into the early stages of statistical behavior through period doubling. Moreover, δ of Eq. (3) is *again* the rate of onset of complexity, and α of Eq. (31) is again the rate at which the spacing of adjacent attractor points is vanishing. Indeed, the one-dimensional theory determines all behavior of Eq. (61) in the onset regime.

In fact, dimensionality is irrelevant. The same theory, the same numbers, etc. also work for iterations in N dimensions, provided that the system goes through period doubling. The basic process, wherever period doubling occurs *ad infinitum*, is functional composition from one level to the next. Accordingly, a modification of Eq. (29) is at the heart of the process, with composition on functions from N dimensions to N dimensions. Should the specific iteration function contract N-dimensional volumes (a dissipative process), then in general there is one direction of slowest contraction, so that after a number of iterations the process is effectively one-dimensional. Put differently, the one-dimensional solution to Eq. (29) is always a solution to its N-dimensional analogue. It is the relevant fixed point of the analogue if the iteration function is contractive.

Universal Behavior in Differential Systems

The next step of generalization is to include systems of differential equations. A prototypic equation is Duffing's oscillator, a driven damped anharmonic oscillator,

$$\ddot{x} + k\dot{x} + x^3 = b \sin 2\pi t \,. \qquad (62)$$

The periodic drive of period 1 determines a natural time step. Figure 12a depicts a period 1 attractor, usually referred to as a *limit cycle*. It is an attractor because, for a range of initial conditions, the solution to Eq. (62) settles down to the cycle. It is period 1 because it repeats the same curve in every period of the drive.

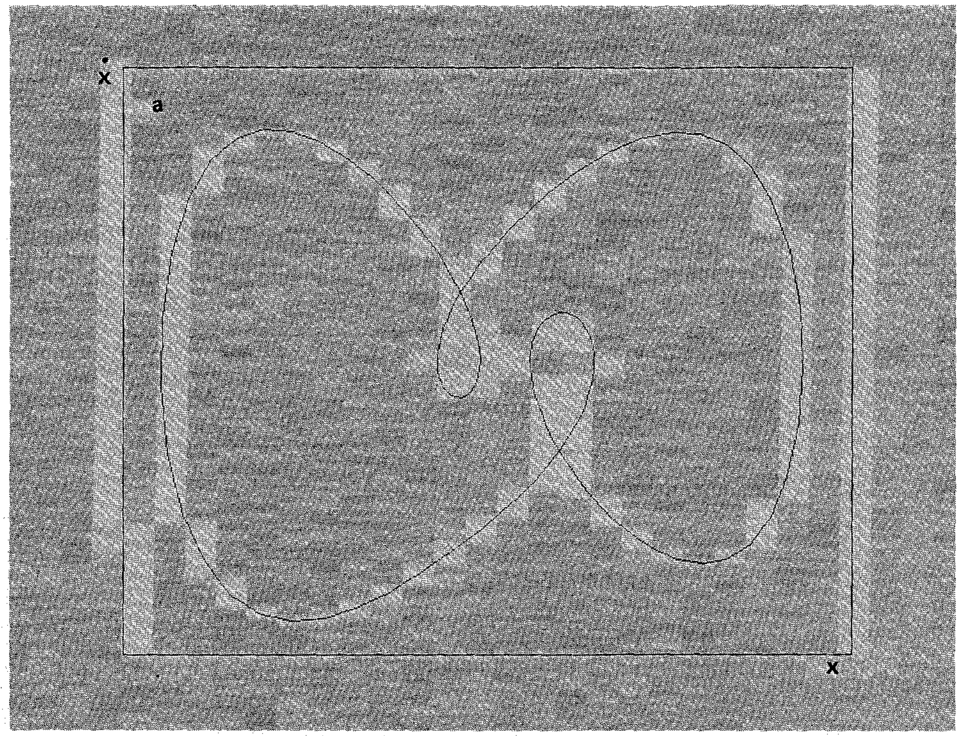

Fig. 12a. The most stable 1-cycle of Duffing's equation in phase space (x,ẋ).

