the momentum flux tensor, we need to assume that each region in the gas has a local Maxwell-Boltzmann distribution. With this assumption one can show that the momentum flux tensor in Eq. 7 has the following form:

$$\Pi_{ik} = \rho v_i v_k + \delta_{ik} p,$$

where p is the pressure. This form of Π_{ik} gives the same Euler equation that we found by general continuum arguments. (We will see in Part II that the form of Π_{ik} for the totally discrete fluid is not so simple but depends upon the geometry of the underlying lattice. Again by assuming a form for the local distribution function (the appropriate form will turn out to be Fermi-Dirac rather than Boltzmann), Π_{ik} will reduce to a form that gives the lattice Euler equation.)

Recovering the Navier-Stokes Equation. The derivation of the Navier-Stokes equation from the kinetic theory picture is more involved and requires us to face the full Boltzmann equation. Hilbert accomplished this through a beautiful argument that relies on a spatial-gradient perturbation expansion around some single-particle distribution function f_l assumed to be given at t_0. In "The Hilbert Contraction" we discuss the main outline of his argument emphasizing the assumptions involved and their limitations. Here we will summarize his argument. Hilbert was able to show that the evolution of f for times $t > t_0$ is given in terms of its initial data at t_0 by the first three moments of f, namely the familiar macroscopic variables ρ (density), $\mathbf{v}$ (mean velocity), and T (temperature). In other words, he was able to contract this many-degree-of-freedom system down to a low-dimensional descriptive space whose variables are the same as those used in the usual hydrodynamical description. The beauty of Hilbert's proof is that it is constructive. It explicitly displays a recursive closed tower of constraint relations on the moments of f that come directly from the Boltzmann equation. The proof also shows that such a contracted description is unique—a very powerful result.

THE HILBERT CONTRACTION

The Boltzmann equation is a microscopic equation for colliding-gas evolution valid in a very tight regime. It is first order in time and so requires a complete description of the one-particle distribution function at one time, say $t = 0$, after which its functional form is completely fixed by the Boltzmann transport equation.

Describing the one-particle distribution function completely is a hopeless procedure, since the amount of information is too large. However, one wants to recover hydrodynamics, which is essentially a partial differential equation for a macroscopic description of the fluid at long times and distances compared to molecular scales. So there must exist a contraction mechanism that reduces the number of degrees of freedom required to describe the solution to the Boltzmann transport equation at such long times and distances. It is not obvious how that can happen, but Hilbert gave a proof that is central to understanding that it must happen and in a rather surprising way. We will call this process the Hilbert contraction. All analyses of the Boltzmann equation are based on this contraction. We would like to give it in detail because it is a beautiful argument, but space forbids this, so we outline how Hilbert reasoned.

Since we don't know what else to do when faced with such a highly nonlinear system, we construct a perturbation expansion in a small variable around some distribution function f, assumed to be given to us at t_0. Under some very mild assumptions, and assuming the existence of such a general perturbation expansion in some parameter δ, Hilbert was able to show that the evolution of f for $t > t_0$ is given in terms of its initial data at t_0 by the first three moments of f, namely $\rho, \mathbf{v}$, and T. The system has contracted down to a low-dimensional descriptive manifold whose coordinates are the same variables used by the hydrodynamic description. The beauty of Hilbert's proof is that it is constructive. It explicitly displays a recursive closed tower of constraint relations on the moments off that come directly from the Boltzmann equation. The proof also shows that such a contracted description is unique—a very powerful result.

It must be pointed out that Hilbert's construction is on the time-evolved solution to the Boltzmann transport equation, not on the equation itself, which still requires a complete specification of f. It amounts to a hard mathematical statement on an effective field-theory description for times much greater than elementary collision times, but with space gradients still smooth enough to entertain a serious gradient perturbation expansion. As such, it says nothing about the turbulent regime, for example, where all these assumptions fail.

In standard physics texts one can read all kinds of plausibility arguments as to why this contraction process should exist, but they lack force, for, by arguing tightly, one can make the conclusion go the other way. This is why the Hilbert contraction is important. It is really a powerful and mathematically unexpected result about a highly nonlinear integro-differential equation of very special form. Beyond Hilbert's theorem and within the Boltzmann transport picture, we can say nothing more about the contraction of descriptions.

The construction of towers of moment constraints, coupled to a perturbation expansion that Hilbert developed for his proof of contraction, was used in a somewhat different form by Chapman and Enskog. Their main purpose was to devise a perturbation expansion with side constraints in such a way as to pick off the values of the coupling constants-which are called transport coefficients in standard terminology-for increasingly more sophisticated forms of macrodynamical equations.

