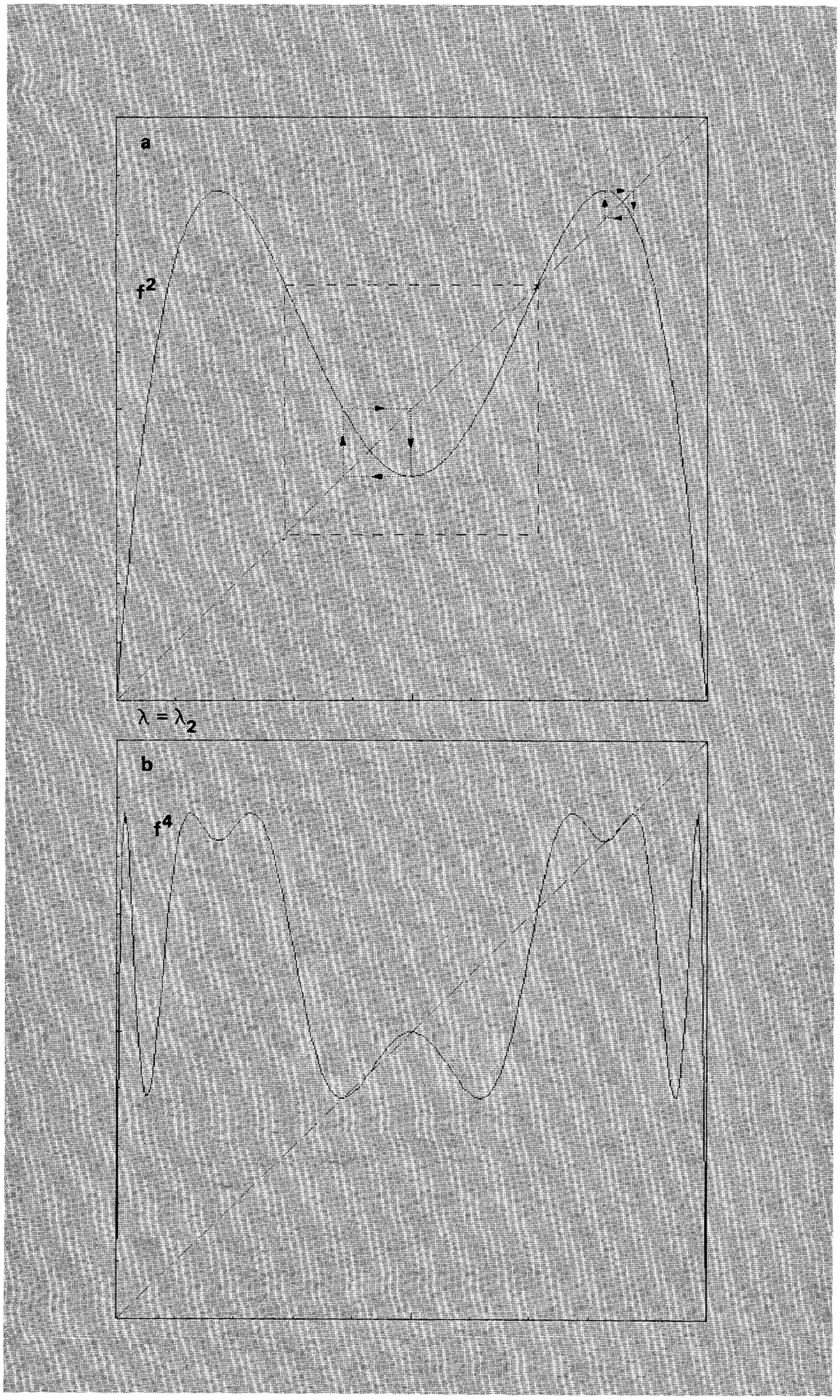

Fig. 7. $\lambda = \lambda_2$. A superstable 4-cycle. The region within the dashed square in (a) should be compared with all of Fig. 5a.

