Knowing α, we can predict through Eq. (28) a definite scaling law binding on the iterates of any scheme possessing period doubling. The law has, indeed, been amply verified experimentally. By Eq. (29), we see that the relevant operation upon functions that underlies period doubling is functional composition followed by magnification, where the magnification is determined by the fixed-point condition of Eq. (29) with the function g the fixed point in this space of functions. However, Eq. (29) does not describe a stable fixed point because we have not incorporated in it the parameter increase from λ_n to λ_{n+1}. Thus, g is not the limiting function of the curves in the circulation squares, although it is intimately related to that function. The full theory is described in the next section. Here we merely state that we can determine the limiting function and thereby can *determine the location of the actual elements of limiting 2^n-cycles.* We also have established that g is an unstable fixed point of functional composition, where the rate of divergence away from g is precisely δ of Eq. (3) and so is computable. Accordingly, there is a full theory that determines, in a precise quantitative way, the aperiodic limit of functional iterations with an *unspecified* function f.

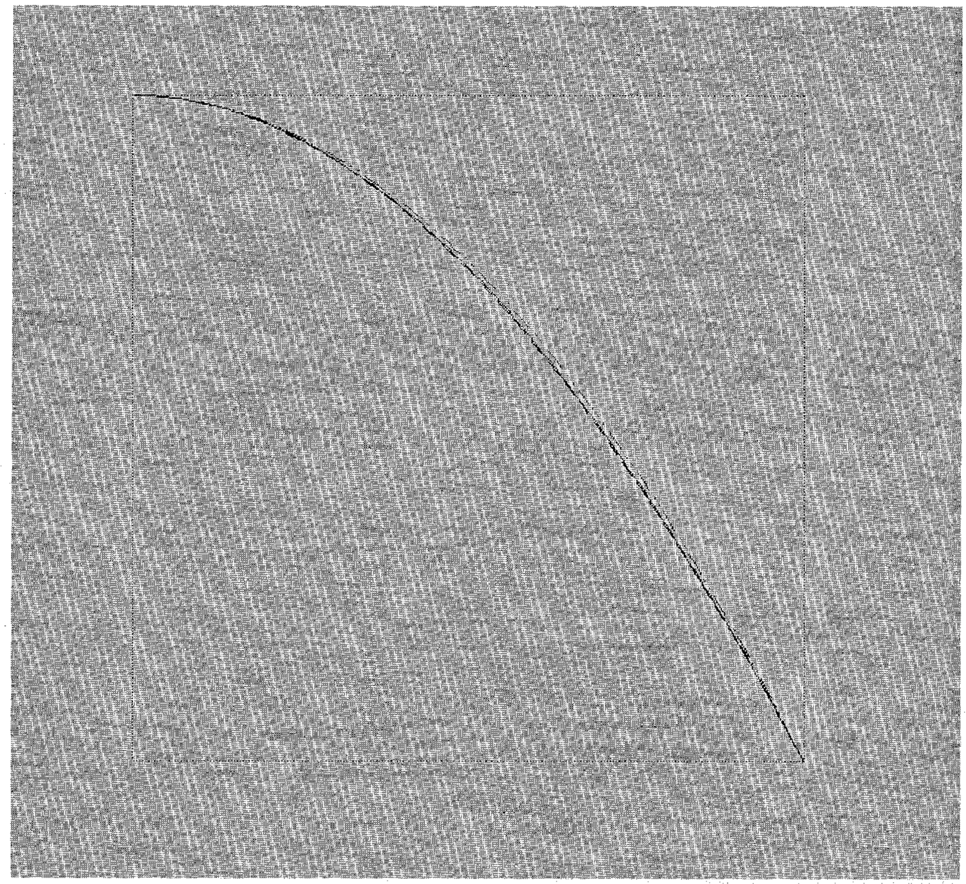

Fig. 8. The superposition of the suitably magnified dotted squares of $f^{2^{n-1}}$ at λ_n (as in Figs. 5a, 7a, ...).

