Quantum Feedback Control

How can we control quantum systems without disturbing them?

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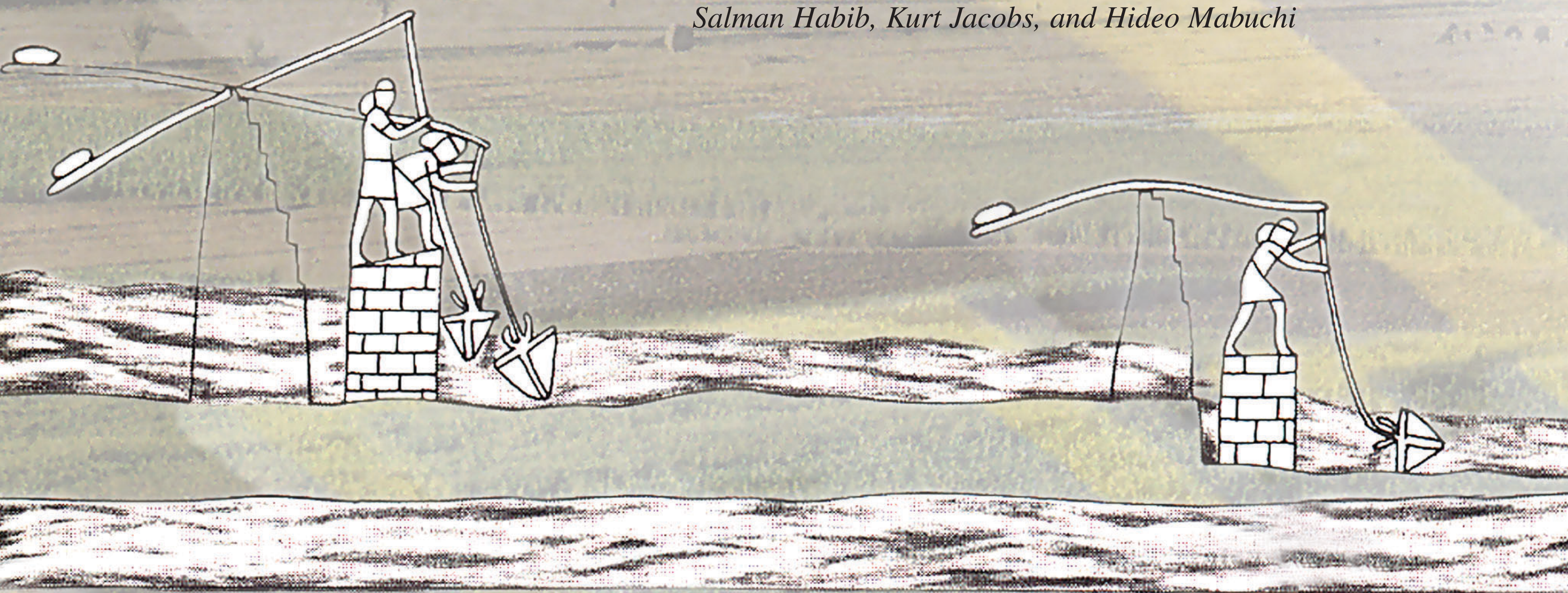

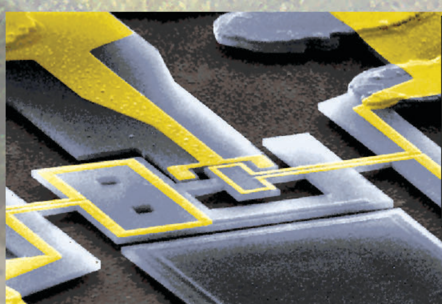

The nanomechanical electrometer shown here was built in Michael Roukes' group at Caltech. It has a demonstrated sensitivity below a single electron charge per unit bandwidth and should ultimately reach sensitivities of the order of parts per million. Its operation is based on the movement of a torsional resonator that carries a detection electrode placed in an external magnetic field. The gate electrode is seen on one side of the resonator.

Ever since Niels Bohr's first attempt at understanding the hydrogen atom, the fundamental cautionary lesson of quantum mechanics has been driven home time after time: Processes in the microworld transpire according to laws and principles that directly contradict those governing the macroworld of human experience. This radical shift in understanding is now almost a century old and has been definitively confirmed by numerous experiments. It might seem likely that the strange behaviors of quantum systems would be familiar by now and practical devices harnessing those behaviors would be commonplace. For the most part, however, we have remained mere spectators of the microphysical realm, where quantum mechanics holds sway, being forced to observe naturally occurring phenomena rather than being able to control and manipulate them. In the coming decade, however, this situation may be reversed.

Recent advances in quantum and atomic optics and condensed matter physics are providing tools to engineer practical quantum devices and perhaps even modestly complex networks of these devices. Quantum information processing, precision measurement, and development of ultrasensitive sensors are driving the present development of quantum technologies. If quantum technologies are ever to achieve the complexity of classically engineered systems such as jet aircraft and the Internet, a quantum analog of classical feedback control must be developed, since feedback control is at the heart of the stability and predictability underlying complex engineered systems.