Figures 7a and b depict this situation for f^2 and f^4, respectively. When λ increases further, the maximum of f^4 at $x = \frac{1}{2}$ now moves up, developing a fixed point with negative slope. Finally, at Λ_3 when the slope of this fixed point (as well as the other three) is again -1, each fixed point will split into a pair giving rise to an 8-cycle, which is now stable. Again, $f^8 = f^4 \circ f^4$, and f^4 can be viewed as fundamental. We define λ_3 so that $x = \frac{1}{2}$ again is a fixed point, this time of f^8. Then at Λ_4 the slopes are -1, and another period doubling occurs. Always,

$$f^{2^{n+1}} = f^{2^n} \circ f^{2^n} . \qquad (27)$$

Provided that a constraint on the range of λ does not prevent it from decreasing the slope at the appropriate fixed point past -1, this doubling must recur *ad infinitum.*

Basically, the mechanism that f^{2^n} uses to period double at Λ_{n+1} is the same mechanism that $f^{2^{n+1}}$ will use to double at Λ_{n+2}. The function $f^{2^{n+1}}$ is constructed from f^{2^n} by Eq. (27), and similarly $f^{2^{n+2}}$ will be constructed from $f^{2^{n+1}}$. Thus, there is a definite operation that, by acting on functions, creates functions; in particular, the operation acting on f^{2^n} at Λ_{n+1}, (or better, f^{2^n} at λ_n) will determine $f^{2^{n+1}}$ at λ_{n+1}. Also, since we need to keep track of f^{2^n} only in the interval including the fixed point of f^{2^n} closest to $x = \frac{1}{2}$ and since this interval becomes increasingly small as λ increases, the part of f that generates this region is also the restriction of f to an increasingly small interval about $x = \frac{1}{2}$. (Actually, slopes of f at points farther away also matter, but these merely set a "scale," which will be eliminated by a rescaling.) The behavior of f away from $x = \frac{1}{2}$ is immaterial to the period-doubling behavior, and in the limit of large n only the *nature of f's maximum* can matter. This means that in the infinite period-doubling limit, all functions with a quadratic extremum will have identical behavior. [$f''(\frac{1}{2}) \neq 0$ is the generic circumstance.] Therefore, the operation on functions will have a *stable fixed point* in the space of functions, which will be the common universal limit of high iterates of any specific function. To determine this universal limit we must enlarge our scope vastly, so that the role of the starting point, x_0, will be played by an arbitrary *function*; the attracting fixed point will become a universal function obeying an equation implicating only itself. The role of the function in the equation $x_0 = f(x_0)$ now must be played by an *operation* that yields a new function when it is performed upon a function. In fact, the heart of this operation is the functional composition of Eq. (27). If we can determine the exact operator and actually can solve *its* fixed-point problem, we shall understand why a special number, such as δ of Eq. (3), has emerged independently of the specific system (the starting function) we have considered.

The Universal Limit of High Iterates

In this section we sketch the solution to the fixed-point problem. In Fig. 7a, a dashed square encloses the part of f^2 that we must focus on for all further period doublings. This square should be compared with the unit square that comprises all of Fig. 5a. If the Fig. 7a square is reflected through $x = \frac{1}{2}$, $y = \frac{1}{2}$ and then *magnified* so that the circulation squares of Figs. 4a and 5a are of equal size, we will have in each square a piece of a function that has the same kind of maximum at $x = \frac{1}{2}$ and falls to zero at the right-hand lower corner of the circulation square. Just as f produced this second curve of f^2 in the square as λ increased from λ_1 to λ_2, so too will f^2 produce another curve, which will be similar to the other two when it has been magnified suitably and reflected twice. Figure 8 shows this superposition for the first *five* such functions; at the resolution of the figure, observe that the last three

A DISCOVERY

The inspiration for the universality theory came from two sources. First, in 1971 N. Metropolis, M. Stein, and P. Stein (all in the LASL Theoretical Division) discovered a curious property of iterations: as a parameter is varied, the behavior of iterates varies in a fashion independent of the particular function iterated. In particular for a large class of functions, if at some value of the parameter a certain cycle is stable, then as the parameter increases, the cycle is replaced successively by cycles of doubled periods. This period doubling continues until an infinite period, and hence erratic behavior, is attained.

Second, during the early 1970s, a scheme of mathematics called dynamical system theory was popularized, largely by D. Ruelle, with the notion of a "strange attractor." The underlying questions addressed were (1) how could a purely causal equation (for example, the Navier-Stokes equations that describe fluid flow) come to demonstrate highly erratic or statistical properties and (2) how could these statistical properties be computed. This line of thought merged with the iteration ideas, and the limiting infinite "cycles" of iteration systems came to be viewed as a possible means to comprehend turbulence. Indeed, I became inspired to study the iterates of functions by a talk on such matters by S. Smale, one of the creators of dynamical system theory, at Aspen in the summer of 1975.