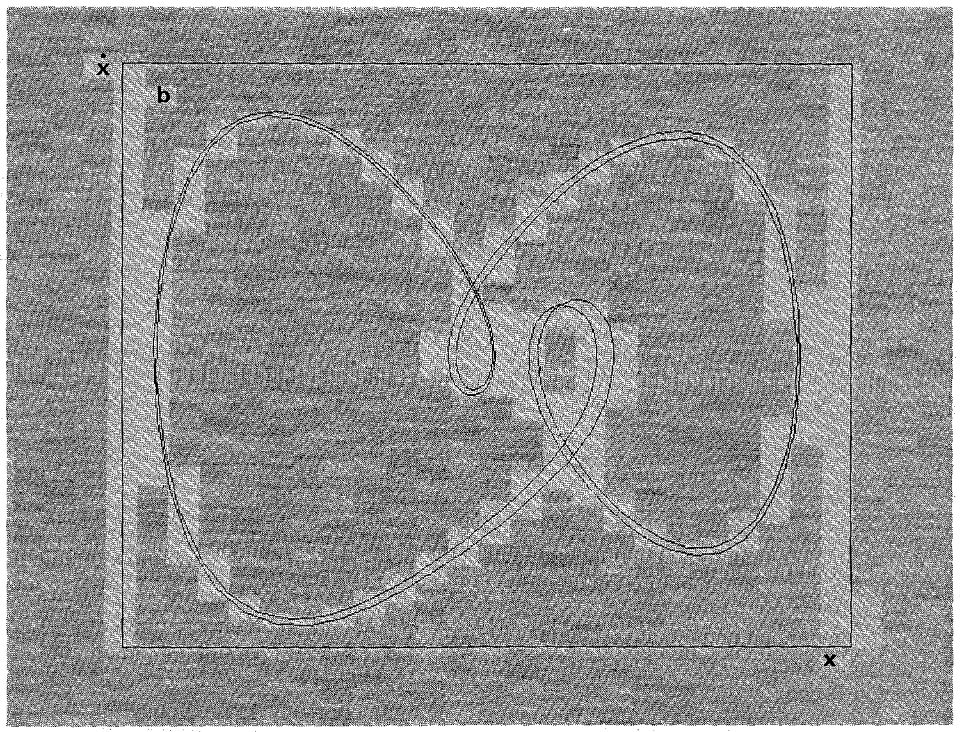

Fig. 12b. The most stable 2-cycle of Duffing's equation. Observe that it is two displaced copies of Fig. 12a.

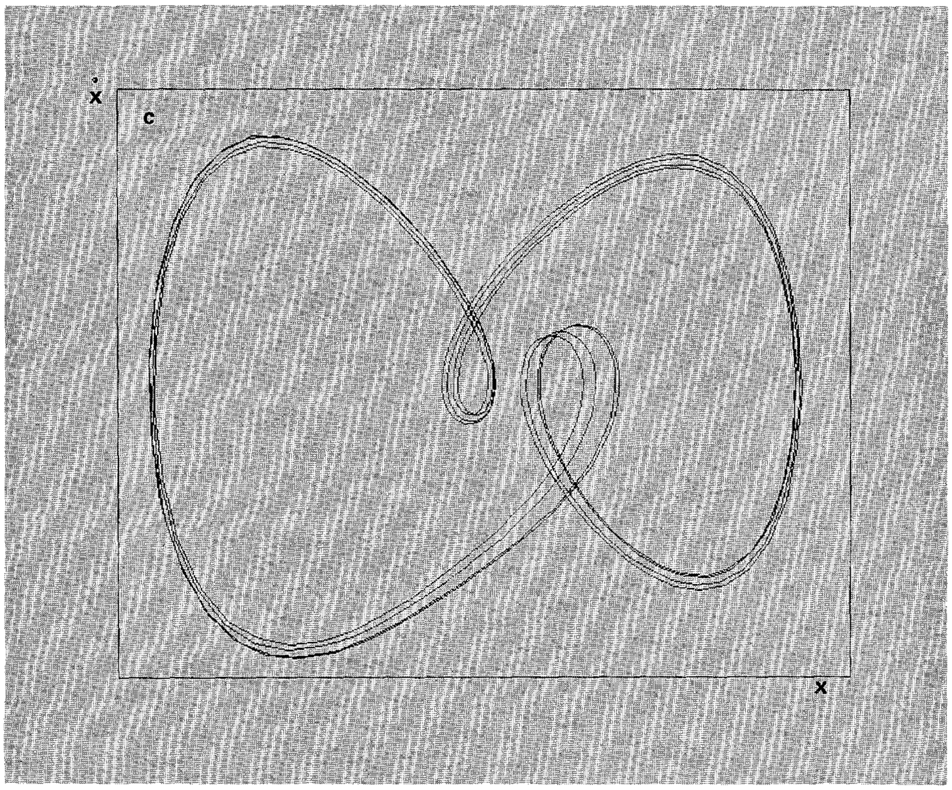

Fig. 12c. The most stable 4-cycle of Duffing's equation. Observe that the displaced copies of Fig. 12b have either a broad or a narrow separation.

