0.75, and similarly in Figs. 4a and b at $\lambda = 0.785$. (Observe the exceptionally slow convergence to x^* at $\lambda = 0.75$, where iterates approach the fixed point not geometrically, but rather with deviations from x^* inversely proportional to the square root of the number of iterations.) Since x_1^* and x_2^*, the new fixed points of f^2, are *not* fixed points of f, it must be that f sends one into the other:

$$x_1^* = f(x_2^*)$$

and

$$x_2^* = f(x_1^*) \, .$$

Such a *pair of points*, termed a *2-cycle*, is depicted by the limiting unwinding circulating square in Fig. 4a. Observe in Fig. 4b that the slope of f^2 is in excess of 1 at the fixed point of f and so is an unstable fixed point of f^2, while the two new fixed points have slopes smaller than 1, and so are *stable*; that is, every two iterates of f will have a point attracted toward x_1^* if it is sufficiently close to x_1^* or toward x_2^* if it is sufficiently close to x_2^*. This means that the sequence under f,

$$x_0, x_1, x_2, x_3, \ldots ,$$

eventually becomes arbitrarily close to the sequence

$$x_1^*, x_2^*, x_1^*, x_2^*, \ldots ,$$

so that this is a stable 2-cycle, or an *at-*

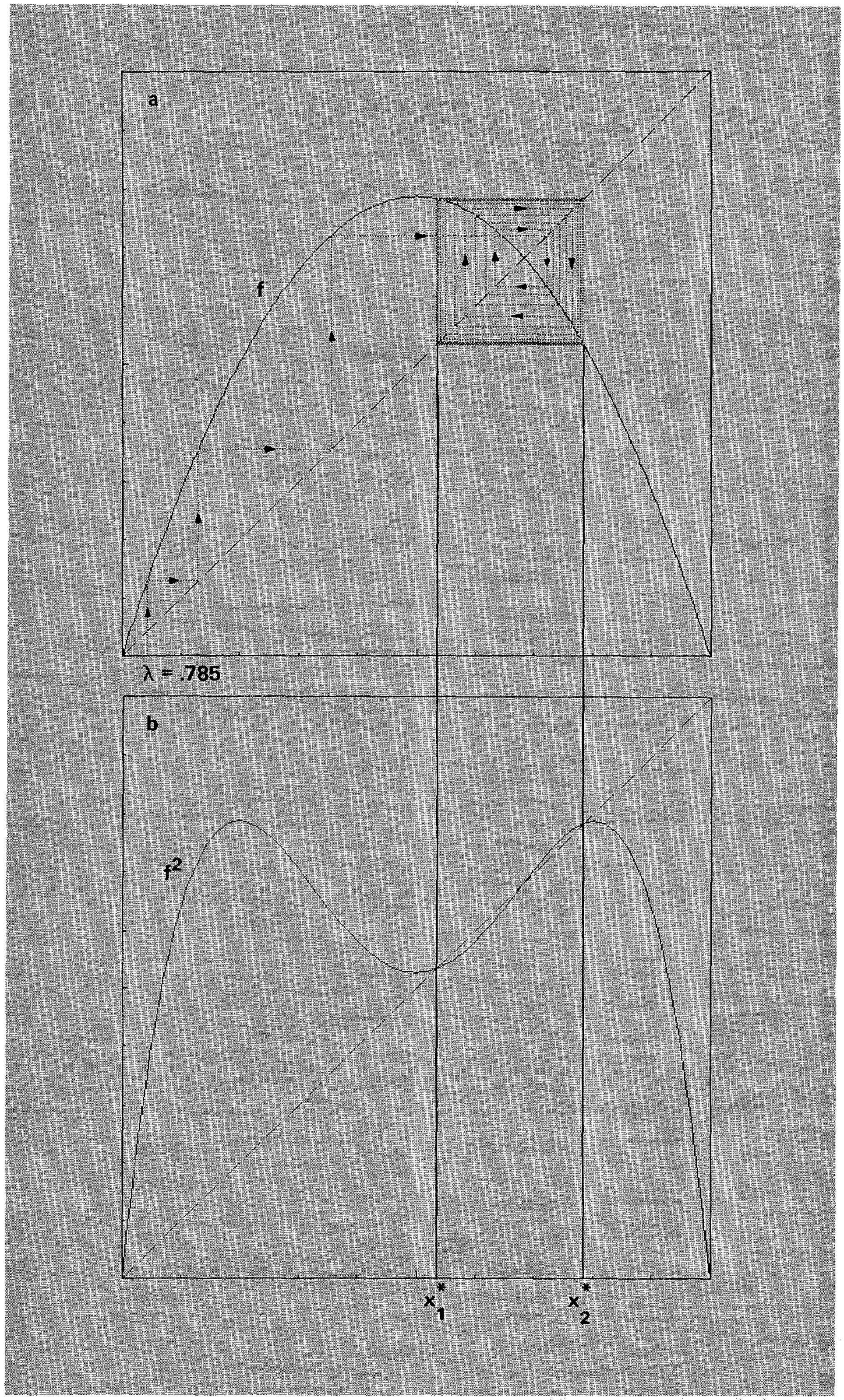

Fig. 4. $\lambda = 0.785$. (a) shows the outward spiralling to a stable 2-cycle. The elements of the 2-cycle, x_1^* and x_2^*, are located as fixed points in (b).

Fig. 5. $\lambda = \lambda_1$. A superstable 2-cycle. x_1^* and x_2^* are at extrema of f^2.

tractor of period 2. Thus, we have observed for Eq. (15) the first period doubling as the parameter λ has increased.

There is a point of paramount importance to be observed; namely, f^2 has the same slope at x_1^* and at x_2^*. This point is a direct consequence of Eq. (20), since if $x_0 = x_1^*$, then $x_1 = x_2^*$, and vice versa, so that the product of the slopes is the same. More generally, if $x_1^*, x_2^*, ..., x_n^*$ is an n-cycle so that

$$x_{r+1}^* = f(x_r^*) \qquad r = 1, 2, ..., n-1$$

and

$$x_1^* = f(x_n^*)\,, \tag{24}$$

then *each* is a fixed point of f^n with identical slopes:

$$x_r^* = f^n(x_r^*) \qquad r = 1, 2, ..., n \tag{25}$$

and

$$f^{n\prime}(x_r^*) = f'(x_1^*) \ldots f'(x_n^*)\,. \tag{26}$$

From this observation will follow period doubling *ad infinitum.*