Along these lines, recent theoretical results on error correction in quantum computation and on the dynamics of open quantum systems may be viewed as first steps in developing a theoretical formalism for practical quantum feedback control (see the articles "Introduction to Quantum Error Correction" on page 188 and "Realizing a Noiseless Subsystem" on page 260). Indeed, feedback control represents a promising new approach to mitigating quantum noise and decoherence in both quantum computation and precision measurement. If we are to apply the concepts and methods of feedback control theory to quantum dynamical systems, we must not only extend classical control concepts to new regimes but also analyze quantum measurement in a way that is useful for control systems.

The Evolution of Control Theory

Controlling natural phenomena through macroscopic engineering goes back thousands of years. Consider for a moment the ingenious ways in which early human civilizations controlled irrigation. In Mesopotamia (2000 BC), where rainfall was poor and the Tigris and Euphrates Rivers were the main sources of water, engineers constructed an elaborate canal system with many diversion dams (see the drawing to the left). In that system, the Euphrates served as a source and the Tigris as a drain. In a similar vein, the ancient Egyptians used water from the Nile and thereby allowed their civilization to flourish. On a smaller scale, machines using feedback control were developed in the Greco-Roman period, and methods for the automatic operation of windmills date back to the Middle Ages.

Perhaps the best-known example of feedback control in the industrial era is the Watt governor, which stabilizes steam engine speeds under fluctuating loads. James Clerk Maxwell provided the first dynamical analysis of this system based on differential equations. His work, which was published in 1868, founded the field of mathematics now known as control theory. In the early part of the 20th century, the idea of self-regulating machinery continued to be pushed in various directions, notably in electronic amplification. Control concepts were further developed for industrial, navigational, and military applications.

After World War II, control systems progressed to a new level of complexity. Up until that time, feedback control systems had been largely single loop, taking the feedback signal from one point and connecting the correction signal to a different point. Multiloop control systems and more-sophisticated feedback techniques emerged from progress in optimization theory and dynamical systems theory, as well as from the advent of digital computers.

After 1960, there emerged what is often referred to as "modern" (as opposed to "classical") control theory (Brogan 1990, Zhou et al. 1996), which emphasizes optimization of cost and performance. For the same control goals, it is clear that not all control strategies will be equally effective in terms of cost and performance. Determining the best strategy defines the problem of optimal control; however, optimal algorithms are often unstable to variations in system parameters and the external environment. Theorists then turned to ensuring performance bounds in the presence of uncertainty. This work resulted in the theory of "robust" control (Zhou et al. 1996). Noise in the inputs, extrinsic disturbances in the system under control, measurement errors, and modeling inadequacies—all can render control systems less effective or, in some cases, even lead to catastrophic failures. The role of robust control is to maintain adequate stability and other performance margins given the uncertainties mentioned earlier.

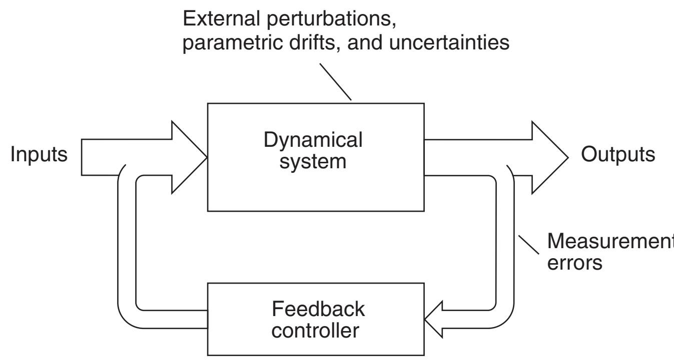

Figure 1. Classical Feedback Control
The classical dynamical system to be controlled has a set of input variables, which are processed by the system dynamics into a set of output variables. Some fraction of the set of input and output variables (possibly different for each case) is available for hookup to the controller. The controller has to perform in the presence of external fluctuations—that is, uncertainties and drifts in the parameters describing the dynamical system—and measurement errors.

Classical Control Systems

Formally speaking, a control system consists of a dynamical system interacting with a controller, a device that influences the state of the dynamical system toward some desired end. The objective may be to regulate the flow of an industrial process, money, energy, information, and so on. In a "closed-loop," or feedback, control system, the controller uses outputs from the dynamical system to monitor and influence its interaction with that dynamical system. For a linear dynamical system, for example, such a situation could be described by the following equation:

$$d\mathbf{x} = A\mathbf{x}dt + Bd\mathbf{W} + C\mathbf{u} \ , \qquad (1)$$

where $\mathbf{x}$ is a vector describing the state of the system, $d\mathbf{W}$ is a vector of Gaussian noise sources, and $\mathbf{u}$ is the vector of inputs determined by the controller. The matrix A gives the system's deterministic motion, and B and C describe, respectively, how the noise and input vectors are coupled into the system. A separate equation, namely,

$$d\mathbf{y} = H\mathbf{x}dt + Rd\mathbf{V} \ , \qquad (2)$$

describes the continuous measurement of system outputs by the controller. In each small time interval dt, the controller obtains the measurement result dy. That result is directly related to the true state of the system by some linear transformation H, but it also includes a Gaussian noise process $\mathbf{V}$, which serves to represent imperfections in the measurement.

Examples of control systems can be found in many applications. For instance, servomechanisms are control systems that use small control inputs to produce changes in large mechanical systems. In effect, the larger systems are "slaved" to the output of the servomechanisms (for example, liquid levels in reservoirs are controlled by float valves). Feedback circuits are used in ingenious ways in electronic amplification to manipulate input and output impedances and to improve the linearity, distortion, and frequency bandwidth of the output signal relative to the input signal.