Some Details of the Full Theory

Returning to Eq. (28), we are in a position to describe theoretically the universal scaling of high-order cycles and the convergence to a universal limit. Since d_n is the distance between $x = \frac{1}{2}$ and the element of the 2^n-cycle at λ_n nearest to $x = \frac{1}{2}$ and since this nearest element is the 2^{n-1} iterate of $x = \frac{1}{2}$ (which is true because these two points were coincident before the n^{th} period doubling began to split them apart), we have

$$d_n = f^{2^{n-1}}(\lambda_n, \tfrac{1}{2}) - \tfrac{1}{2}\,. \qquad (32)$$

For future work it is expedient to perform a coordinate translation that moves $x = \frac{1}{2}$ to $x = 0$. Thus, Eq. (32) becomes

$$d_n = f^{2^{n-1}}(\lambda_n, 0)\,. \qquad (33)$$

Equation (28) now determines that the rescaled distances,

$$r_n \equiv (-\alpha)^n\, d_{n+1}\,.$$

will converge to a definite finite value as $n \rightarrow \infty$. That is,

$$\lim_{n\rightarrow\infty} (-\alpha)^n f^{2^n}(\lambda_{n+1}, 0) \qquad (34)$$

must exist if Eq. (28) holds.

However, from Fig. 8 we know something stronger than Eq. (34). When the n^{th} iterated function is *magnified* by $(-\alpha)^n$, it converges to a definite function. Equation (34) is the value of this function at $x = 0$. After the magnification, the convergent functions are given by

$$(-\alpha)^n f^{2^n}(\lambda_{n+1}, x/(-\alpha)^n)\,.$$

Thus,

$$g_1(x) \equiv \lim_{n\rightarrow\infty} (-\alpha)^n f^{2^n}(\lambda_{n+1}, x/(-\alpha)^n) \qquad (35)$$

Fig. 9. The function g_1. The squares locate cycle elements.

is the limiting function inscribed in the square of Fig. 8. The function $g_1(x)$ is, by the argument of the restriction of f to increasingly small intervals about its maximum, the *universal* limit of all iterates of all f's with a quadratic extremum. Indeed, it is numerically easy to ascertain that g_1 of Eq. (35) is always the same function independent of the f in Eq. (32).

What is this universal function good for? Figure 5a shows a crude approximation of g_1 [n = 0 in the limit of Eq. (35)], while Fig. 7a shows a better approximation (n = 1). In fact, the extrema of g_1 near the fixed points of g_1 support circulation squares each of which contains two points of the cycle. (The two squares shown in Fig. 7a locate the four elements of the cycle.) That is, g_1 determines the location of elements of high-order 2^n-cycles near x = 0. Since g_1 is *universal*, we now have the amazing result that the location of the actual elements of highly doubled cycles is universal! The reader might guess this is a *very* powerful result. Figure 9 shows g_1 out to x sufficiently large to have 8 circulation squares, and hence locates the 15 elements of a 2^n-cycle nearest to x = 0. Also, the universal value of the scaling parameter α, obtained numerically, is

$$\alpha = 2.502907875 \ldots . \qquad (36)$$

Like δ, α is a number that can be *measured* [through an experiment that observes the d_n of Eq. (28)] in any phenomenon exhibiting period doubling.

If g_1 is universal, then of course its iterate g_1^2 also is universal. Figure 7b depicts an early approximation to this iterate. In fact, let us define a new universal function g_0, obtained by scaling g_1^2:

$$g_0(x) \equiv -\alpha g_1^2(-x/\alpha) . \qquad (37)$$

(Because g_1 is universal and the iterates of our quadratic function are all symmetric in x, both g_1 and g_0 are symmetric functions. Accordingly, the minus sign within g_1^2 can be dropped with impunity.) From Eq. (35), we now can write

$$g_0(x) = \lim_{n\to\infty} (-\alpha)^n f^{2^n}(\lambda_n, x/(-\alpha)^n) . \qquad (38)$$

[We introduced the scaling of Eq. (37) to provide one power of α per period doubling, since each successive iterate of f^{2^n}

reduces the scale by α].

In fact, we can generalize Eqs. (35) and (38) to a *family* of universal functions g_r:

$$g_r(x) = \lim_{n\to\infty}(-\alpha)^n f^{2^n}(\lambda_{n+r}, x/(-\alpha)^n) . \quad (39)$$