As λ is increased further, the minimum at $x = \frac{1}{2}$ will drop as the slope of f^2 through the fixed point of f increases. At some value of λ, denoted by λ_1, $x = \frac{1}{2}$ will become a fixed point of f^2. Simultaneously, the right-hand maximum will also become a fixed point of f^2. [By Eq. (26), both elements of the 2-cycle have slope 0.] Figures 5a and b depict the situation that occurs at $\lambda = \lambda_1$.

Period Doubling *Ad Infinitum*

We are now close to the end of this story. As we increase λ further, the minimum drops still lower, so that both x_1^* and x_2^* have negative slopes. At some parameter value, denoted by Λ_2, the slope at *both* x_1^* and x_2^* becomes equal to -1. Thus at Λ_2 the same situation has developed for f^2 as developed for f at $\Lambda_1 = 3/4$. This transitional case is depicted in Figs. 6a and b. Accordingly, just as the fixed point of f at Λ_1 issued into being a 2-cycle, so too does *each* fixed point of f^2 at Λ_2 create a 2-cycle, which in turn is a 4-cycle of f. That is, we have now encountered the second period doubling.

The manner in which we were able to follow the creation of the 2-cycle at Λ_1 was to anticipate the presence of period 2, and so to consider f^2, which would resolve the cycle into a pair of fixed points. Similarly, to resolve period 4 into fixed points we now should consider f^4. Beyond being the fourth iterate of f, Eq. (8) tells us that f^4 can be computed from f^2:

$$f^4 = f^2 \circ f^2 .$$

From this point, we can abandon f itself, and take f^2 as the "fundamental" function. Then, just as f^2 was constructed by iterating f with itself we now iterate f^2 with itself. The manner in which f^2 reveals itself as being an iterate of f is the slope equality at the fixed points of f^2, which we saw imposed by the chain rule. Since the operation of the chain rule is "automatic," we actually needed to consider only the fixed point of f^2 nearest to $x = 1/2$; the behavior of the other fixed point is slaved to it. Thus, at the level of f^4, we again need to focus on only the fixed point of f^4 nearest to $x = 1/2$: the other *three* fixed points are similarly slaved to it. Thus, a recursive scheme has been unearthed. We now increase λ to λ_2, so that the fixed point of f^4 nearest to $x = 1/2$ is again at $x = 1/2$ with slope 0.