In an "open-loop" control system, the controller does not monitor the output of the dynamical system. A dynamical model for the system is assumed, and control is applied with the idea that the desired outcome will actually be achieved. Open-loop strategies are useful in situations in which the system dynamics are known precisely and vary only slowly. Processes with long measurement dead times are sometimes better suited to open-loop control methods than to feedback methods. Open-loop control strategies are applied in situations as diverse as the maximization of returns from financial investments, optimal determination of aircraft flight paths, and controlled dissociation of molecules.

Figure 1 shows how to implement closed-loop control for a dynamical system. One must be able to measure some of the dynamical variables of the system under control (the outputs) and use them to influence some other variables (the inputs). In other words, given the output variables, the controller implements a particular control strategy to influence the state of the dynamical system by appropriately varying the inputs. Robust controllers take into account variations in system parameters and fluctuations from the external environment to produce control strategies with guaranteed stability bounds.

Control systems can involve many different interacting physical systems

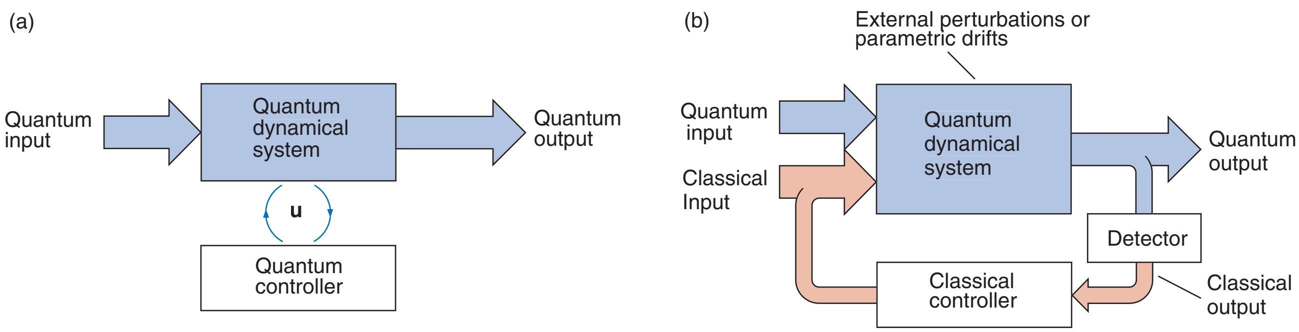

Figure 2. Directly and Indirectly Coupled Quantum Feedback

(a) Both the dynamical system and the controller are quantum systems coupled through a unitary interaction. A quantum variable is coupled to the quantum controller, and a quantum input path from the controller goes directly back to the quantum system. (b) A quantum dynamical system can be viewed as having two sets of inputs, one relating to the variation in the classical parameters describing the Hamiltonian and the other representing fully quantum inputs. Similarly, the output channel can be divided into a quantum and a classical channel. The classical channel is, in fact, a piece of the quantum channel that has become classical after observation. The controller analyzes the classical record to form an estimate of the dynamical system's state and uses this information to implement the appropriate control.

with a large number of sequential, parallel, and nested control loops that are both open and closed. For example, closed- and open-loop strategies can be combined as in the fast closed-loop systems used to stabilize the slower, inherently unstable open-loop dynamics of modern fighter aircraft.

Developing Control in Quantum Systems

The general picture of control systems outlined in the previous section appears to be extendable to quantum systems. Certainly, open-loop control problems are conceptually straightforward in the quantum context. One begins with the time evolution operator of the quantum system—the Schrödinger equation for the wave function, the Liouville equation for the density matrix, or more complicated dynamical evolution equations for the density matrix characterizing a system coupled to an environment. A theory for time-dependent variations in the evolution operator is then developed in such a way that the wave function or the density operator at some time is close to some target value. This target value does not have to be unique, nor in fact is the time evolution to that value unique. The approach just outlined applies equally well to classical probabilistic evolutions: Although quantum and classical systems are dynamically distinct, the principles for open-loop control are in fact very similar.

Controlling chemical reactions by laser-produced electromagnetic fields that are time dependent is a well-known open-loop quantum control problem. In the frequency-resolved approach to control, the quantum interference between different evolutionary paths is being manipulated; in the time-resolved approach, the dynamics of wave packets produced by ultrafast laser pulses leads to control. For some specific control of the chemical reactions, one can optimize the temporal and spectral structure of those laser pulses (Shi et al. 1988).

The fundamental differences between classical and quantum systems become real issues, however, in the field of closed-loop control. Quantum systems can have two distinct types of feedback control: directly and indirectly coupled quantum feedback (see Figure 2). As illustrated in Figure 2(a), in a system with directly coupled quantum feedback, a quantum variable of the system is coupled to the quantum controller, and a quantum input path from the controller goes directly back to the quantum system. When the quantum feedback is indirect, as shown in Figure 2(b), the quantum dynamical system under control is an observed system. It therefore generates a classical output, also known as the measurement record, which the controller may analyze to provide a best estimate of the original quantum state of the system. The controller then feeds back a classical signal to vary parameters in the quantum evolution operator in accord with the chosen control strategy. Hybrid couplings using both direct and indirect quantum feedback channels are easy to envisage: The channel from the system output to the controller input may be directly coupled whereas the channel from the controller output to the system input may be coupled indirectly through a classical path.