To understand this, observe that g_0 locates the cycle elements as the fixed points of g_0 at extrema; g_1 locates the same elements by determining two elements per extremum. Similarly, g_r determines 2^r elements about each extremum near a fixed point of g_r. Since each f^{2^n} is always magnified by $(-\alpha)^n$ for each r, the scales of all g_r are the same. Indeed, g_r for $r > 1$ looks like g_1 of Fig. 9, except that each extremum is slightly higher, to accommodate a 2^r-cycle. Since each extremum must grow by convergently small amounts to accommodate higher and higher 2^r-cycles, we are led to conclude that

$$g(x) = \lim_{r\to\infty} g_r(x) \quad (40)$$

must exist. By Eq. (39),

$$g(x) = \lim_{n\to\infty}(-\alpha)^n f^{2^n}(\lambda_\infty, x/(-\alpha)^n) . \quad (41)$$

Unlike the functions g_r, g(x) is obtained as a limit of f^{2^n}'s at a *fixed value* of λ. Indeed, this is the special significance of λ_∞; it is an isolated value of λ at which repeated iteration and magnification lead to a convergent function.

We now can write the equation that g satisfies. Analogously to Eq. (37), it is easy to verify that all g_r are related by

$$g_{r-1}(x) = -\alpha g_r(g_r(-x/\alpha)) . \quad (42)$$

By Eq. (40), it follows that g satisfies

$$g(x) = -\alpha g(g(x/\alpha)) . \quad (43)$$

The reader can verify that Eq. (43) is invariant under a magnification of g. Thus, the theory has nothing to say about absolute scales. Accordingly, we must fix this by hand by setting

$$g(0) = 1 . \quad (44)$$

Also, we must specify the nature of the maximum of g at x = 0 (for example, quadratic). Finally, since g is to be built by iterating $a - x^2$, it must be both smooth and a function of x through x^2. With these specifications, Eq. (43) has a *unique* solution. By Eqs. (44) and (43),

$$g(0) = 1 = -\alpha g(g(0)) = -\alpha g(1) ,$$

so that

$$\alpha = -1/g(1) . \quad (45)$$

Accordingly, Eq. (43) determines α together with g.

Let us comment on the nature of Eq. (43), a so-called functional equation. Because g is smooth, if we know its value at a finite number of points, we know its value to some approximation on the interval containing these points by any sufficiently smooth interpolation. Thus, to some degree of accuracy, Eq. (43) can be replaced by a finite coupled system of nonlinear equations. Exactly then, Eq. (43) is an infinite-dimensional, nonlinear vector equation. Accordingly, we have obtained the solution to one-dimensional period doubling through our infinite-dimensional, explicitly universal problem. Equation (43) must be infinite-dimensional because it must keep track of the infinite number of cycle elements demanded of any attempt to solve the period-doubling problem. Rigorous mathematics for equations like Eq. (43) is just beyond the boundary of present mathematical knowledge.

At this point, we must determine two items. First, where is δ? Second, how do we obtain g_1, the real function of interest for locating cycle elements? The two problems are part of one question. Equation (42) is itself an iteration scheme. However, unlike the elements in Eq. (4), the elements acted on in Eq. (42) are *functions*. The analogue of the function of f in Eq. (4) is the operation in function space of functional composition followed by a magnification. If we call this operation T, and an element of the function space ψ, Eq. (42) gives

$$T[\psi](x) = -\alpha\psi^2(-x/\alpha) . \quad (46)$$

In terms of T, Eq. (42) now reads

$$g_{r-1} = T[g_r] , \quad (47)$$

and Eq. (43) reads

$$g = T[g] . \quad (48)$$

Thus, g is precisely the fixed point of T. Since g is the limit of the sequence g_r, we can obtain g_r for large r by linearizing T about its fixed point g. Once we have g_r in the linear regime, the exact repeated application of T by Eq. (47) will provide g_1. Thus, we must investigate the stability of T at the fixed point g. However, it is obvious that T is *unstable* at g: for a large enough r, g_r is a point arbitrarily close to the fixed point g; by Eq. (47), successive iterates of g_r under T move away from g. How unstable is T? Consider a one-parameter family of functions f_λ, which means a "line" in the function space. For each f, there is an isolated parameter value λ_∞, for which repeated applications of T lead to convergence towards g [Eq. (41)]. Now, the function space can be "packed" with all the lines corresponding to the various f's. The set of all the points on these lines specified by the respective λ_∞'s determines a "surface" having the property that repeated applications of T to any point on it will converge to g. This is the surface of stability of T (the "stable manifold" of T through g). But through each point of this surface issues out the corresponding line, which is one-dimensional since it is parametrized by a single parameter, λ. Accordingly, T is *unstable* in only *one* direction in function space. Linearized about g, this line of instability can be written as the one-parameter family