Figures 12b and c depict attractors of periods 2 and 4 as the friction or damping constant k in Eq. (62) is reduced systematically. The parameter values $k = \lambda_0, \lambda_1, \lambda_2, ...,$ are the damping constants corresponding to the most stable 2^n-cycle in analogy to the λ_n of the one-dimensional functional iteration. Indeed, this oscillator's period doubles (at least numerically!) *ad infinitum.* In fact, by $k = \lambda_5$, the δ_3 of Eq. (2) has converged to 4.69. Why is this? Instead of considering the entire trajectories as shown in Fig. 12, let us consider only where the trajectory point is located every 1 period of the drive. The 1-cycle then produces only one point, while the 2-cycle produces a pair of points, and so forth. This *time-one map* [if the trajectory point is $(x,\dot{x})$ now, where is it one period later?] is by virtue of the differential equation a smooth and invertible function in two dimensions. Qualitatively, it looks like the map of Eq. (61). In the present state of mathematics, little can be said about the analytic behavior of time-one maps; however, since our theory is universal, it makes no difference that we don't know the explicit form. We still can determine the complete quantitative behavior of Eq. (62) in the onset regime where the motion tends to aperiodicity. If we already know, by measurement, the precise form of the trajectory after a few period doublings, we can compute the form of the trajectory as the friction is reduced throughout the region of onset of complexity by carefully using the full power of the universality theory to determine the spacings of elements of a cycle.

Let us see how this works in some detail. Consider the time-one map of the Duffing's oscillator in the superstable 2^n-cycle. In particular, let us focus on an element at which the scaling function σ (Fig. 10) has the value σ_0, and for which the next iterate of this element also has the scaling σ_0. (The element is not at a big discontinuity of σ.) It is then intuitive that if we had taken our time-one examination of the trajectory at values of time displaced from our first choice, we would have seen the same scaling σ_0 for this part of the trajectory. That is, the differential equations will extend the map-scaling function continuously to a function along the entire trajectory so that, if two successive time-one elements have scaling σ_0, then the entire stretch of trajectory over this unit time interval has scaling σ_0. In the last section, we were motivated to construct σ as a function of t along an interval precisely towards this end.

To implement this idea, the first step is to define the analogue of d_n. We require the spacing between the trajectory at time t and at time $T_n/2$ where the period of the system in the 2^n-cycle is

$$T_n \cong 2^n T_0 \, . \tag{63}$$

That is, we define

$$d_n(t) \equiv x_n(t) - x_n(t + T_n/2) \, . \tag{64}$$

(There is a d for each of the N variables for a system of N differential equations.) Since σ was defined as periodic of period 1, we now have

$$d_{n+1}(t) \sim \sigma(t/T_{n+1}) d_n(t) \, . \tag{65}$$

The content of Eq. (65), based on the n-dependence arising solely through the T_n in σ, and not on the detailed form of σ, already implies a strong scaling prediction, in that the ratio

$$\frac{d_{n+1}(t)}{d_n(t)} \, ,$$

when plotted with t scaled so that $T_n =$

1, is a function *independent* of n. Thus if Eq. (65) is true for *some* σ, whatever it might be, then knowing $x_n(t)$, we can compute $d_n(t)$ and from Eq. (65) $d_{n+1}(t)$. As a consequence of periodicity, Eq. (64) for $n \rightarrow n + 1$ can be solved for $x_{n+1}(t)$ (through a Fourier transform). That is, if we have measured any chosen coordinate of the system in its 2^n-cycle, we can compute its time dependence in the 2^{n+1}-cycle. Because this procedure is recursive, we can compute the coordinate's evolution for all higher cycles through the infinite period-doubling limit. If Eq. (65) is true and σ not known, then by measurement at a 2^n-cycle and at a 2^{n+1}-cycle, σ could be *constructed* from Eq. (65), and hence all higher order doublings would again be determined. Accordingly, Eq. (65) is a very powerful result. However, we know much more. The universality theory tells us that period doubling is universal and that there is a *unique* function σ which, indeed, we have computed in the previous section. Accordingly, by *measuring* x(t) in some chosen 2^n-cycle (the higher the n, the more the number of effective parameters to be determined empirically, and the more precise are the predictions), we now can compute the entire evolution of the system on its route to turbulence.