In both classical and quantum contexts, the main goal of closed-loop control is to enhance system performance in the presence of noise from both the environment and the uncertainty in the system parameters. To limit the effects of noise, the controller must perform an irreversible operation. Noise generates a large set of undesirable evolutions, and the controller's task is to map this large set to a much smaller one of more desirable evolutions. Mapping from the larger to the smaller set is by definition irreversible. In other words, noise is a source of entropy for the system. To control the system, the controller must extract the entropy from the system under control and put it somewhere else. The controller must therefore have enough degrees of freedom to respond conditionally upon the noise realization. In indirect quantum feedback control, the measurement process, coupled with the conditional response of the controller, is the source of entropy reduction. In direct quantum feedback control, the evolution of the system is fully unitary, or quantum mechanical. The quantum controller provides a large Hilbert space of quantum mechanical states. That is precisely where the entropy generated by the noise may be put (or where the history of the effect of the noise on the system may be stored). The quantum controller then reacts conditionally to this quantum record, keeping the entropy of the quantum dynamical system low, while the entropy of the storage location grows continually.

Inherent Noise Generation in Quantum Feedback Control

Unlike classical systems, quantum systems may be easily disturbed when information about them is extracted. Measurement disturbs a quantum system through the following intrinsic property of quantum mechanics: Obtaining accurate knowledge about one observable of a quantum system necessarily limits the information about an observable conjugate to the first. For example, particle position and momentum are conjugate observables, and the uncertainties inherent in the knowledge of both are codified by the famous Heisenberg uncertainty relation. If the chosen feedback-control strategy involves measurement, one must take into account the effects of the measurement on the evolution of the quantum system. A generally applicable model for including those effects is that of a continuous quantum measurement. This model was developed for quantum optics (Carmichael 1993), a field in which such measurements have been realized experimentally, and it was also derived in the mathematical physics literature with the help of more abstract reasoning (Barchielli 1993). In this volume, the model of a continuous quantum measurement is presented in the article "The Emergence of Classical Dynamics in a Quantum World" on page 110.

Quantum measurements may introduce unwanted noise in three more-or-less distinct ways. First, one may measure an observable conjugate to the real variable of interest and thereby introduce more uncertainty in the latter variable. More generally, one may attempt to obtain information inconsistent with the state under control. For example, to preserve a state that is the superposition of two position states, position measurements must be avoided because they will destroy the superposition. Thus, in quantum mechanics, the type of measurement chosen must be consistent with the control objectives. This condition is unnecessary in classical feedback control. Second, if trying to control the values of observables (Doherty et al. 2000), one must consider that the time evolutions of different observables necessarily affect each other over time. Observables whose values are uncertain at one time will cause other observables (perhaps more accurately known) to become uncertain at a later time. For example, a very accurate measurement of the particle position at one time introduces uncertainty into the value of the particle momentum. Because the value of momentum determines the position of the particle at a later time, the momentum uncertainty makes the future position of the particle more uncertain, hence introducing noise into the quantity that is being measured. This mechanism for introducing noise is usually referred to as the back action of a quantum measurement.

The third kind of noise involves the randomness of the measurement results. Because the state of the observed system after a measurement depends upon the outcome of the measurement, the more the result fluctuates, the more noise there is in the evolution of the system. For classical measurements, fluctuations in measurement results cannot be any more than the entropy of the system before measurement; that is, the measurement does not introduce any additional noise into the system. In quantum mechanics, however, even if the system state is known precisely, one can still make measurements that change the state in a random way, thereby actually injecting noise into the system. This observation is particularly relevant when the overall state of the system, rather than a specific observable, is being controlled. The situation is further complicated by the fact that, for certain classes of measurements, there is actually a tradeoff between the noise injected by the measurement and the information gained by the observer (Doherty et al. 2001). As a result, designing measurement strategies is far from being a trivial activity.

Strategies for Quantum Feedback Control

The differences between classical and quantum measurements profoundly affect the design of feedback control algorithms. A classical controller extracts as much information from the system as possible. In quantum control, irreducible disturbances are inherent to any measurement, and therefore the measurement strategy becomes a significant part of the feedback algorithm. For example, just as the inputs to the system change with time, the measurements too may need to be varied with time so that the best control should be achieved.

Adaptive measurement, or altering the measurement as it proceeds, was first introduced by Howard Wiseman (1995), not for control but for accuracy. The result was a more accurate measurement of some aspect of the quantum state. Nevertheless, this approach has a unique bearing on quantum feedback control algorithms. Knowing that quantum measurements can disturb the state being measured, one may want to start a continuous measurement process by measuring in a way that is not necessarily optimal but is sufficiently weak to cause minimal disturbance to the aspect of interest. As the measurement proceeds, one uses the continuously obtained information about the state to make the measurement increasingly close to optimal.

For example, consider measuring the oscillation amplitude of a harmonic oscillator when the phase of the oscillation is unknown but the oscillator is known to be in an amplitude-squeezed state; that is, the uncertainty in amplitude or energy is much smaller than the uncertainty in phase, the conjugate variable (see Figure 3). In this case, an accurate measurement of amplitude is given by a measurement of position at the moment when the particle is at its maximum spatial extent, or maximum distance from $x = 0$. On the other hand, at the moment when the particle has the most momentum (at position $x = 0$), the ideal quantity to measure is momentum. Thus, for a continuous measurement of the oscillation amplitude, a linear combination of position and momentum should be measured and the relative weighting of those two variables should be allowed to oscillate in time. However, without knowing the mean phase of oscillation, one cannot know which variable should have the most weighting in the measurement at what time. Using an adaptive measurement procedure, one can start by assuming the oscillator to have a particular phase and then adjust the relative weights of position and momentum to more desirable values as information about the phase is obtained.