$$f_\lambda(x) = g(x) - \lambda\, h(x)\,, \tag{49}$$

which passes through g (at $\lambda = 0$) and deviates from g along the unique direction h. But f_λ is just one of our transformations [Eq. (4)]! Thus, as we vary λ, f_λ will undergo period doubling, doubling to a 2^n-cycle at Λ_n. By Eq. (41), λ_∞ for the family of functions f_λ in Eq. (49) is

$$\lambda_\infty = 0\,. \tag{50}$$

Thus, by Eq. (1)

$$\lambda_n \sim \delta^{-n}\,. \tag{51}$$

Since applications of T by Eq. (47) iterate in the opposite direction (diverge away from g), it now follows that the rate of instability of T along h must be precisely δ.

Accordingly, we find δ and g_1 in the following way. First, we must linearize the operation T about its fixed point g. Next, we must determine the stability directions of the linearized operator. Moreover, we expect there to be precisely one direction of instability. Indeed, it turns out that infinitesimal deformations (conjugacies) of g determine *stable* directions, while a unique unstable direction, h, emerges with a stability rate (eigenvalue) precisely the δ of Eq. (3). Equation (49) at λ_r is precisely g_r for asymptotically large r. Thus g_r is known asymptotically, so that we have entered the sequence g_r and can now, by repeated use of Eq. (47), step down to g_1. All the ingredients of a full description of high-order 2^n-cycles now are at hand and evidently are universal.

Although we have said that the function g_1 universally locates cycle elements near $x = 0$, we must understand that it doesn't locate all cycle elements. This is possible because a finite distance of the scale of g_1 (for example, the location of the element nearest to $x = 0$) has been magnified by α^n for n diverging. Indeed, the distances from $x = 0$ of all elements of a 2^n-cycle, "accurately" located by g_1, are reduced by $-\alpha$ in the 2^{n+1}-cycle. However, it is obvious that some elements have no such scaling: because $f(0) = a_n$ in Eq. (13), and $a_n \to a_\infty$, which is a definite nonzero number, the distance from the origin of the element of the 2^n-cycle farthest to the right certainly has not been reduced by $-\alpha$ at each period doubling. This suggests that we must measure locations of elements on the far right with respect to the farthest right point. If we do this, we can see that these distances scale by α^2, since they are the images through the quadractic maximum of f at $x = 0$ of elements close to $x = 0$ scaling with $-\alpha$. In fact, if we image g_1 through the maximum of f (through a quadratic conjugacy), then we shall indeed obtain a new universal function that locates cycle elements near the right-most element. The correct description of a highly doubled cycle now emerges as one of universal local clusters.

We can state the scope of universality for the location of cycle elements precisely. Since $f(\lambda_1, x)$ exactly locates the two elements of the 2^1-cycle, and since $f(\lambda_1, x)$ is an approximation to g_1 [$n = 0$ in Eq. (35)], we evidently can locate both points exactly by appropriately sealing g_1. Next, near $x = 0$, $f^2(\lambda_2, x)$ is a better approximation to g_1 (suitably scaled). However, in general, the more accurately we scale g_1 to determine the smallest 2-cycle elements, the greater is the error in its determination of the right-most elements. Again, near $x = 0$, $f^4(\lambda_3, x)$ is a still better approximation to g_1. Indeed, the suitably scaled g_1 now can determine several points about $x = 0$ accurately, but determination of the right-most elements is still worse. In this fashion, it follows that g_1, suitably scaled, can determine 2^r points of the 2^n-cycle near $x = 0$ for $r \ll n$. If we focus on the neighborhood of one of these 2^r points at some definite distance from $x = 0$, then by Eq. (35) the larger the n, the larger the *scaled* distance of this region from $x = 0$, and so, the poorer the approximation of the location of fixed points in it by g_1. However, just as we can construct the version of g_1 that applies at the right-most cycle element, we also can construct the version of g_1 that applies at this chosen neighborhood. Accordingly, the universal description is set through an acceptable tolerance: if we "measure" f^{2^n} at some definite n, then we can use the actual location of the elements as foci for 2^n versions of g_1, each applicable at one such point. For all further period doubling, we determine the new cycle elements through the g_1's. In summary, the *more accurately we care to know the locations* of arbitrarily high-order cycle elements, the *more parameters we must measure* (namely, the cycle elements at some chosen order of period doubling). This is the sense in which the universality theory is asymptotic. Its ability to have serious predictive power is the fortunate consequence of the high convergence rate $\delta(\sim 4.67)$. Thus, typically after the first two or three period doublings, this asymptotic theory is already accurate to within several percent. If a period-doubling system is *measured* in its 4- or 8-cycle, its behavior throughout and symmetrically beyond the period-doubling regime also is determined to within a few percent.