How well does this work? The empirically determined σ [for Eq. (62)] of Eq. (65) is shown for n = 3 in Fig. 13a and n = 4 in Fig. 13b. The figures were constructed by plotting the ratios of d_{n+1} and d_n scaled respective to T = 16 in Fig. 13a and T = 32 in Fig. 13b. Evidently the scaling law Eq. (65) is being obeyed. Moreover, on the same graph Fig. 14 shows the empirical σ for n = 4 and the recursion theoretical σ of Fig. 10. The reader should observe the detail-by-detail agreement of the two. In fact, if we use Eq. (65) and the theoretical σ with n = 2 as empirical input, the n = 5 frequency spectrum agrees with the empirical n = 5 spectrum to

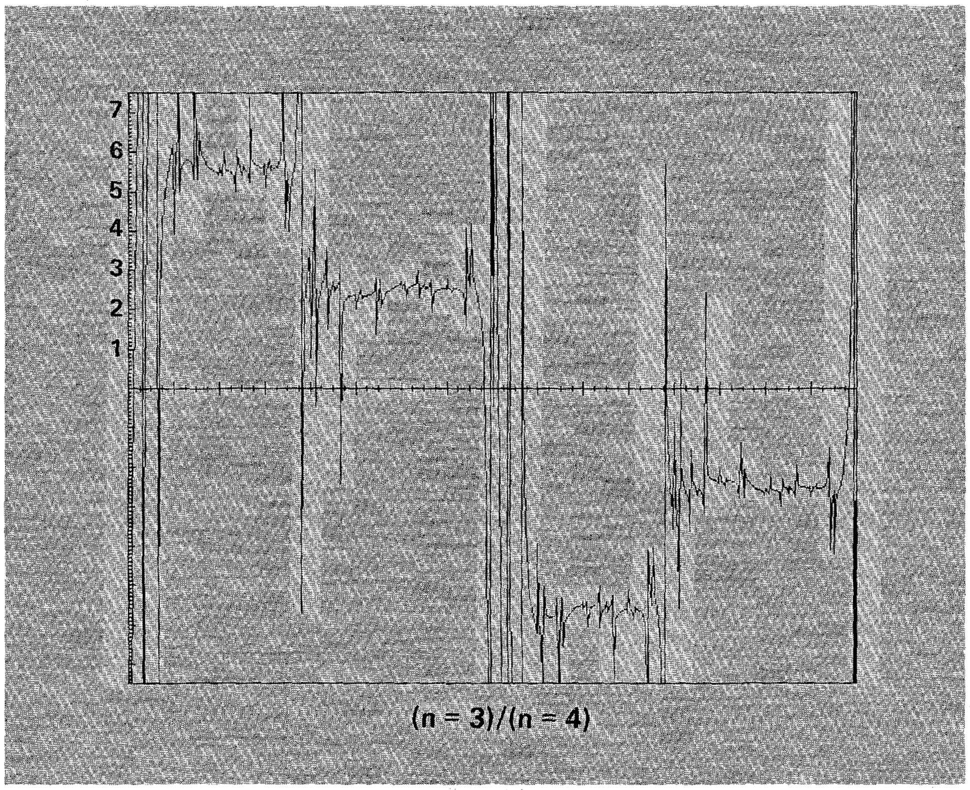

Fig. 13a. The ratio of nearest copy separations in the 8-cycle and 16-cycle for Duffing's equation.