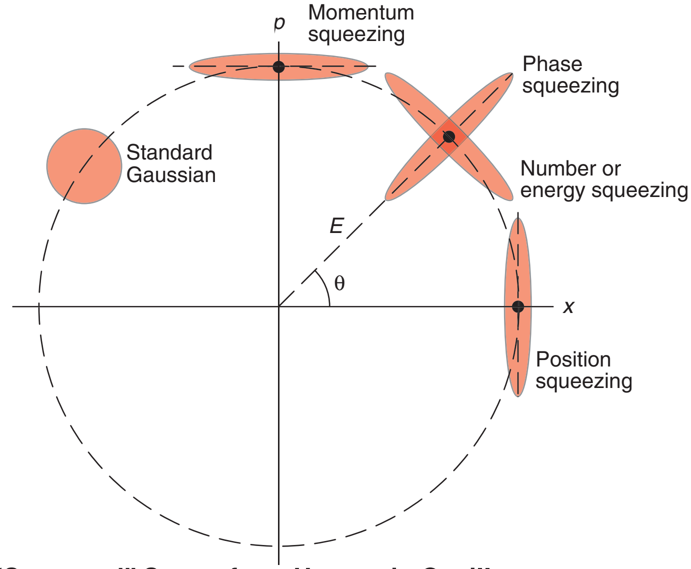

Figure 3. "Squeezed" States for a Harmonic Oscillator
Squeezing may be illustrated by considering phase-space plots of a Gaussian wave function. For a standard Gaussian state, the uncertainties in the *x*- and *p*-direction are equal, and the uncertainty ellipse takes the shape of a circle, provided appropriate position and momentum scalings are made. When states are "squeezed," the area of the uncertainty ellipse remains constant, but the ellipse is rotated and squeezed as shown. Squeezing momentum, for example, means reducing the uncertainty in momentum. The constant energy surface is the dashed circle, and the position on the circle can be specified by the angle. Squeezing phase and energy again refers to changes in shape of the uncertainty ellipse for the wave function.

Applications of Quantum Control

Atomic optics is one field in which it should be possible to test quantum feedback control in the near future. It has already been demonstrated that a single atom can be trapped inside an ultralow-loss optical cavity (mirror reflectivity is $R = 0.9999984$ in experiments at Caltech) in the strong-coupling quantum regime (Mabuchi et al. 1999). Figure 4 illustrates the experimental setup used at Caltech. The strong coupling occurs between the atom and the radiation field in the cavity and is proportional to the induced atomic dipole moment and the single-photon cavity field. Continuous measurements and real-time feedback could be used to cool such an atom to the "ground" state of the quantized mechanical potential produced by several photons in the cavity. The average number of photons circulating inside such a cavity can be kept very low (from 1 to 10 photons) if one uses a weak driving laser that barely balances the slow rate at which individual photons leak out. If the cavity mode volume is

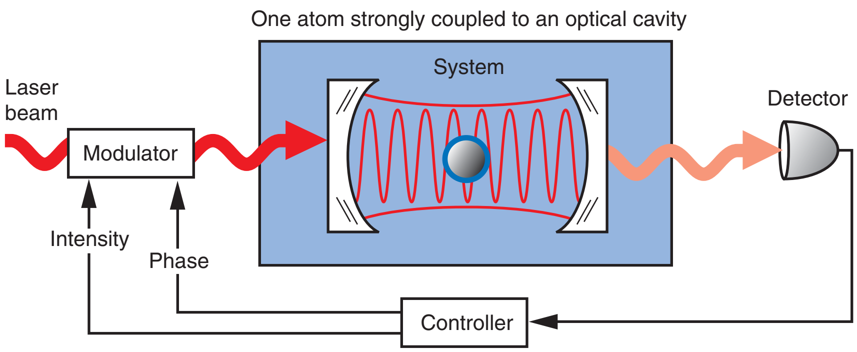

Figure 4. Quantum Feedback in a Cavity Quantum Electrodynamics Application
The dynamics between the atom and the photon field in the cavity can be modified by continuous measurement of the light transmitted through the cavity (which bears information about the evolving system state) and by continuous adjustment of the amplitude/phase of the driving laser in a manner that depends on the measurement results. Control objectives of fundamental interest include active cooling of the motion of an individual atom, feedback-stabilized quantum state synthesis, and active focusing of atomic beams for applications such as direct-write lithography.