To make precise dynamical predictions, we do not have to construct all the local versions of g_1; all we really need to know is the local *scaling* everywhere along the attractor. The scaling is $-\alpha$ at $x = 0$ and α^2 at the right-most element. But what is it at an arbitrary point? We can determine the scaling law if we order elements not by their location on the x-axis, but rather by their order as iterates of $x = 0$. Because the time sequence in which a process evolves is precisely this ordering, the result will be of immediate and powerful predictive value. It is precisely this scaling law that allows us to compute the spectrum of the onset of turbulence in period-doubling systems.

What must we compute? First, just as the element in the 2^n-cycle nearest to $x = 0$ is the element halfway around the cycle from $x = 0$, the element nearest to an arbitrarily chosen element is precisely the one halfway around the cycle from it. Let us denote by $d_n(m)$ the distance between the m^{th} cycle element (x_m) and the element nearest to it in a 2^n-cycle. [The d_n of Eq. (28) is $d_n(0)$]. As just explained,

$$d_n(m) = x_m - f^{2^{n-1}}(\lambda_n, x_m) . \qquad (52)$$

However, x_m is the m^{th} iterate of $x_0 = 0$. Recalling from Eq. (6) that powers commute, we find

$$d_n(m) = f^m(\lambda_n, 0) - f^m(\lambda_n, f^{2^{n-1}}(\lambda_n, 0)) . \qquad (53)$$

Let us, for the moment, specialize to m of the form 2^{n-r}, in which case

$$\begin{aligned} d_n(2^{n-r}) &= f^{2^{n-r}}(\lambda_n, 0) - f^{2^{n-r}}(\lambda_n, f^{2^{n-1}}(\lambda_n, 0)) \\ &= f^{2^{n-r}}(\lambda_{(n-r)+r}, 0) - f^{2^{n-r}}(\lambda_{(n-r)+r}, f^{2^{n-1}}(\lambda_n, 0)) . \end{aligned} \qquad (54)$$

For $r \ll n$ (which can still allow $r \gg 1$ for n large), we have, by Eq. (39),

$$d_n(2^{n-r}) \sim (-\alpha)^{-(n-r)}[g_r(0) - g_r((-\alpha)^{n-r} f^{2^{n-1}}(\lambda_n, 0))]$$

or

$$d_n(2^{n-r}) \sim (-\alpha)^{-(n-r)}[g_r(0) - g_r((-\alpha)^{-r+1} g_1(0))] . \qquad (55)$$

The object we want to determine is the local scaling at the m^{th} element, that is, the ratio of nearest separations at the m^{th} iterate of $x = 0$, at successive values of n. That is, if the scaling is called σ,

$$\sigma_n(m) \equiv \frac{d_{n+1}(m)}{d_n(m)} . \qquad (56)$$

[Observe by Eq. (28), the definition of α, that $\sigma_n(0) \sim (-\alpha)^{-1}$.] Specializing again to $m = 2^{n-r}$, where $r \ll n$, we have by Eq. (55)

$$\sigma(2^{n-r}) \sim \frac{g_{r+1}(0) - g_{r+1}((-\alpha)^{-r} g_1(0))}{g_r(0) - g_r((-\alpha)^{-r+1} g_1(0))} . \qquad (57)$$

Finally, let us rescale the axis of iterates so that all 2^{n+1} iterates are within a unit interval. Labelling this axis by t, the value of t of the m^{th} element in a 2^n-cycle is

$$t_n(m) = m/2^n . \qquad (58)$$

In particular, we have

$$t_n(2^{n-r}) = 2^{-r} . \qquad (59)$$

Defining σ along the t-axis naturally as

$$\sigma(t_n(m)) \sim \sigma_n(m) \qquad (\text{as } n \to \infty) ,$$

we have by Eqs. (57) and (59),

$$\sigma(2^{-r-1}) = \frac{g_{r+1}(0) - g_{r+1}((-\alpha)^{-r} g_1(0))}{g_r(0) - g_r((-\alpha)^{-r+1} g_1(0))} . \qquad (60)$$

