in ε_a, and the linear term then requires that T_a be an antisymmetric matrix. Moreover, $\exp(i\varepsilon^a T_a)$ must be a real matrix so that $\varphi(x)$ remains real after the transformation. This implies that all elements of the T_a are imaginary. These constraints are solved by the three imaginary antisymmetric 3-by-3 matrices with elements

$$(T_a)_{ij} = -i\varepsilon_{aij}\,, \tag{24}$$

where $\varepsilon_{123} = +1$ and ε_{abc} is antisymmetric under the interchange of any two indices (for example, $\varepsilon_{321} = -1$). (It is a coincidence in this example that the number of fields is equal to the number of independent symmetry generators. Also, the parameter ε_a with one index should not be confused with the tensor ε_{abc} with three indices.)

The conditions on $U(\varepsilon)$ imply that it is an orthogonal matrix; 3-by-3 orthogonal matrices can also describe rotations in three spatial dimensions. Thus, the three components of φ_i transform in the same way under isospin rotation as a spatial vector $\mathbf{x}$ transforms under a rotation. Since the rotational symmetry is SU(2), so is the isospin symmetry. (Thus "isospin" is like spin.) The T_a matrices satisfy the SU(2) commutation relations

$$[T_a, T_b] \equiv T_aT_b - T_bT_a = i\varepsilon_{abc}T_c\,. \tag{25}$$

Although the explicit matrices of Eq. 24 satisfy this relation, the T_a can be generalized to be quantum-mechanical operators. In the example of Eqs. 21 and 22, the isospin multiplet has three fields. Drawing on angular momentum theory, we can learn other possibilities for isospin multiplets. Spin-J multiplets (or representations) have $2J + 1$ components, where J can be any nonnegative integer or half integer. Thus, multiplets with isospin of ½ have two fields (for example, neutron and proton) and isospin-3/2 multiplets have four fields (for example, the Δ^{++}, Δ^{+}, Δ^{0}, and Δ^{-} baryons of mass $\sim 1232\ \text{GeV}/c^2$).

The basic structure of all continuous symmetries of the standard model is completely analogous to the example just developed. In fact, part of the weak symmetry is called weak isospin, since it also has the same mathematical structure as strong isospin and angular momentum. Since there are many different applications to particle theory of given symmetries, it is often useful to know about symmetries and their multiplets. This mathematical endeavor is called group theory, and the results of group theory are often helpful in recognizing patterns in experimental data.

Continuous symmetries are defined by the algebraic properties of their generators. Group transformations can always be written in the form of Eq. 21. Thus, if Q_a ($a = 1, \ldots, N$) are the generators of a symmetry, then they satisfy commutation relations analogous to Eq. 25:

$$[Q_a, Q_b] = if_{abc}Q_c\,, \tag{26}$$

where the constants f_{abc} are called the structure constants of the Lie algebra. The structure constants are determined by the multiplication rules for the symmetry operations, $U(\varepsilon_1)U(\varepsilon_2) = U(\varepsilon_3)$, where ε_3 depends on ε_1 and ε_2. Equation 26 is a basic relation in defining a Lie algebra, and Eq. 21 is an example of a Lie group operation. The Q_a, which generate the symmetry, are determined by the "group" structure. The focus on the generators often simplifies the study of Lie groups. The generators Q_a are quantum-mechanical operators. The $(T_a)_{ij}$ of Eqs. 24 and 25 are matrix elements of Q_a for some symmetry multiplet of the symmetry.

The general problem of finding all the ways of constructing equations like Eq. 25 and Eq. 26 is the central problem of Lie-group theory. First, one must find all sets of f_{abc}. This is the problem of finding all the Lie algebras and was solved many years ago. The second problem is, given the Lie algebra, to find all the matrices that represent the generators. This is the problem of finding all the representations (or multiplets) of a Lie algebra and is also solved in general, at least when the range of values of each ε_a is finite. Lie group theory thus offers an orderly approach to the classification of a huge number of theories.

Once a symmetry of the Lagrangian is identified, then sets of n fields are assigned to n-dimensional representations of the symmetry group, and the currents and charges are analyzed just as in Note 2. For instance, in our example with three real scalar fields and the Lagrangian of Eq. 22, the currents are

$$J^a_\mu(x) = \varepsilon^{aij}(\partial_\mu\varphi_i)\varphi_j \tag{27}$$

and, if $m^2 > 0$, the global symmetry charge is

$$Q^a = \int d^3\mathbf{x}\,\varepsilon^{aij}\frac{\partial\varphi_i}{\partial t}\varphi_j\,, \tag{28}$$

where the quantum-mechanical charges Q_a satisfy the commutation relations

$$[Q_a, Q_b] = i\varepsilon_{abc}Q_c\,. \tag{29}$$

(The derivation of Eq. 29 from Eq. 28 requires the canonical commutation relations of the quantum $\varphi_i(x)$ fields.)

The three-parameter group SU(2) has just been presented in some detail. Another group of great importance to the standard model is SU(3), which is the group of 3-by-3 unitary matrices with unit determinant. The inverse of a unitary matrix U is $U^\dagger$, so $U^\dagger U = I$. There are eight parameters and eight generators that satisfy Eq. 26 with the structure constants of SU(3). The low-dimensional representations of SU(3) have 1, 3, 6, 8, 10, . . . fields, and the different representations are referred to as **1**, **3**, $\bar{\mathbf{3}}$, **6**, $\bar{\mathbf{6}}$, **8**, **10**, $\overline{\mathbf{10}}$, and so on.