One makes the usual kinetic assump-

THE HILBERT CONTRACTION (continued)

tions: The gas reaches local equilibrium in a collision time or so; the one-particle distribution function has a local Maxwell-Boltzmann form (or whatever form is appropriate), call it f_L; a second time scale is assumed where space gradients are still small, but collective modes develop at large distances and at times much greater than molecular collision times. Then one assumes a general functional perturbation expansion exists of the form

$$f = f_L(1 + \xi^{(1)} + \xi^{(2)} + \cdots),$$

which turns out to be explicitly a spatial gradient expansion:

$$f = f_L(1 + c_1(\mathrm{v})(\lambda\nabla) + c_2(\mathrm{v})(\lambda\nabla)^2 + \cdots)$$

where λ is the mean free path in the system and v is the macrovelocity.

The perturbation expansion is set up so that at nth order, the correction to f_L obeys an integral equation of the form $f_L C(\xi^{(n)}) = L_n$, where C is the Boltzmann collision operator and L_n is an operator that depends only on lower order spatial derivatives. This generates a recursive tower of relations $\xi^{(n)}$ whose solubility conditions at order n are the $(n-1)$th-order hydrodynamical equations.

For example, assume

$$f = f_L\left(1 + \xi^{(1)}\right);$$

that is, we keep only 1st order in ξ. Then in the Boltzmann collision term keep consistently only order $\xi^{(1)}$ and in the streaming operator put $\xi^{(0)} \equiv f_L$. So we get

$$(\partial_t + \mathrm{v}_\alpha\partial_\alpha + a_\alpha\frac{\partial}{\partial \mathrm{v}_\alpha})f_L = f_L C(\xi^{(1)}),$$

which is of the form

$$f_L C(\xi^{(1)}) = L_1.$$

The solubility conditions for this are that L_1 must be orthogonal to the five zero eigenmodes of $C(\xi^{(1)}) = 0$ (the solutions are $1, \mathbf{v}$, and $\mathbf{v}^2$). These solubility conditions are the Euler equation for ρ, v, and T and the ideal gas equation of state. In this way one derives a sequence of hydrodynamical equations with explicit forms for the transport coefficients. Order 0 gives the Euler equation, order 1 gives the Navier-Stokes equations, order 2 and greater give the generalized hydrodynamical equations, which have some validity only in special situations. The expansion is an asymptotic functional expansion, so going beyond Navier-Stokes takes one away from ordinary fluids rather than closer to them. Solving explicitly for the various $\xi^{(n)}$ gives a way to evaluate the transport quantities (viscosity, etc.).

There are many other ways to do the same thing—multiple time expansions, dispersion methods, etc. We have developed everything so far within the conceptual frame of the Boltzmann transport equation. Within that framework the problem of deriving macrodynamical equations and associated transport coefficients reduces to tedious but straightforward linear algebra that has absorbed the best efforts of excellent technical people since the turn of the century. It is a problem best suited to a computer but only recently have algebraic processors of sufficient power been available.

This asymptotic perturbation expansion is a way to compute measurable quantities from microdynamical properties, but the physical insight one gains from doing it is small. The other methods mentioned, especially correlation-function techniques, are much more revealing. All of these comments and approaches carry over directly to the discrete case of the lattice gas. Nothing conceptually new arises in the totally discrete case, but explicit calculations are a great deal easier. ■

Boltzmann equation. The zero-order relation gives the Euler equations and the second-order relation gives the Navier-Stokes equations. However, Hilbert's method is an asymptotic functional expansion, so that the higher order terms take one away from ordinary fluids rather than closer to them. Nevertheless, solving explicitly for the terms in the functional expansion provides a way of evaluating transport coefficients such as viscosity. (See the "Hilbert Contraction" for more discussion.)

Summary of the Kinetic Theory Picture. Our review of the kinetic theory description of fluids introduced a number of important concepts: the idea of local thermal equilibrium; the characterization of an equilibrium state by a few macroscopic observable; the Boltzmann transport equation for systems of many identical objects (with ordinary statistics) in Collision; and the fact that a solution to the Boltzmann transport equation is an ensemble of equilibrium states. In "The Hilbert Contraction" we introduced the linear approximation to the Boltzmann equation with which one can derive the Navier-Stokes equations for systems not too far (in an appropriate sense) from equilibrium in terms of these same macroscopic observable (density, pressure. temperature, etc.). We then outlined a method for calculating the coupling constants in the Navier-Stokes system—that is, the strengths of the nonlinear terms-as a function of any particular mi**crodynamics.**

This review was intended to give a flavor for the chain of reasoning involved. We will use this chain again in the to tally discrete lattice world. However, just as important as understanding the kinetic theory viewpoint is keeping in mind its limitations. In particular, notice that perturbation theory was the main tool used for going from the exact Boltzrnann transport equation to the Navier-Stokes equations. We did not discover more pow-