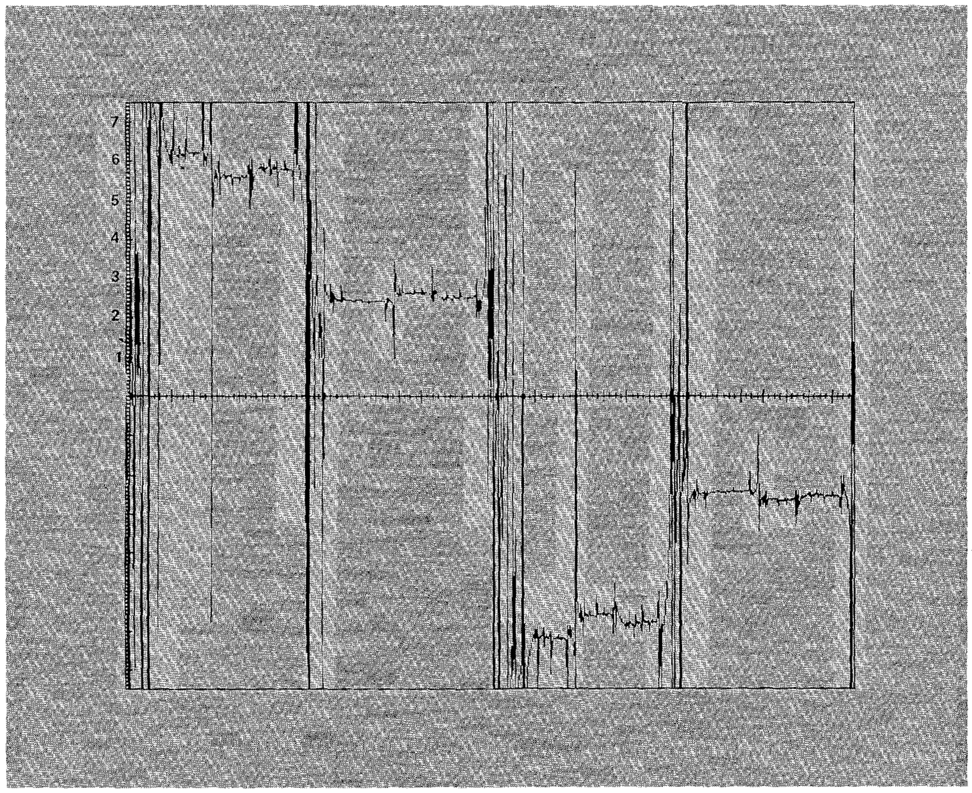

Fig. 13b. The same quantity as in Fig. 13a, but for the 16-cycle and 32-cycle. Here, the time axis is twice as compressed.