It is not much more difficult to obtain σ for all t. This is done first for rational t by writing t in its binary expansion:

$$t_{r_1 r_2 r_3 \ldots} = 2^{-r_1} + 2^{-r_2} + \ldots .$$

In the 2^n-cycle approximation we require σ_n at the $2^{n-r_1} + 2^{n-r_2} + \ldots$ iterate of the origin. But, by Eq. (8),

$$f^{2^{n-r_1} + 2^{n-r_2} + \cdots} = f^{2^{n-r_1}} \circ f^{2^{n-r_2}} \circ \ldots .$$

It follows by manipulations identical to those that led from Eq. (54) to Eq. (60) that σ at such values of t is obtained by replacing the individual g_r terms in Eq. (60) by appropriate iterates of various g_r's.

There is one last ingredient to the computation of σ. We know that $\sigma(0) = -\alpha^{-1}$. We also know that $\sigma_n(1) \sim \alpha^{-2}$. But, by Eq. (59),

$$t_n(1) = 2^{-n} \to 0 .$$

Thus σ is discontinuous at $t = 0$, with $\sigma(0 - \varepsilon) = -\alpha^{-1}$ and $\sigma(0 + \varepsilon) = \alpha^{-2}(\varepsilon \to 0^+)$. Indeed, since $x_{2^{n-r}}$ is always very close to the origin, each of these points is imaged quadratically. Thus Eq. (60) actually determines $\sigma(2^{-r-1} - \varepsilon)$, while $\sigma(2^{-r-1} + \varepsilon)$ is obtained by replacing each numerator and denominator g_r by its square. The same replacement also is correct for each multi-g_r term that figures into σ at the binary expanded rationals.

Altogether, we have the following results. $\sigma(t)$ can be computed for all t, and it is *universal* since its explicit computation depends only upon the universal functions g_r. σ is *discontinuous* at all the rationals. However, it can be established that the *larger* the number of terms in the binary expansion of a rational t, the smaller the discontinuity of σ. Lastly, as a finite number of iterates leaves t unchanged as $n \to \infty$, σ must be *continuous* except at the rationals. Figure 10 depicts $1/\sigma(t)$. Despite the pathological nature of σ, the reader will observe that basically it is constant half the time at α^{-1} and half the time at α^{-2} for $0 < t < \frac{1}{2}$. In a succeeding approximation, it can be decomposed in each half into two slightly different quarters,

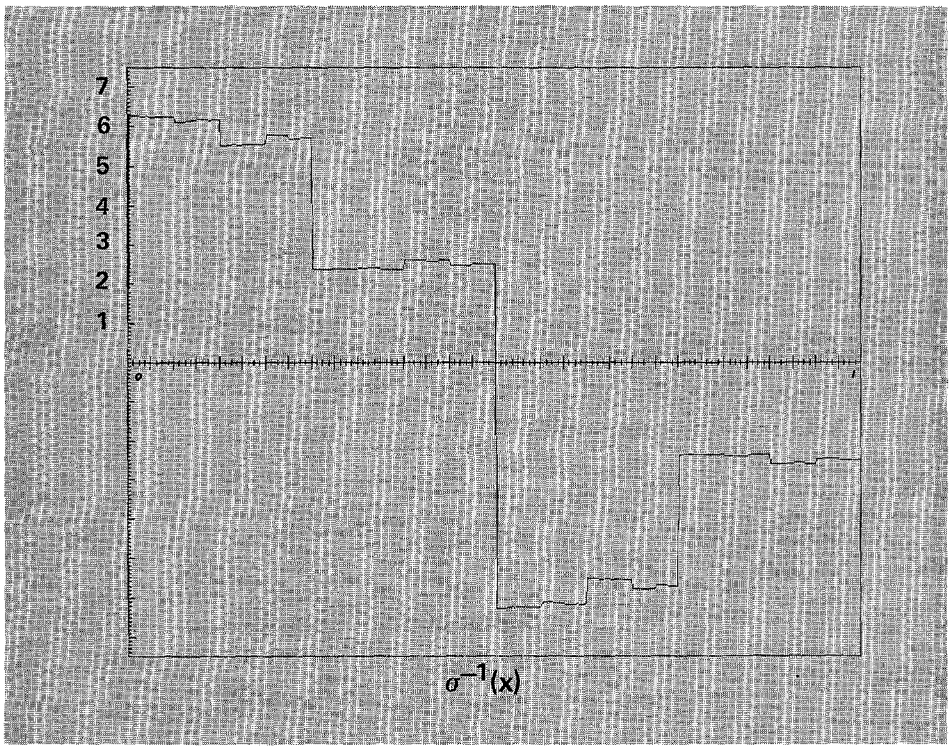

Fig. 10. The trajectory scaling function. Observe that $\sigma(x + 1/2) = -\sigma(x)$.