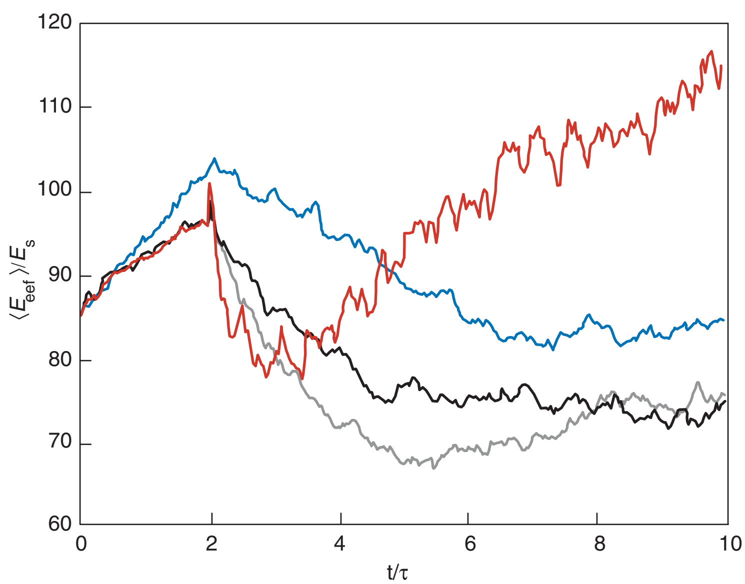

Figure 5. Simulating a Feedback Algorithm to Cool Atomic Motion in an Optical Cavity
In this simulated experiment, the light forms an effective sinusoidal potential for the atom, and the controller switches this potential between a high and a low value (separated by some ΔV) to cool the atomic motion. In this simulation, the feedback is turned on at $t = 2$, and the expected value of the atomic motion energy is plotted here as a function of time for four different values of ΔV. Although these results are still preliminary, they indicate that the effectiveness of the feedback algorithm is highly dependent on ΔV.

sufficiently small, just a few photons can give rise to dipole (alternating-current Stark shift) forces that are strong enough to bind an atom near a local maximum of the optical field distribution. At the same time, the atomic motion can be monitored in real time by phase-sensitive measurements of the light leaking out of the cavity. To a degree determined by the fidelity of these phase measurements, the information gained can be used continually to adjust the strength of the driving laser (and hence the depth of the optical potential) in a manner that tends to remove kinetic energy from the motion of the atomic center of mass.

In order to perform such a task in real time, however, it is essential to develop approximate techniques for continuously estimating the state of the atomic motion. Approximations are needed because integrating a stochastic conditioned-evolution equation to obtain a continuous estimate of the density matrix is far too complex a task to be performed in real time. While this experiment remains to be carried out, we have developed an approximate estimation algorithm[1] and used it in combination with an experimentally realizable feedback algorithm (see Figure 5).

Feedback cooling ideas can also be applied to condensed-matter systems. Some of our recent calculations predict that feedback control can be used to cool a nanoresonator below the limits set by refrigeration. This method would reduce thermal fluctuations to approximately the quantum energy level spacing of the resonator. These findings are important because nanoscale devices are interesting from a more fundamental perspective than merely sensing and actuation applications. Provided they can be cooled to

[1] This algorithm is described in a yet unpublished paper by Salman Habib, Kurt Jacobs, Hideo Mabuchi, and Daniel Steck.

sufficiently low temperatures, low-loss nanomechanical resonators would be excellent candidates for the first observation of quantum dynamics in mechanical mesoscopic systems. Yet, as mentioned above, in order achieve this goal, we must reduce thermal fluctuations to approximately the quantum energy level spacing of the resonator, a task which requires temperatures in the range of millikelvins.

To cool the position coordinate of the nanoresonator, one needs a suitable scheme for continuous position measurement. One practical method of performing a continuous measurement of a nanoresonator's position is to use a single-electron transistor (SET)—see Figure 6. To make the measurement, one locates the resonator next to the central island of the SET. When the resonator is charged and the SET is biased so that current flows through it, changes in the resonator's position modify the energy of the central island, which produces changes in the SET current. The current therefore provides a continuous measurement of the position of the resonator, a requirement for implementing a linear feedback cooling algorithm. A feedback force can be applied to the resonator by varying the voltage on a "feedback electrode," which is capacitively coupled to the resonator (see Figure 6). The applied voltage is adjusted so as to damp the amplitude of oscillation.

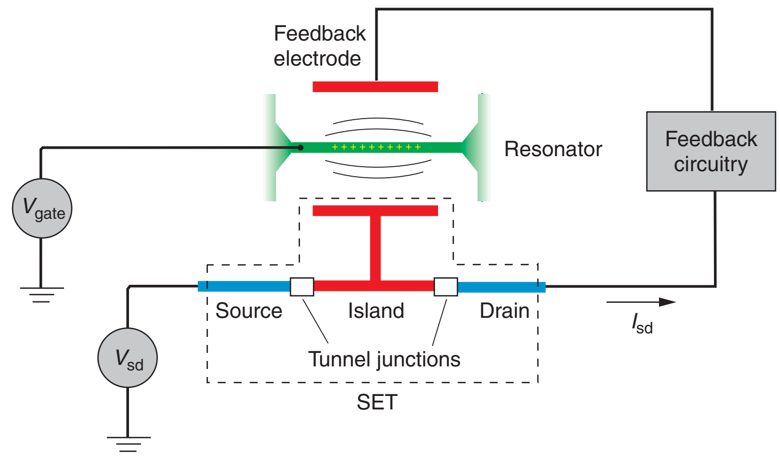

Figure 6. Cooling a Nanomechanical Resonator
This schematic diagram illustrates a concept for cooling a nanomechanical resonator to millikelvin temperatures, at which we can possibly observe quantum dynamics. An SET measures the position of the resonator, and a feedback mechanism damps (cools) the resonator's motion. The resonator, which is charged by the voltage source V_{gate} acts as the SET gate electrode. The resonator is also capacitively coupled to the SET island (red) and feedback electrode. As it moves back and forth relative to the SET island, the current I_{sd} flowing through the SET changes. Information about the changing current is used by the feedback circuitry to charge the feedback electrode. A force is generated that damps the resonator's oscillations.