My first effort at understanding this problem was through the complex analytic properties of the generating function of the iterates of the quadratic map

$$x_{n+1} = \lambda x_n(1 - x_n) .$$

This study clarified the mechanism of period doubling and led to a rather different kind of equation to determine the values of λ at which the period doubling occurs. The new equations were intractable, although approximate solutions seemed possible. Accordingly, when I returned from Aspen, I numerically determined some parameter values with an eye toward discerning some patterns. At this time I had never used a large computer—in fact my sole computing power resided in a programmable pocket calculator. Now, such machines are very slow. A particular parameter value is obtained iteratively (by Newton's method) with each step of iteration requiring 2^n iterates of the map. For a 64-cycle, this means 1 minute per step of Newton's method. At the same time as n increased, it became an increasingly more delicate matter to locate the desired solution. However, I immediately perceived the λ_n's were converging geometrically. This enabled me to predict the next value with increasing accuracy as n increased, and so required just one step of Newton's method to obtain the desired value. To the best of my knowledge, this observation of geometric convergence has never been made independently, for the simple reason that the solutions have always been performed automatically on large and fast computers!

That a geometric convergence occurred was already a surprise. I was interested in this for two reasons: first, to gain insight into my theoretical work, as already mentioned, and second, because a convergence rate is a number invariant under all smooth transformations, and so of mathematical interest. Accordingly, I spent a part of a day trying to fit the convergence rate value, 4.669, to the mathematical constants I knew. The task was fruitless, save for the fact that it made the number memorable.

At this point I was reminded by Paul Stein that period doubling isn't a unique property of the quadratic map, but also occurs, for example, in

$$x_{n+1} = \lambda \sin \pi x_n .$$

However, my generating function theory rested heavily on the fact that the nonlinearity was simply quadratic and not transcendental. Accordingly, my interest in the problem waned.

Perhaps a month later I decided to determine the λ's in the transcendental case numerically. This problem was even slower to compute than the quadratic one. Again, it became apparent that the λ's converged geometrically, and altogether amazingly, the convergence rate was the same 4.669 that I remembered by virtue of my efforts to fit it.

Recall that the work of Metropolis, Stein, and Stein showed that precise qualitative features are independent of the specific iterative scheme. Now I learned that precise quantitative features also are independent of the specific function. This discovery represents a complete inversion of accustomed ritual. Usually one relies on the fact that similar equations will have qualitatively similar behavior, but quantitative predictions depend on the details of the equations. The universality theory shows that qualitatively similar equations have the identical *quantitative* behavior. For example, a system of differential equations naturally determines certain maps. The computation of the actual analytic form of the map is generally well beyond present mathematical methods. However, should the map exhibit period doubling, then precise quantitative results are available from the universality theory because the theory applies independently of which map it happens to be. In particular, certain fluid flows have now been experimentally observed to become turbulent through period doubling (subharmonic bifurcations). From this one fact we know that the universality theory applies—and indeed correctly determines the precise way in which the flow becomes turbulent, without any reference to the underlying Navier-Stokes equations.

curves are coincident. Moreover, the scale reduction that f^2 will determine for f^4 is based solely on the functional composition, so that if these curves for f^{2^n}, $f^{2^{n+1}}$, *converge* (as they obviously do in Fig. 8), the scale reduction from level to level will *converge to a definite constant.* But the width of each circulation square is just the distance between $x = \frac{1}{2}$ when it is a fixed point of f^{2^n} and the fixed point of f^{2^n} next nearest to $x = \frac{1}{2}$ (Figs. 7a and b). That is, asymptotically, *the separation of adjacent elements of period-doubled attractors is reduced by a constant value from one doubling to the next.* Also from one doubling to the next, this next nearest element *alternates* from one side of $x = \frac{1}{2}$ to the other. Let d_n denote the algebraic distance from $x = \frac{1}{2}$ to the nearest element of the attractor cycle of period 2^n, in the 2^n-cycle at λ_n. A positive number α scales this distance down in the 2^{n+1}-cycle at λ_{n+1}:

$$\frac{d_n}{d_{n+1}} \sim -\alpha . \qquad (28)$$

But since rescaling is determined only by functional composition, there is some function that composed with itself will *reproduce* itself reduced in scale by $-\alpha$. The function has a quadratic maximum at $x = \frac{1}{2}$, is symmetric about $x = \frac{1}{2}$, and can be scaled by hand to equal 1 at $x = \frac{1}{2}$. Shifting coordinates so that $x = \frac{1}{2} \rightarrow x = 0$, we have

$$-\alpha g(g(x/\alpha)) = g(x) . \qquad (29)$$

Substituting $g(0) = 1$, we have

$$g(1) = -\frac{1}{\alpha} . \qquad (30)$$

Accordingly, Eq. (29) is a definite equation for a function g depending on x through x^2 and having a maximum of 1 at $x = 0$. There is a unique smooth solution to Eq. (29), which determines

$$\alpha = 2.502907875 \ldots . \qquad (31)$$