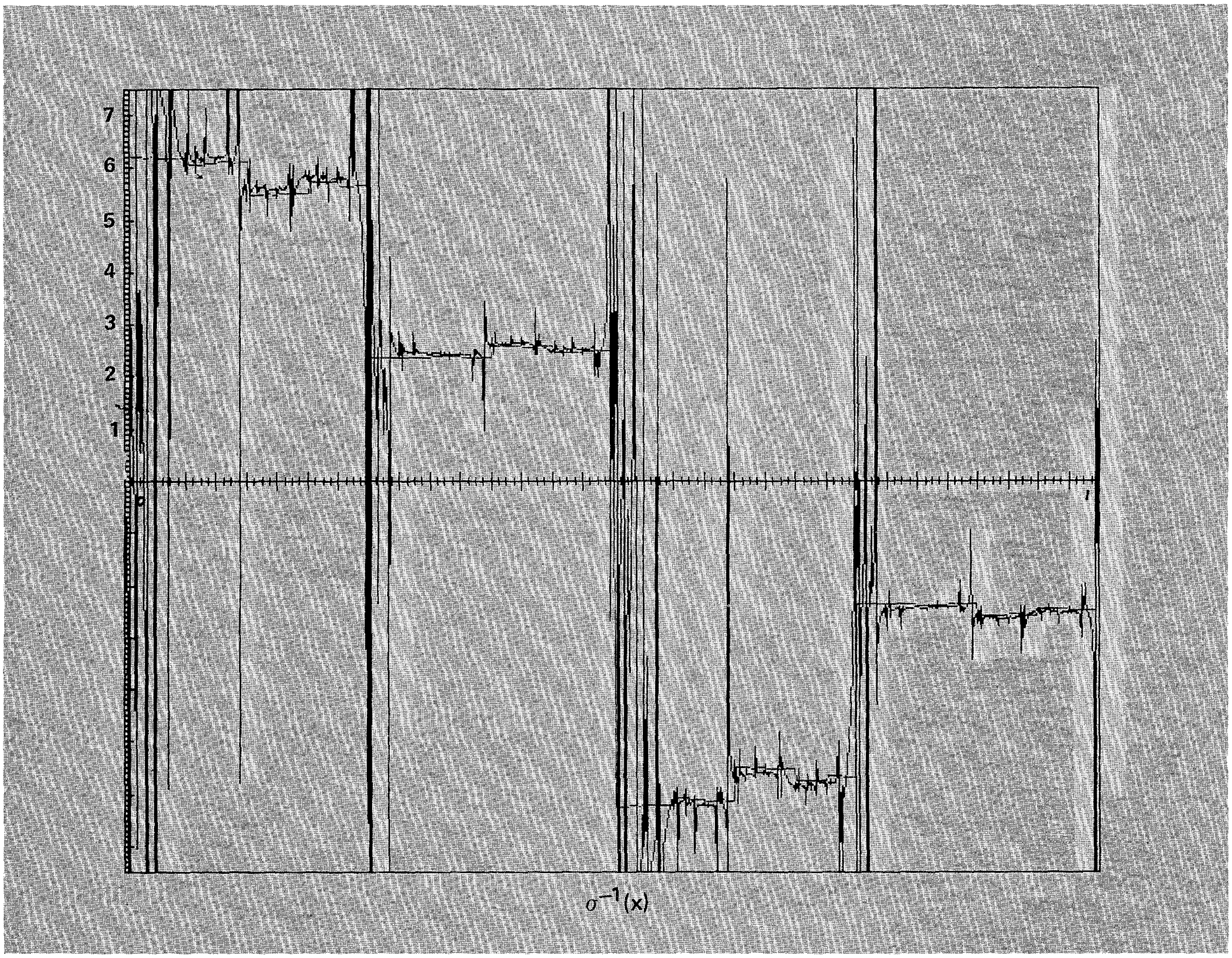

Fig. 14. Figure 13b overlayed with Fig. 10 compares the universal scaling function σ with the empirically determined scaling of nearest copy separations from the 16-cycle to the 32-cycle for Duffing's equation.

within 10%. (The n = 4 determines n = 5 to within 1%.) Thus the asymptotic universality theory is correct *and* is already well obeyed, even by n = 2!

Equations (64) and (65) are solved, as mentioned above, through Fourier transforming. The result is a recursive scheme that determines the Fourier coefficients of $x_{n+1}(t)$ in terms of those of $x_n(t)$ and the Fourier transform of the (known) function $\sigma(t)$. To employ the formula accurately requires knowledge of the entire spectrum of x_n (amplitude *and* phase) to determine each coefficient of x_{n+1}. However, the formula enjoys an approximate local prediction, which roughly determines the amplitude of a coefficient of x_{n+1} in terms of the amplitudes (alone) of x_n near the desired frequency of x_{n+1}.

What does the spectrum of a period-doubling system look like? Each time the period doubles, the fundamental frequency halves; period doubling in the continuum version is termed half-subharmonic bifurcation, a typical behavior of coupled nonlinear differential equations. Since the motion *almost* reproduces itself every period of the drive, the amplitude at this original frequency is high. At the first subharmonic halving, spectral components of the odd halves of the drive frequency come in. On the route to aperiodicity they saturate at a certain amplitude. Since the motion more nearly reproduces itself every two periods of drive, the next saturated subharmonics, at the odd fourths of the original frequency, are smaller still than the first ones, and so on, as each set of odd 2^nths comes into being. A crude approximate prediction of the theory is that whatever the system, the saturated amplitudes of each set of successively lower half-frequencies define a smooth interpolation located 8.2 dB *below* the smooth interpolation of the previous half-frequencies. [This is shown in Fig. 15 for Eq. (62).] After subharmonic bifurcations *ad infinitum*, the system is now no longer periodic; it has developed a continuous broad spectrum down to zero frequency with a definite internal distribution of the energy. That is, the system emerges from this process having developed the beginnings of broad-band noise of a determined nature. This process also occurs in the onset of turbulence in a fluid.

The Onset of Turbulence

The existing idea of the route to turbulence is Landau's 1941 theory. The idea is that a system becomes turbulent through a succession of instabilities, where each instability creates a new degree of freedom (through an indeterminate phase) of a time-periodic nature with the frequencies successively higher and incommensurate (*not* harmonics); because the resulting motion is the superposition of these modes, it is quasi-periodic.