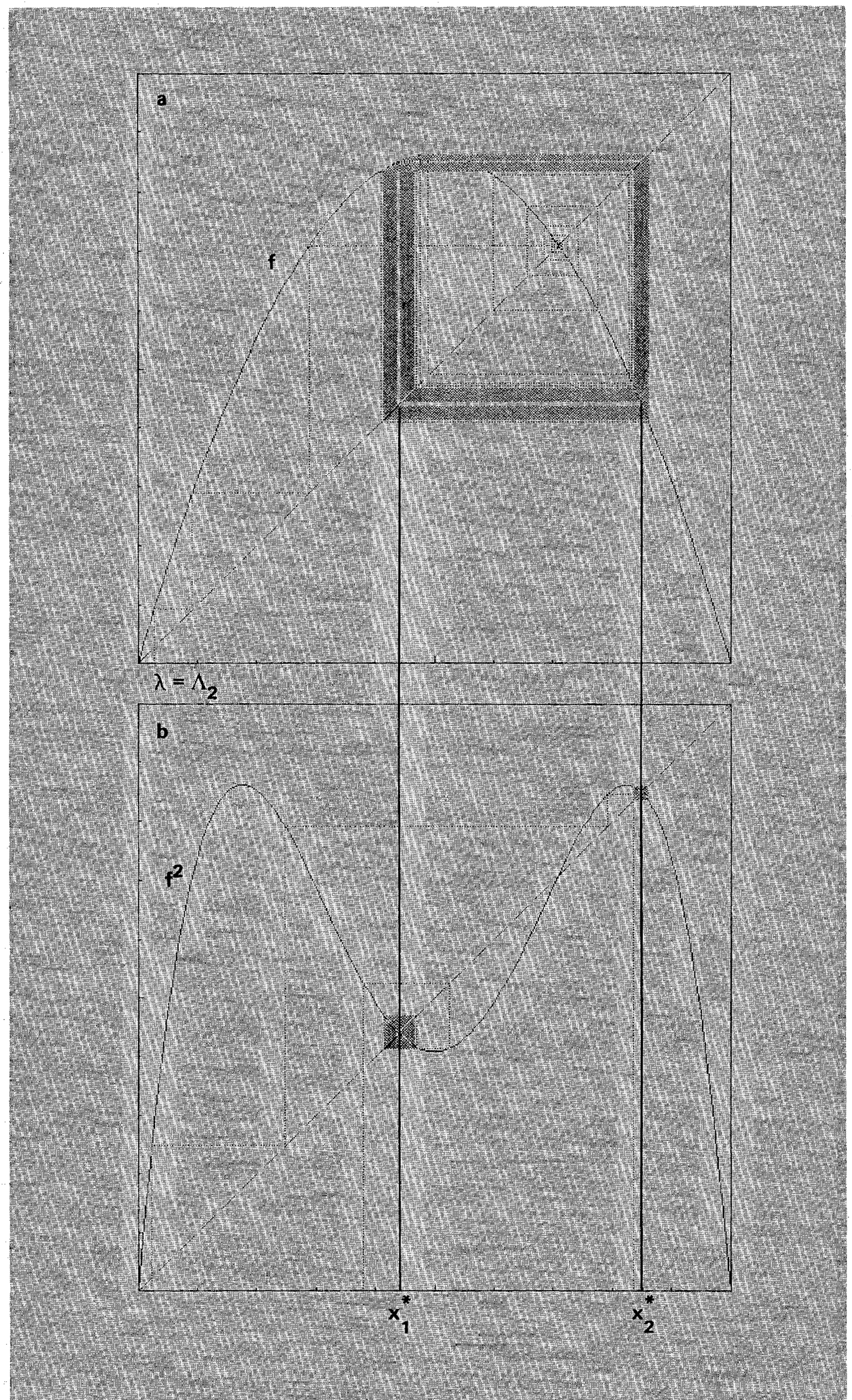

***Fig. 6.** $\lambda = \Lambda_2$. x_1^* and x_2^* in (b) have the same slow convergence as the fixed point in Fig. 3a.*