Experiments on nanomechanical oscillators observed with SETs currently start at temperatures near 100 millikelvins. These oscillators have fundamental frequencies f_0 on the order of 1 to 100 megahertz. As a concrete example, consider a practical oscillator with f_0 = 10 megahertz, a length of 2 micrometers, and the other two dimensions on the order of 100 nanometers. The effective mass of such an oscillator is roughly 10^{-19} kilograms. An achievable quality factor, Q, is about 10^4. In order to observe discrete quantum passage from one oscillator energy level to another, the thermal energy should be on the order of the level spacing, that is, $k_B T \sim hf_0$, which corresponds to an effective temperature T = .24 millikelvin. Habib, Jacobs, Asa Hopkins, and Keith Schwab have shown that feedback cooling applied to this system at an initial temperature T = 100 millikelvins can yield a final temperature of T = 0.35 millikelvin. At this temperature, the aggregate occupation number lies between zero, the ground state, and one, the first excited state of the nanomechanical resonator. In other words, the system is cold enough to allow observation of quantum "jumps." Although our calculations are based on certain idealized assumptions, those assumptions are close enough to reality that experimentalists can hope to achieve similar results.

Another, seemingly paradoxical, application of quantum feedback control techniques might be in suppressing quantum dynamical effects such as tunneling. A classical memory device can be viewed as a two-state system with the two states separated by a finite energy barrier. At low temperatures, there is a finite probability of coherent or incoherent tunneling from one minimum to the other. Tunneling generates random memory errors, but continuous measurement, coupled with feedback, can suppress it. One such scheme is described and demonstrated in Andrew Doherty et al. (2000). The Hamiltonian for the double well is taken to be

$$H = \frac{1}{2}p^2 - Ax^2 + Bx^4 \,, \tag{3}$$

where x and p are dimensionless position and momentum. Choosing A = 2 and B = 1/9 puts the minima of the wells at ±3 and gives a barrier height of approximately 13.5. The controller is allowed to continuously observe the position of the particle and to apply a

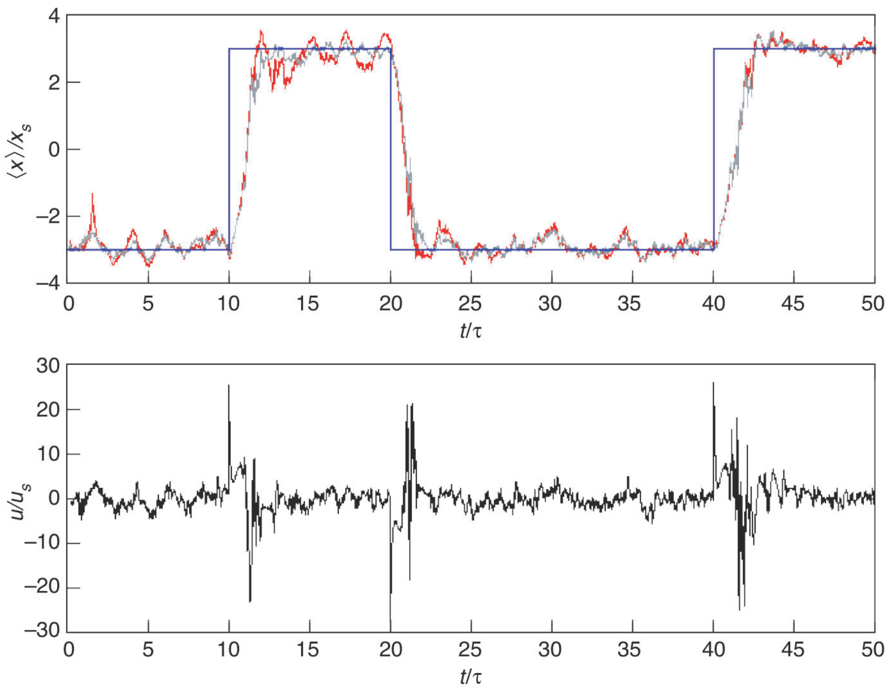

Figure 7. Particle in a Double Well Controlled by an Estimation-Feedback Scheme

(a) Shown here, as a function of time, are the target position (blue line), the 'true" mean position (red line) obtained with the stochastic master equation (in which the measurement strength k equals 0.3 and the thermal heating rate β equals 0.1), and the position obtained with the Gaussian estimator (gray line). (b) The control strength (size of applied force) is shown as a function of time.

linear force in addition to the "double-well" potential already present. The continuous observation is described by the equation

$$dq = \langle x \rangle dt + \frac{dV}{\sqrt{8k}} \quad , \tag{4}$$

where dq is the measurement result in the time interval dt and k is a constant characterizing the accuracy, or strength, of the measurement.
The system is also driven by a thermal heat bath in the high-temperature limit. The effect of that bath is, in fact, the same as that of a continuous quantum measurement of position that ignores the measurement result. When the bath is described in this way, it is the strength of the fictitious measurement that gives the rate of thermal heating, and we will denote this constant by β.