In fact, it is experimentally clear that quasi-periodicity is incorrect. Rather, to produce the observed noise of rapidly decaying correlation the spectrum must become *continuous* (broad-band noise) down to zero frequency. The defect can be eliminated through the production of successive half-subharmonics, which then emerge as an allowable route to turbulence. If the general idea of a succession of instabilities is maintained, the new modes do *not* have indeterminate phases. However, only a small number of modes need be excited to produce the required spectrum. (The number of modes participating in the transition is, as of now, an open experimental question.) Indeed, knowledge of the phases of a small number of amplitudes at an early stage of period doubling suffices to determine the phases of the transition

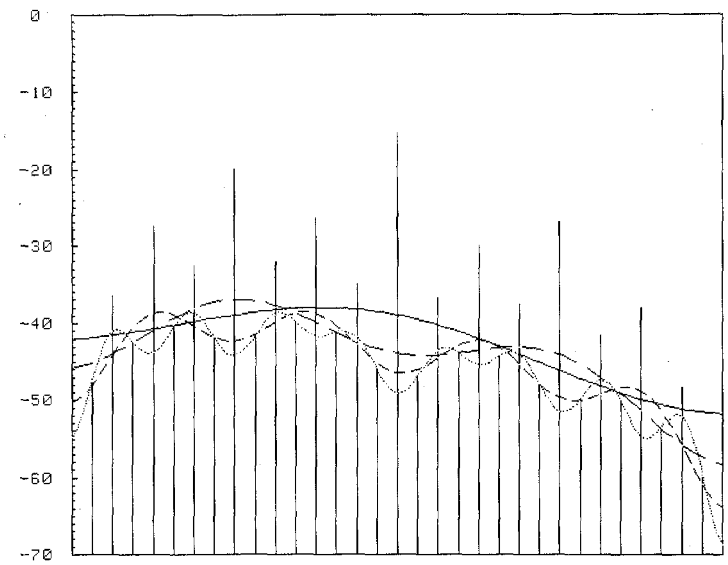

Fig. 15. The subharmonic spectrum of Duffing's equation in the 32-cycle. The dotted curve is an interpolation of the odd 32nd subharmonics. The shorter dashed curve is constructed similarly for the odd 16th subharmonics, but lowered by 8.2 dB. The longer dashed curve of the 8th subharmonics has been dropped by 16.4 dB, and the solid curve of the 4th subharmonics by 24.6 dB.

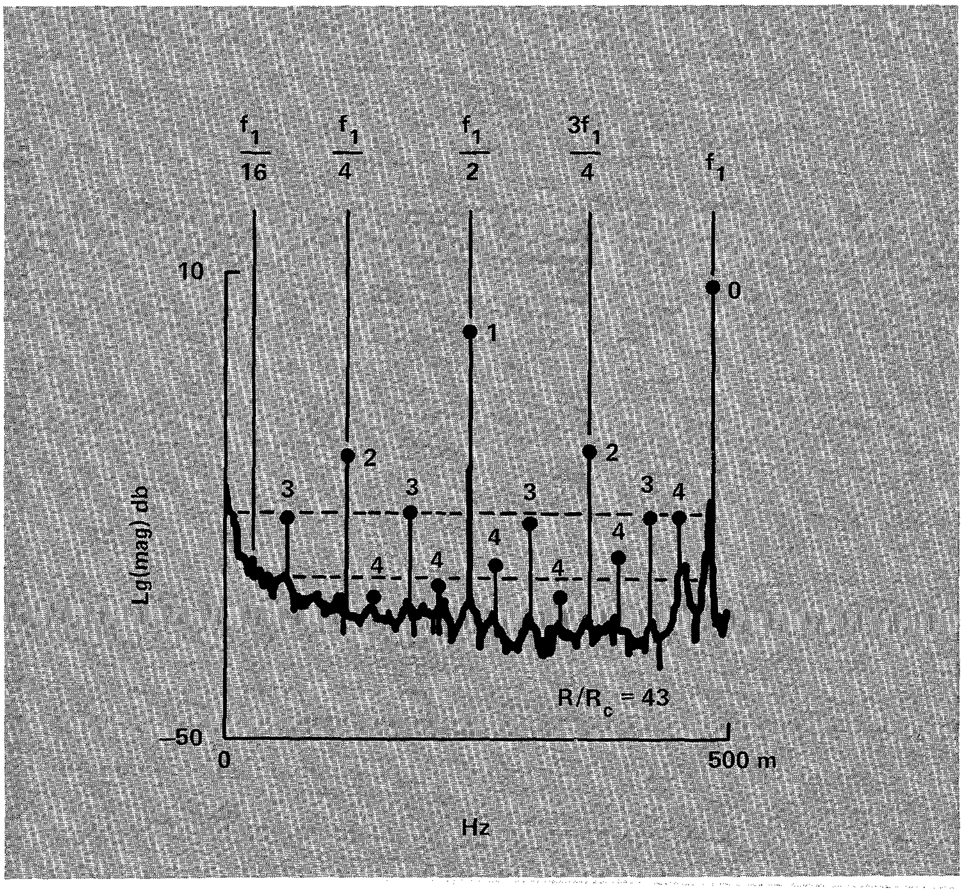

Fig. 16. The experimental spectrum (redrawn from Libchaber and Maurer) of a convecting fluid at its transition to turbulence. The dashed lines result from dropping a horizontal line down through the odd 4th subharmonics (labelled 2) by 8.2 and 16.4 dB.