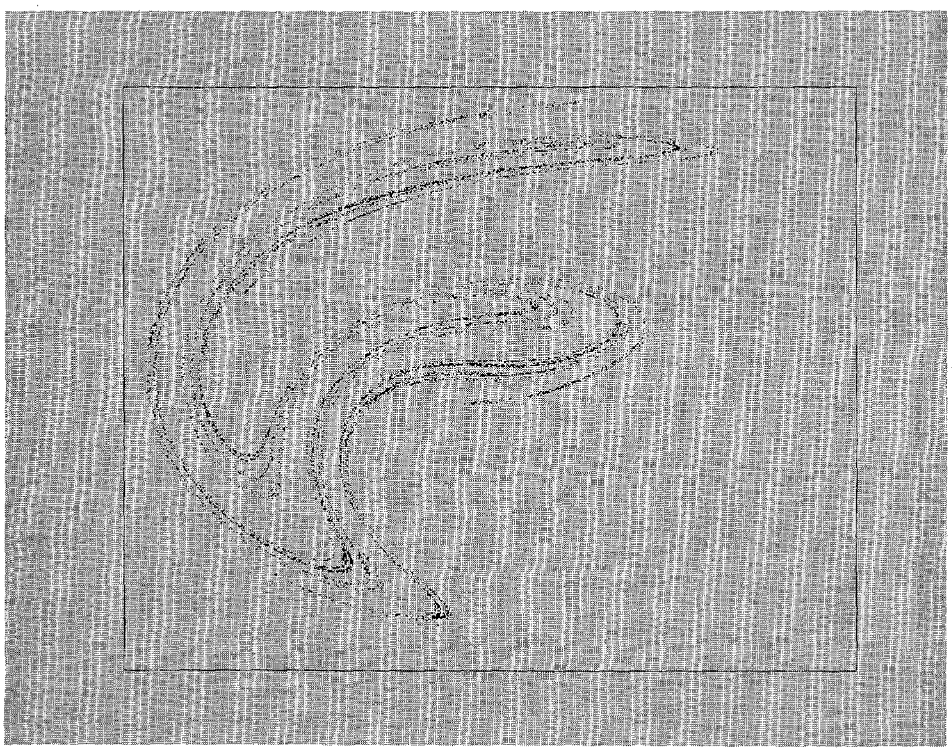

Fig. 11. The plotted points lie on the "strange attractor" of Duffing's equation.

and so forth. [It is easy to verify from Eq. (52) that σ is periodic in t of period 1, and has the symmetry

$$\sigma(t + \tfrac{1}{2}) = -\sigma(t) .$$

Accordingly, we have paid attention to its first half $0 < t < \tfrac{1}{2}$.] With σ we are at last finished with one-dimensional iterates per se.

Universal Behavior in Higher Dimensional Systems

So far we have discussed iteration in *one* variable; Eq. (15) is the prototype. Equation (14), an example of iteration in two dimensions, has the special property of preserving areas. A generalization of Eq. (14),

$$x_{n+1} = y_n - x_n^2$$

and

$$y_{n+1} = a + bx_n \qquad (61)$$

with $|b| < 1$, contracts areas. Equation (61) is interesting because it possesses a so-called *strange attractor*. This means an attractor (as before) constructed by folding a curve repeatedly upon itself (Fig. 11) with the consequent property that two initial points very near to one another are, in fact, very far from each other when the distance is measured along the folded attractor, which is the path they follow upon iteration. This means that after some iteration, they will soon be far apart in actual distance as well as when measured along the attractor. This general mechanism gives a system highly sensitive dependence upon its initial conditions and a truly statistical character: since very small differences in initial conditions are magnified quickly, unless the initial conditions are known to *infinite precision*, all known knowledge is eroded rapidly to future ignorance. Now, Eq. (61) enters