Integrating a stochastic master equation gives the observer's state of knowledge as a result of the continuous measurement. However, since this is a differential equation for the density matrix of the single particle, it is numerically expensive to integrate. For practical purposes, one requires a simplified means for calculating a state estimate. To achieve this goal, we note that, as a result of the continuous observation, even though the dynamics are nonlinear, the density matrix remains approximately Gaussian. When a Gaussian approximation is used, the stochastic master equation reduces to a set of five equations (for all the moments of x and p up to quadratic order), and so it provides us with a practical method for obtaining a continuous state estimate. In practice, this Gaussian estimator can be shown to work quite well; that is, mean values from the approximate estimator agree very well with mean values derived from exact numerical solutions of the stochastic master equation—see Figure 7(a).

In addition to a state estimation procedure, we also require a feedback algorithm. If the system were linear, one could apply the optimal techniques of modern control theory to find a feedback algorithm. Because attempting an optimal control solution for the full nonlinear problem is computationally intractable, the idea is to linearize the system dynamics around the present estimate of the state with the further assumption that the probability density, conditioned on the measurement record, remains Gaussian. As long as position measurements are sufficiently strong, this last condition is satisfied. The importance of this condition is twofold: Having a Gaussian approximation does not only mean that a small number of moments (five) are needed to describe the distribution but also that the quantum propagator is very close to the classical propagator at each time step (for exactly Gaussian states, the two are identical), and hence techniques borrowed from classical control have an excellent chance of working. The control can fail if the measurement is too weak to maintain a localized Gaussian distribution or if it is too strong. In the latter case, the state is Gaussian, but the measurement noise is too large.

The Gaussian state estimate is now used to set the value of the feedback term in the Hamiltonian (the sign and the magnitude of the coefficient of the linear feedback term in the potential). By choosing appropriate strengths for the measurement and the feedback strength, one can show that the feedback scheme is effective in controlling whether the particle is in the desired minimum—see Figure 7(b). For this plot, the measurement strength is $k = 0.3$, and the thermal heating rate is $d\langle E \rangle/dt = \beta = 0.1$.

This scheme has limitations arising from unwanted heating due to the measurement. Although some of the heating derives directly from having to keep the state close to Gaussian, a more general limitation also contributes to heating: The measurement must be sufficiently strong to provide enough information for control to be effective. Developing new estimation and feedback schemes that can reduce the measurement-induced heating rate is an important area for future research.

Outlook for the Future

Most likely, ideas in quantum feedback control will first be tested in condensed matter physics and in quantum and atomic optics. Experiments in atomic optics have already furnished the cleanest tests and demonstrations of quantum mechanics in the last several decades. These include violations of the Bell inequalities, quantum teleportation, quantum state tomography, quantum cryptography, and single-atom interference. The ability to compare experimental results with precise theoretical benchmarks is a hallmark of these tests. As these experiments become increasingly sophisticated and complex, one can envisage a passage from "toy" demonstrations to real applications such as feedback control. The more strongly coupled systems of condensed matter physics are less amenable to accurate theoretical prediction. Nevertheless, experiments are becoming comparable in quality to early atomic optics experiments, and the time is ripe for active interaction between these two fields: Theoretical development in quantum optics, such as continuous measurement and quantum control, can be taken over to condensed matter contexts, most notably in nanotechnology. As the size of the smallest structures that can be fabricated by lithographic techniques decreases, the need for quantum mechanics becomes inevitable. Since lithography is the only way we know to create very complex systems at reasonable cost, it follows that a fundamental and predictive understanding of quantum dynamics applicable to these systems (whether coherent or incoherent) will be required. It is also clear that, for these systems to be designable and to function reliably in an engineering sense, further development of quantum control theory will be necessary.

From a "more algorithmic" perspective, the Holy Grail is the development of optimal and robust control algorithms that are generally applicable. So far, apart from the trivial case in which the system dynamics are linear and the measurement strategy is considered fixed (Doherty and Jacobs 1999), no such optimal algorithms have been found for quantum feedback control. In classical control theory, optimal and robust control algorithms exist for linear systems, but only very few for nonlinear systems despite the best effort of control theorists in the past few decades. Nonlinear classical optimal control is a very difficult problem indeed, and probably intractable in most cases. Systematic numerical search algorithms for optimal strategies exist, but these also become intractable for systems of reasonable size. Because the dynamics of noisy and measured quantum systems is inherently nonlinear, the quantum control problem may also be intractable (Doherty et al. 2000). However, in quantum dynamics, nonlinearity is of a restricted kind, and the possibility of obtaining general analytic results providing optimal and robust algorithms for the feedback control of quantum systems remains an open problem. ■

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Hideo Mabuchi received an A.B. in physics from Princeton University and a Ph.D. from Caltech. Upon graduating, he became Assistant Professor of Physics at Caltech but spent his first year on leave at Princeton University as a visiting fellow in chemistry. Since returning to Caltech, he has been named an A. P. Sloan Research Fellow, Office of Naval Research Young Investigator, and a MacArthur Fellow. His current research focuses on quantum measurement, quantum feedback control, control-theoretic approaches to the theory of multiscale phenomena, and optical measurement techniques for molecular biophysics.

For biographies of Salman Habib and Kurt Jacobs, see page 125.