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Mitchell J. Feigenbaum is a mathematical physicist and inventor of the universality theory. He earned his bachelor of electrical engineering degree from the City College of New York in 1964 and his Ph.D. in elementary particle physics from Massachusetts Institute of Technology in 1970. He was a Research Associate at Cornell University and Virginia Polytechnic Institute before becoming a Staff Member in the Theoretical Division at LASL in 1974. In May, 1980, he received a LASL Distinguished Performance Award for his universality theory and development of the first quantitative understanding of the onset of turbulence. He is presently writing a monograph on this work for the series *Progress in Physics* published by Birkhauser Boston, Inc. His other technical interests include field theories and functional integrals, the renormalization group and phase transitions, and nonlinear dynamics generally. His peers at LASL often turn to him as the resident consultant on related technical matters.

spectrum. What is important is that a purely causal system can and does possess essentially statistical properties. Invoking *ad hoc* statistics is unnecessary and generally incompatible with the true dynamics.

A full theoretical computation of the onset demands the calculation of successive instabilities. The method used traditionally is perturbative. We start at the static solution and add a small time-dependent piece. The fluid equations are linearized about the static solution, and the stability of the perturbation is studied. To date, only the first instability has been computed analytically. Once we know the parameter value (for example, the Rayleigh number) for the onset of this first time-varying instability, we must determine the correct form of the solution after the perturbation has grown large *beyond* the linear regime. To this solution we add a new time-dependent perturbative mode, again linearized (now about a time-varying, nonanalytically available solution) to discover the new instability. To date, the second step of the analysis has been performed only numerically. This process, in principle, can be repeated again and again until a suitably turbulent flow has been obtained. At each successive stage, the computation grows successively more intractable.

However, it is just at this point that the universality theory solves the problem; it works only after enough instabilities have entered to reach the asymptotic regime. Since just two such instabilities already serve as a good approximate starting point, we need only a few parameters for each flow to empower the theory to complete the hard part of the infinite cascade of more complex instabilities.

Why should the theory apply? The fluid equations make up a set of coupled field equations. They can be spatially Fourier-decomposed to an infinite set of coupled ordinary differential equations. Since a flow is viscous, there is some smallest spatial scale below which no significant excitation exists. Thus, the equations are effectively a finite coupled set of nonlinear differential equations. The number of equations in the set is completely irrelevant. The universality theory is generic for such a dissipative system of equations. Thus it is possible that the flow exhibits period doubling. If it does, then our theory applies. However, to prove that a given flow (or any flow) actually should exhibit doubling is well beyond present understanding. All we can do is experiment.

Figure 16 depicts the experimentally measured spectrum of a convecting liquid helium cell at the onset of turbulence. The system displays measurable period doubling through four or five levels; the spectral components at each set of odd half-subharmonics are labelled with the level. With $n = 2$ taken as asymptotic, the dotted lines show the crudest interpolations implied for the $n = 3$, $n = 4$ component. Given the small amount of *amplitude* data, the interpolations are perforce poor, while ignorance of higher odd multiples prevents construction of any significant interpolation at the right-hand side. Accordingly, to do the crudest test, the farthest right-hand amplitude was dropped, and the oscillations were smoothed away by averaging. The experimental results, −8.3 dB and −8.4 dB, are in surprisingly good agreement with the theoretical 8.2!

From this good experimental agreement and the many period doublings as the clincher, we can be confident that the measured flow has made its transition according to our theory. A measurement of δ from its fundamental definition would, of course, be altogether convincing. (Experimental resolution is insufficient at present.) However, if we work backwards, we find that the several percent agreement in 8.2 dB is an *experimental observation* of α in the system to the same accuracy. Thus, the present method has provided a theoretical calculation of the actual dynamics in a field where such a feat has been impossible since the construction of the Navier-Stokes equations. In fact, the scaling law Eq. (65) transcends these equations, and applies to the *true* equations, whatever they may be.